University of Reading

School of Mathematics

American Spread Option Pricing

by

Aanand Venkatramanan

August 2005

———————————————————

This dissertation is submitted to the Department of Mathematics in partial

fulfillment of the requirements for the degree of Master of Science.

American Spread Option Pricing

Aanand Venkatramanan

August 21, 2005

Abstract

We discuss the pricing of American spread options on correlated assets when the marginal distribution of each asset return is a mixture of normal distributions. Leaning upon the existing pricing models a substantial time is devoted in extending the Bivariate normal mixture(BNM) model to price American spread options calibrated to both volatility smiles and the correlation frown. Firstly we calibrate the Univariate normal mixture(UNM) model and then the Bivariate normal mixture model to the market prices of European options. Since the Black-Scholes(BS) model over different volatilities is equivalent to the market model we use the BS model instead. These calibrated models will be used to find the American option price using a 3-D binomial tree approach.

Declaration

I confirm that this is my own work, and the use of all material from other sources has been properly and fully acknowledged.

Contents

List of Figures

iv

Notations

Throughout this report we follow the usual notations given below:

K - strike price

T - time to expiry / residual maturity

S_i - price of the stock i

σ_i - volatility of the stock i

ρ - correlation between two stocks

r - risk-free interest rate

q - dividend of the stock

p - option price

f - option price function

Chapter 1

Introduction

Spread options are very simple in nature, but the difficulty in obtaining a closed form solution for pricing them forces one to adopt numerical techniques. In simple words, they can defined as a derivative product on several underlying assets. An exposition of spreads in different markets is presented in Carmona and Durrleman (2003), along with specific examples from energy markets and spread option pricing techniques.

Many models that price ordinary single asset options [1] assume that the asset price is log-normally distributed and follows a Geometric Brownian motion (GBM). This assumption is what leads to an analytic solution (for the price) although we start with a stochastic process.

In the case of spread options, the assumption no longer helps as a linear combination of log-normal processes is not log-normal. Moreover, assuming the spread itself to have dynamics governed by univariate diffusion process would not take into account the correlation between the underlying assets. The difficulty in obtaining their price is quite contrary to their simpler nature.

---

[1] also called vanilla options

In addition to this, the early-exercise feature of American options makes it even worse. Although there has been extensive research in this field, no efficient and accurate pricing model has yet been developed.

Some of the work done so far for pricing American spread options can be found in Broadie and Detemple (1997), Broadie and Glasserman(1997,2004), Longstaff and Schwartz(2001) and others. Broadie and Detemple characterize the optimal exercise regions and provide valuation formulas for a number of American option contracts on multiple underlying assets with convex/non-convex payoff functions. Broadie and Glasserman(1997), and Longstaff and Schwartz(2001) use simulation methods while Broadie and Glasserman(2004) discuss a stochastic mesh method for pricing high-dimensional (multiple underlying assets) American options.

In this project, we explore the possibility of extending few of the existing (tested) vanilla and spread option pricing models to achieve the same. This new model would be an amalgam of both analytical and numerical models that are consistent with the market data.

The following chapter introduces the basics of vanilla and spread options and familiarizes the jargon used in this report. It also discusses two of the most important parameters, *implied volatility and implied correlation*, in pricing spread options.

In chapter 3, we discuss in brief the various types of pricing models available for pricing vanilla and spread options and detail those used in this project. The models that we shall see later are Black-Scholes model, Univariate normal mixture model, and Binomial tree model that are used to price vanilla options and Kirk's model, Bivariate normal mixture model for

spread options.

We then move on to price American spread options in chapter 4, by introducing the 3-D binomial tree model. How these models mentioned so far are extended to price American spread options, consistent with the market model [2] is explained in sec. 4.2.

The implementation of these approaches using C++ and the various modules in the program are explained in chapter. 5.2. The last two chapters deal with the output and analysis of the program results where we do a comparative study of the performance of the code and the market models.

---

[2]A model that is consistent with the market data and behaviour

Chapter 2

Spread Options

2.1 Option basics

An *option* is a contract that provides the holder the right to buy or sell a particular stock on or before a specified date. A *call/put* is an option to buy/sell an underlying security for a fixed price on or before a certain date. The fixed price upon the contract is the *exercise price/strike price* (K) and enforcing the contract is to *exercise the option*. The *expiration date* is the last date on which the option is still valid.

Options can be broadly classified into Vanilla and Exotic options. Vanilla options are plain ordinary options with no special features - calls and puts, whereas exotic options are not ordinary options. They are also classified based on when they can be exercised. American options are those that can be exercised on any date up to the expiry date while European options can be exercised only on the expiry date. This option of *American options* is what makes them more expensive than their European counterparts. A *payoff* is the value of option at exercise given by $max[0, (S_T - K)]$ *or* $max[0, (K - S_T)]$

for a call or put respectively. Since the payoff depends only on the stock price for a particular strike, tracking the stock price movement would be a useful tool in option pricing. Hence we have *volatility* as a measure of dispersion in a stochastic process like stock price. The *moneyness* of an option is the potential yield (profit or loss) of an option if exercised immediately. An option may be an in-the-money, at-the-money or out-of-the-money options. In-the-money options are profitable when exercised, while out-of-the-money are not. At-the-money options yield neither profit nor loss when exercised.

Calibration is the means of fixing the model parameters like volatility and correlation to the market prices. Once calibrated the model can be expected to produce results close to markets expectations.

2.2 Implied Volatility

Implied volatility is the forecast of the average volatility in the underlying price dynamics over the residual maturity T of the option that is implicit in a market price of an option. estimated volatility of the price of the underlying asset.

It is specific to an option pricing model because we obtain implied volatility by equating a model price to an observed market price. It is computed by solving an inverse linear problem(LP) for a given stock price and parameters.

$$p = LP(\sigma) \qquad \rightarrow \qquad LP^{-1}(p) = \sigma$$

When we speak of implied volatility, by default we mean the Black-Scholes implied volatility. In the Black-Scholes world the option prices are calculated by assuming that the volatility is constant for different strikes and
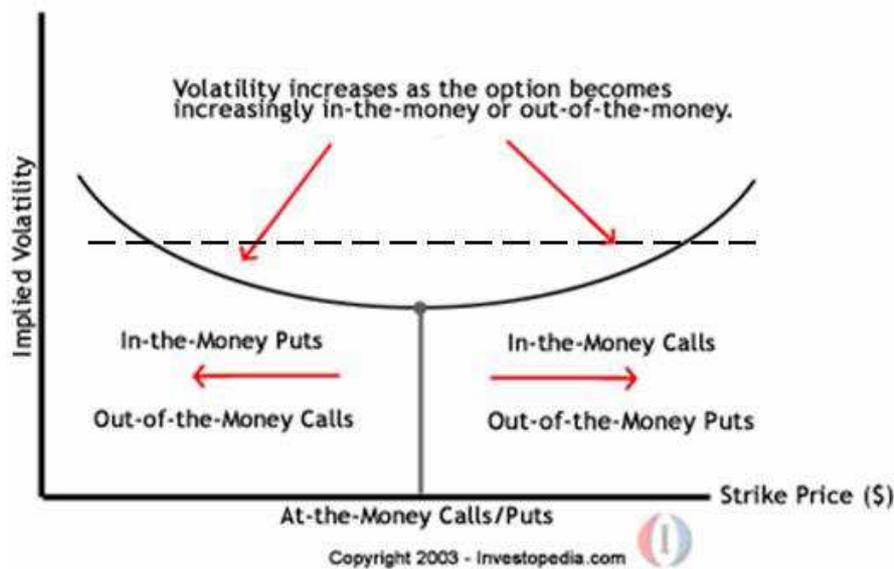
Figure 2.1: Volatility smile (source: www.investopedia.com)

maturity. Although the Black-Scholes(BS) model performs well the assumption is proved to be flawed. When the volatility is computed (implied) by a model for a set of market prices for different strikes, the volatility is not observed to be a constant rather it is skewed. In the case of currency option markets the implied volatility of in-the-money and out-of-the-money options is greater than the at-the-money options as shown in the figure. Hence the volatility smiles in this case! This is explained by the fact that traders speculate a larger price movement than is assumed in the BS model. Since every other parameter is a constant in the BS model the disparity in the computed and market prices can be explained only by increasing the volatility.

2.3 Spread options

Spread options are derivative products on two or more assets. Most often they are referred to those written on the difference between the values of two indexes. For example, a European call spread on two underlying assets with prices S_1 and S_2 will have a pay off function $[S_1 - S_2 - K]^+$. The + superscript denotes that payoff can only be positive, for any negative value it equals zero. On a broader perspective it includes all forms of options written as a linear combination of a finite set of indexes. Spread options occur in different markets, to name a few - currency and fixed income markets, agricultural futures markets, equity markets and energy markets. They are also classified on what parameters the price difference of the underlying asset is based upon. For example, in the commodity markets, spread options are based on the difference between the prices of the same commodity at two different location (location spread) or at two different points of time (calendar spread). Spread options also come as an American, European or Bermudan option.

2.4 Implied Correlation

The most important characteristic of a spread option is the correlation between the underlying assets. *Correlation* can be defined as a parameter that gauges the extent to which a price movement in one asset influences the other. It serves as a major instrument for trading correlation opening up a new dimension of trading.

Just like implied volatility, when the correlation is backed out of an spread option pricing model for a given pair of stock prices and volatility, we obtain

implied correlation. It is similar to the former in all respects except that it frowns and does not smile! That is, the implied correlation is lesser for in-the-money and out-of-the-money options when compared with at-the-money options. Lower (or more negative) the correlation higher the moneyness of the spread option.

In general the sum of two normally distributed variates is itself a normal variate. Since we consider the difference between two stock prices S_1 *and* S_2 which are both log-normal stochastic processes, we have $log(S_1) - log(S_2) = log(S_1/S_2)$ a normal variate.

Hence we have, the volatility of the ratio of S_1 *and* S_2

$$\sigma_{1/2}^2 \;=\; var[ln(S_t^1/S_t^2)] \;=\; \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1^2\sigma_2^2$$

The negative sign in front of the correlation component in the above equation implies that if the volatility of $ln(S_t^1/S_t^2)$ were to increase (a smile) the correlation has to decrease (a frown).

Smile and frown consistency implies that the model captures the market behaviour and hence the prices computed by the model would be consistent with the market model.

Chapter 3

Option Pricing

3.1   A brief overview

The three most commonly found techniques to price spread options are Partial differential equation (PDE) solvers, tree methods and Monte Carlo methods. While the PDE solvers, as their name suggests, are solutions to the PDE, the tree and Monte Carlo are Numerical (and probabilistic) approximations to the price of spread options. In general, no analytic formulae exist for pricing spread options, with non-zero strikes in particular, until now. This is mainly due to the fact that a linear combination of correlated log-normals is not log-normal. The reader is referred to Carmona & Durrleman for a detailed and informed survey of most of the known methods available for pricing spread options.

As already mentioned, we will be extending the BNM model using 3-D tree approach in order to price American spread options. Before we see how this works it is essential to understand the models that form a part of it. The next few sections provide a brief overview of these methods. For a more

detailed illustration the reader is advised to refer to the works suggested therein.

3.2 Vanilla Option pricing

3.2.1 Black-Scholes(BS) model

The most earliest and powerful tool to compute the price of European options was discovered by Black and Scholes(1973). Even thirty years later it remains to be one of the most preferred model and serves as the basis for many others in the world of options theory. It states that the price of a call option at a time t is given by the solution of the backward parabolic partial differential equation

$$\frac{\delta p_0}{\delta T} = (r-q)S_0\frac{\delta p_0}{\delta S_0} + \frac{1}{2}\sigma^2 S_0^2\frac{\delta^2 p_0}{\delta S_0^2} - rp_0 \qquad (3.1)$$

with terminal conditions

$p(T,x) = (x-K)^+$

$lim_{S_0\to 0}p(S_0,T) \to 0$

$lim_{S_0\to\infty}p(S_0,T) \to S_0 e^{-qT} - Ke^{-rT}$

The stock price is assumed to follow the stochastic process

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dW_i dt, \qquad i = 1,2 \qquad (3.2)$$

where dZ is a normally distributed random variable, μ and σ are functions of S and t. The Black Scholes formula gives a value for p when $S(T)$ has a log-normal distribution under no-arbitrage condition. In general,

$$p = \mathbf{E}\{e^{-rT}(S(T) - K)^+\}$$

where r is the short rate of interest. We then have,

$$p = S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2) \tag{3.3}$$

where

$$d_1 = \frac{\ln\left(\frac{S(0)e^{rT}}{K}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \quad and \quad d_2 = d_1 - \sigma\sqrt{T} \tag{3.4}$$

Here $\Phi(x)$ represents the cumulative distribution function of the standard normal $N(0,1)$ distribution, i.e.,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-u^2}{2}}\,du \tag{3.5}$$

In the case of spread option pricing, according to the Black-Scholes model, the price of the spread option is given by the risk-neutral expectation[1] :

$$p = e^{-rT}\mathbf{E}_Q\{(S_2(T) - S_1(T) - K)^+\} \tag{3.6}$$

3.2.2   Univariate normal mixture model

The Univariate normal mixture model was introduced by Brigo and Mercurio (2001). It is a no-arbitrage single asset option valuation, consistent with log-normal mixture asset price dynamics. In this approach the marginal distributions of each asset return is assumed to be a mixture of normal distributions. The volatility σ can be expressed as a weighted sum of two different volatilities σ_C and σ_T, each representing the core and tail volatilities of the log-normal distribution of the stock prices. We have, due to the linearity of the density function,

$$f_{1,t}(s_1) = \lambda_1\Phi(s_1;\mu_{11},\nu_{11}(\tau)) + (1-\lambda_1)\Phi(s_1;\mu_{12},\nu_{12}(\tau))$$

---

[1]Expectation when the probability measure Q does not allow for any arbitrage

The UNM price expressed as a weighted sum of two prices corresponding to each of the volatilities is given by,

$$f_{UNM}(\sigma) \quad = \quad \lambda f_{BS}(\sigma_C) \; + \; (1 - \lambda)f_{BS}(\sigma_T) \tag{3.7}$$

and

$$\sigma \quad = \quad \lambda \sigma_C \; + \; (1 - \lambda)\sigma_T \tag{3.8}$$

The Bivariate normal mixture model which is a two-dimensional extension of this approach will be discussed in section 3.3.2 along with the advantages of adopting this approach. The reader is referred to Brigo and Mercurio(2001) for a detailed research on this model.

3.2.3   Binomial tree approach

The binomial tree approach works by capturing the randomness of the price movement of a stock. This model is based on the random walk which is discussed in detail by James(2002) in his book option theory. The model converges to the log-normal distribution for stock price movements, when the number of steps is large, i.e., the computed price converges to the analytic formulas based on a log-normal assumption for the stock price movements. The reader is referred to Hull(2000) for further reading.

Consider a random walk with forward and backward step lengths U and D and respective probabilities p and $1 - p$. If x_n is the distance travelled after n steps of the random walk, then

- $\mathbf{E}[x_N] \; = \; N\{pU - (1 - p)D\}$

- $var[x_N] \; = \; Np(1 - p_(U + D)^2$

- The distribution of x_N is a binomial distribution which approaches the normal distribution as $N \to \infty$

James(2002) starts by describing a single step binomial process and then the binomial network. We shall now see how this approach can be used to price European (vanilla) calls. For the sake of simplicity we assume that the interest and the dividend terms are in continuous form. Let u and d be constant multiplicative factors, for an upward or downward movement with probability p and $(1-p)$ respectively. Then the price of the stock at the n^t time step is given by

$$S_0 \;=\; e^{-r\delta t}(pS_u + (1-p)S_d)$$

and the option price

$$f_0 = \; e^{-r\delta t}(pf_u + (1-p)f_d)$$

It is important to note that the p is the risk-neutral probability measure satisfying the no-arbitrage principle, given by

$$p \;=\; \frac{S_0 e^{-(r-q)\delta t} - S_d}{S_u - S_d} \tag{3.9}$$

We now construct the binomial tree as described in the following few steps:

- Choose suitable values of u, d, and p.

- Calculate the value of the stock at every node in the tree until the last column (N) of nodes corresponding to final time T. We have,
  $$S^{(u)}_{k+1,i+1} = uS_{k,i} \quad and \quad S^{(d)}_{k,i+1} = dS_{k,i}, \;\; where \; i \in [1,N] \; and \; k \in [1,i]$$
  Since we are pricing only European options, it would suffice if we compute the stock values at final time T.

- At every node corresponding to the final time T we must now have a stock price $S_{j,N}$.

- Assuming that the derivative depends only on the final stock price we can calculate the derivative payoff $f_{j,N} = [S_{j,N} - K]^+$, where $j \in [1, N]$

- We can then calculate the price of the derivative at every node using
  $$f_{k,i} = e^{-(r-q)\delta t}(pf_{k+1,i+1} + (1-p)f_{k,i+1})$$

- The price thus obtained at the initial node corresponding to time $t = 0$, is our required European option price.

Binomial trees really come into application when pricing American Options, as there does not exist any accurate closed form analytic solution for pricing them. The procedure to price American options remains the same excepting for the early exercise condition that has to be taken into account. This can be achieved by a small addition in the procedure. Our new procedure looks like as follows:

- Calculate the value of the stock at every node in the tree until the last column (N) of nodes corresponding to final time T. We have,
  $$S^{(u)}_{k+1,i+1} = uS_{k,i} \quad and \quad S^{(d)}_{k,i+1} = dS_{k,i}, \quad where i \in [1, N] \ and \ k \in [1, i]$$

- At every node corresponding to the final time T we must have a stock price $S_{j,N}$.

- Assuming that the derivative depends only on the final stock price we can calculate the derivative payoff $f_{j,N} = [S_{j,N} - K]^+$, where $j \in [1, N]$

- We can then calculate the price of the derivative at every node using
  $$f_{k,i} = e^{-(r-q)\delta t}(pf_{k+1,i+1} + (1-p)f_{k,i+1})$$

- If the calculated option price is lesser then the value of the pay off function $f_{j,i} = [S_{j,i} - K]^+$, $i \in [1, N]$, then we replace the former by
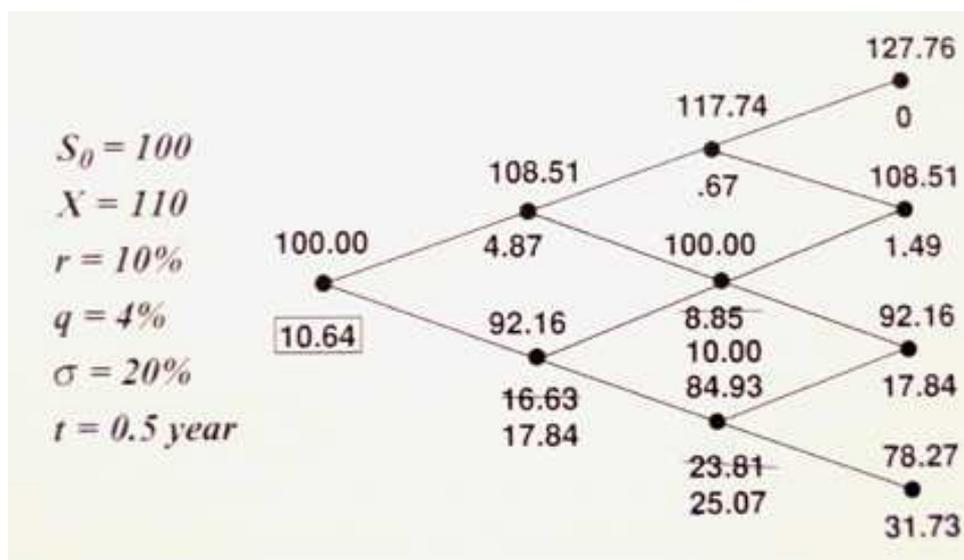
Figure 3.1: 2-D Binomial tree

the latter and continue calculating the option price in the preceding level.

- The price thus obtained at the initial node corresponding to time $t = 0$, is the required American option price.

Figure 3.1 shows how the American option price is calculated.

3.3 Spread option pricing

3.3.1 Kirk's formula

The need for a closed form analytic solution for pricing spread options with non-zero strike was partially fulfilled by Kirk(1995) who recently proposed a closed form approximation for the same. The derivation of the formula

was based on the Black-Scholes price for spread options expressed as an expectation of the payoff function, as in 3.6. The formula is as follows:

$$\hat{p} \;=\; x_2\Phi\left(\frac{\ln\left(\frac{x_2}{x_1+Ke^{-rT}}\right)}{\sigma^K}+\frac{\sigma^K}{2}\right)-(x_1\;+\;Ke^{-rT})\Phi\left(\frac{\ln\left(\frac{x_2}{x_1+Ke^{-rT}}\right)}{\sigma^K}-\frac{\sigma^K}{2}\right)$$

where

$$\sigma^K = \sqrt{\sigma_2^2\;-\;2\rho\sigma_1\sigma_2\frac{x_1}{x_1+K\exp^{-rT}}\;+\;\sigma_1^2\left(\frac{x_1}{x_1+K\exp^{-rT}}\right)^2} \qquad (3.10)$$

Carmona and Durrleman performs a comparative study of how this model performs against other models. A more refined approach can be found in Eydeland and Wolyniec(2003).

3.3.2 Bivariate normal mixture model

As mentioned earlier the Bivariate normal mixture model is a generalisation of the UNM model to two dimensions. A striking feature of this model is it's simplicity although analytic. Alexander and Scourse (2004) start by assuming that each marginal log price density at time T is given by a mixture of two normal components. They start with the Brigo and Mercurio(2001) closed-form solution for each of the two vanilla option prices, and then extend this to a smile and frown consistent model for European spread option that has an analytic approximation that extends that of Kirk(1995). This leads to,

$$f_{1,t}(s_1) \;=\; \lambda_1\Phi(s_1;\mu_{11},\nu_{11}(\tau)) \;+\; (1-\lambda_1)\Phi(s_1;\mu_{12},\nu_{12}(\tau))$$
$$f_{2,t}(s_1) \;=\; \lambda_2\Phi(s_2;\mu_{21},\nu_{21}(\tau)) \;+\; (1-\lambda_2)\Phi(s_2;\mu_{22},\nu_{22}(\tau)) \quad (3.11)$$

where Φ is the bivariate normal density function, $\mu_{ij}'s$ are the mean vectors and ν_{ij} is the τ-period variance of the j_{th} normal component for asset i.

Hence one can express the bivariate normal mixture joint density at T as

$$f_t(s_1, s_2) = \lambda_1\lambda_2\Phi(s_1, s_2; \mu_1(\tau), V_1(\tau)) + (1 - \lambda_1)\lambda_2\Phi(s_1, s_2; \mu_2(\tau), V_2(\tau))$$

$$+\lambda_1(1 - \lambda_2)\Phi(s_1, s_2; \mu_3(\tau), V_3(\tau)) + (1 - \lambda_1)(1 - \lambda_2)\Phi(s_1, s_2; \mu_4(\tau), V_4(\tau))$$

$$(3.12)$$

where $s_1(t)$ and $s_2(t)$ denote the log prices of the two assets at time t and $V_i's$ are the covariance matrices.

The covariance matrices $V_i's$ are defined as follows:

$$\mathbf{V}_1 = \begin{pmatrix} \sigma_{1C}^2 & cov_1 \\ cov_1 & \sigma_{2C}^2 \end{pmatrix}, \quad \mathbf{V}_2 = \begin{pmatrix} \sigma_{1T}^2 & cov_2 \\ cov_2 & \sigma_{2C}^2 \end{pmatrix}$$

$$\mathbf{V}_3 = \begin{pmatrix} \sigma_{1C}^2 & cov_3 \\ cov_3 & \sigma_{2T}^2 \end{pmatrix}, \quad \mathbf{V}_4 = \begin{pmatrix} \sigma_{1T}^2 & cov_4 \\ cov_4 & \sigma_{2T}^2 \end{pmatrix} \qquad (3.13)$$

From this relation one can directly express the price of the option as a combination of prices with different correlations and volatilities. Below we shall see how normal mixture prices of European options are a weighted sum of BS prices based on different volatilities. Note that the 'overall' correlation ρ between the two price processes is given by the weighted average of four correlations.

$$E(\rho) = \lambda_1\lambda_2\rho_{CC} + (1-\lambda_1)\lambda_2\rho_{TC} + \lambda_1(1-\lambda_2)\rho_{CT} + (1-\lambda_1)(1-\lambda_2)\rho_{TT} \quad (3.14)$$

The four correlations correspond to the four components of the bivariate normal mixture that are core, core-tail, tail-core, and tail-tail. Finally, the bivariate normal mixture (BNM) spread option price is expressed as a linear combination of these prices:

$$P_t(\sigma_1, \sigma_2, \rho) = \lambda_1\lambda_2 P_{2GBM}(\sigma_{1C}, \sigma_{2C}, \rho_{CC}) + (1 - \lambda_1)\lambda_2 P_{2GBM}(\sigma_{1T}, \sigma_{2C}, \rho_{TC})$$

$$+\lambda_1(1 - \lambda_2)P_{2GBM}(\sigma_{1C}, \sigma_{2T}, \rho_{CT}) + (1 - \lambda_1)(1 - \lambda_2)P_{2GBM}(\sigma_{1T}, \sigma_{2T}, \rho_{TT})$$

σ_{iC}, σ_{iT} are the volatilities of core and tail normal densities. P_{2GBM} is the price of 2-Geometric Brownian motion model(2GBM). The 2GBM models assume two correlated log-normal diffusions to model European spread options (Ravindran 1993, Shimko 1994, Kirk 1995, James 2002 and others).

The difference here is that the terminal risk neutral density will be a bivariate normal mixture instead of bivariate normal, but the transition probabilities still remains normal. An interesting fact is that although the option price is a linear combination at time t = 0 and T (bivariate normal mixture), at time $t = \upsilon$ one can uniquely identify the price $(P'^{s}_{2GBM} \ - \ as \ in \ A)$, corresponding to a particular probability density that would yield the optimal price.

Chapter 4

Pricing American Spread Options

4.1   3-D tree model

The three dimensional binomial tree model for two asset options is shown in figure 4.1. The space variables used are $x_t = \ln S_t^{(1)}/S_0^{(1)}$ and $y_t = \ln S_t^{(2)}/S_0^{(2)}$ instead of the stock prices themselves. This means that the step sizes are of constant sizes, rather than proportional to the stock prices, hence making it simpler. The first node in the tree has value zero. If the risk-neutral drift of $S_1^{(1)}$ is $r - q_1$, then the drift of x_t is $r - q_1 - \frac{1}{2}\sigma_1^2 = m_x$, and y_t is $r - q_2 - \frac{1}{2}\sigma_2^2 = m_y$. The correlation between the two assets makes it difficult to find the nodal values since the value of $y_{\delta t}$ will depend on the value of $x_{\delta t}$. Without loss of generality, we choose the transition probabilities to each node in the successive level to be $\frac{1}{4}$, and allow $x_\delta t$ to have only two values: x_u and x_d. Here $y_{\delta t}$ takes different values relative to $x_{\delta t}$ in different nodes.

Figure 4.1: 3-D tree structure

The Wiener processes for the two space variables can be written as

$$\delta x_t = m_x \delta t + \sigma_x \sqrt{\delta t} z_1$$

$$\delta y_t = m_y \delta t + \sigma_x sqrt \delta t z_2 = m_y \delta t + \sigma_x \sqrt{\delta t} \{\rho z_1 + \sqrt{1 - \rho^2} z_3\}$$

where z_1 and z_3 are uncorrelated standard normal variates.

Hence the following equations:

$$\delta y_\alpha = m_y \delta t + \sigma_x \sqrt{\delta t} \{\rho + \sqrt{1 - \rho^2}\}$$

$$\delta y_\beta = m_y \delta t - \sigma_x \sqrt{\delta t} \{\rho - \sqrt{1 - \rho^2}\}$$

$$\delta y_\gamma = m_y \delta t - \sigma_x \sqrt{\delta t} \{\rho + \sqrt{1 - \rho^2}\}$$

$$\delta y_\delta = m_y \delta t + \sigma_x \sqrt{\delta t} \{\rho - \sqrt{1 - \rho^2}\} \tag{4.1}$$

For a detailed discussion on 3D tree models see James(2002). A more advanced 3-D tree approach can be found in Boyle(1988).

4.2   Extension of BNM model

We aim to extend the frown consistent[1] bivariate normal mixture model introduced by Alexander and Scourse (2004) for pricing European spread options to American Spread Options. The need for American option prices that are consistent with the market prices of European options requires us to use prices obtained from a *smile consistent model*[2]. We assume that the marginal distribution of each correlated asset return is a mixture of normal distributions.

Leaning upon the existing volatility models a substantial time would be dedicated in extending the Bivariate normal mixture(BNM) model (Alexander and Scourse, 2004) for pricing American spread options calibrated to both volatility smiles and the correlation frown. Alexander and Scourse assume that each asset return density is a mixture of two normal densities and that their joint density is a bivariate normal mixture.

Firstly we calibrate the univariate normal mixture(UNM) model and then the Bivariate normal mixture model to the market prices of European options. These calibrated models will be used to find the American option price using a 3-D binomial tree approach as described in James(2002). Since the BNM model is smile and frown consistent and also the univariate normal mixture model is smile consistent, the American spread option price obtained

---

[1]Computed prices are consistent with the correlation frown

[2]A option pricing model that is consistent with the volatility smile

using these models can be expected to exhibit the same.

Firstly, the univariate normal mixture model will be calibrated to a volatility smile by equating the model prices to Black Scholes prices. This is equivalent to calibrating it to the market prices as the Black-Scholes(BS) model prices are equal to the market prices when the latter is based on different implied volatilities for each option as in 4.2. Calibration was achieved by minimizing the square of the error between the model and BS prices, by using an iterative converging algorithm. In this project, the iterative Newtons method was used, the implementation of which will be discussed in the following section. We try to minimize the square of the error as its curve is smoother and the differential exists at the minima.

$$
\begin{aligned}
f_{bs}(\sigma 1) &= \lambda_1 f_{bs}(\sigma_{11}) + (1 - \lambda_1) f_{bs}(\sigma_{12}) \\
f_{bs}(\sigma 2) &= \lambda_2 f_{bs}(\sigma_{21}) + (1 - \lambda_2) f_{bs}(\sigma_{22})
\end{aligned}
\tag{4.2}
$$

The implied volatilities to which σ_{i1} and σ_{i2} are calibrated should be of the same maturity as the spread option.

Since BS prices are analytic, so are NM prices of European options. Hence the calibration of the univariate models can be done analytically. Similarly, in the bivariate normal mixture model for spread options, the model prices are a weighted sum of European spread option prices based on two correlated Brownian motions (2GBM prices). There is an analytic approximation for these 2GBM prices (Kirk, 1995) and hence also in the bivariate case, the model prices of European options are analytic (approximately).

A European option price surface for each asset (against strike price and maturity) could be interpolated and extrapolated using Finite element methods, from the finite set of values quoted in the market. However, since the

normal mixture model prices are analytic, we do not actually need to do this, as we would if prices were computed using numerical methods. Instead we can calibrate the model parameters based on just the quoted prices of European options in the market.

Once calibrated the model parameters (volatilities and correlations) are used to obtain the prices for European options with any combination of maturity and strike price and - more importantly - the price of American options.

First the 2GBM American option prices will be obtained using the three dimensional tree model as in James(2002) where each node is linked to four other nodes of the successive time step. Each link/branch carries a weight dictated by the correlation of the two assets and the volatility of each asset (or equivalently the transition probabilities). That is, given a covariance matrices $\mathbf{V}$ as defined in (3.13), an American spread option price is obtained using 3-D binomial tree model.

In order to calibrate the correlation (we already know the volatilities from calibration to univariate options) we follow the same procedure that we adopted in the univariate normal mixture case. Minimizing the square of the error between the 2GBM model price and the BNM model price as done while calibrating volatility would give us the required (calibrated) correlations.

Unlike the volatility the uncertainty in the calibrated values of the correlation to be frown consistent arises due to the fact that the implied correlation (to which the model parameters are calibrated) is by itself ill-defined. For, the strikes of the two implied volatilities σ_1 *and* σ_2 have a complex relation with the strike of the spread option. There are finitely many number

of pairs (S_1, S_2) for which $S_1 - S_2 = K$. In order to overcome this it was assumed that the strike convention used to calculate the implied volatility was $K_1 = S_1 - (K - S_1 + S_2)/2$ *and* $K_2 = S_2 - (K - S_1 + S_2)/2$. When the strike is zero they give rise to exchange options, which are more easier to handle. An analytic pricing formula for exchange options was first derived by Margrabe(1978).

For the sake of simplicity, we assume that $\sigma_{i1} > \sigma_i$ *and* $\sigma_{i2} < \sigma_i$. One would expect $\sigma_{i2} \approx \sigma_i$ as that addresses the core volatility of the normal mixture. Without loss of generality we assume that $0 < \lambda < 0.5$. This implies that the higher volatility makes lower contribution and the lower volatility makes higher contribution to the overall volatility.

A similar argument applies for the correlation as well, where $\rho_{CC} <$ ρ *and* $\rho < \rho_{TT}$. We assume that $\rho_{CT} = \rho_{TC}$ and that ρ_{TT} takes values close to twice as that of ρ.

Then, three dimensional binomial trees are constructed using each of the above correlations $\rho_{CC}, \rho_{CT}, \rho_{TC}$, *and* ρ_{TT} and the corresponding volatilities. That is, each of the four covariance matrices, $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$, *and* $\mathbf{V}_4$ are taken in turn.

The linear combination of these prices given by 3.15 would give the required price of an American spread option. As always one would expect this price to be higher than the European spread option prices.

C++ and MATLAB is used to implement the above.

Chapter 5

Description of Code

The program calculates the value of an American spread option given the required set of data using Bivariate normal mixture model (sec. 3.3.2) and the 3-D binomial tree model (sec. 4.1). Calibration forms the heart of the code which involves finding of optimum values of volatilities and correlation and takes substantial amount of the entire computational time.

5.1 Modules

The whole program is divided into 4 modules. Each module is independent to the other with respect to their functionalities but not with data. The list of functions defined is given in 5.3The different modules are:

5.1.1 Univariate normal mixture module

This module uses the functions *blackscholes(), g(), dg() and phi().* The output is the square of the difference between the Univariate normal mixture model price and blackscholes price with a specified tolerance for a European

option on a single underlying asset. The tolerance value was assumed to be
0.05% of the Black-Scholes price for a call option.

The module involves a straight-forward implementation of equation (3.7).
The BS function calculates the price as in equation (3.3) by calling the func-
tion phi to calculate the cumulative density. Phi in turn uses Simpson's rule
to evaluate the line integral. The lower limit of integration is restricted to
-25 instead of $-\infty$ without any significant contribution to the error.

5.1.2   Bivariate normal mixture module

This implements the model described by Alexander and Scourse (2004) to
price European spread options. It uses the functions *g(), dg(), kirk() and
phi().* The calibrated values of the volatilities and lambdas from the UNM
module serve as the input for this module. This module is executed third
chronologically after UNM and calibration module.

This module too involves a straight-forward implementation of equations
3.6 and 3.10. The *kirk()* function calculates the price using (3.10) by calling
the function *phi()* given prices of two stocks.

The output of this module is the square of the difference between the
bivariate normal mixture model price and the Kirk's price.

5.1.3   Calibration module

This is the most important of all the modules. It finds the optimum values
for volatility and correlation for a set of inputs. Moreover, its output can
significantly alter the final output of the program due to the cascading effect
of errors. This module is run twice separately to optimise the volatilities of

two options and then to optimise the correlation of the spread option. A snapshot of the flow diagram is shown below:

Let us recall the assumptions that were made in 4.2:

- $\sigma_{i1} > \sigma_i$ and value of σ_{i1} is close to σ_i

- $\sigma_{i2} < \sigma_i$

- $\lambda_i \in (0, 0.5)$

- $\rho_{CC} < \rho$ and value of ρ_{CC} is close to ρ

- $\rho_{TT} > \rho$

- $\rho_{CT} = \rho_{TC}$

For computational reasons, we further restrict the values of λ_i to (0,0.1). The starting value of ρ_{CC} in Newtons method was chosen to be 1.3 times ρ.

This module is strongly threaded to both the UNM and BNM modules which are called for every iteration. The process of calibration involves the minimisation of the UNM/BNM price (output of UNM/BNM module) using the iterative Newton-Raphson method. The Newton-Raphson method was applied to a function of one variable by reducing f (that depends on $\sigma_1, \sigma_2, and \lambda$) to a univariate function. Firstly, we eliminate σ_2 by expressing it in terms of σ_1 and λ. Since the function f still depends on two variables, an easy way out would be to fix the value of λ to a constant value. We then have,

$$g_1 = [f_{bs}(\sigma_1) \; - \; \lambda_1 f_{bs}(\sigma_{11}) \; + \; (1 - \lambda_1) f_{bs}(c_1)]^2$$
$$g_2 = [f_{bs}(\sigma_2) \; - \; \lambda_2 f_{bs}(\sigma_{21}) \; + \; (1 - \lambda_2) f_{bs}(c_2)]^2$$

$$where \quad c_i = \frac{\sigma_i - \lambda_i \sigma_{i1}}{(1 - \lambda_i)}$$

$$and \ \lambda'_i s \ are \ constants. \tag{5.1}$$

Newtons method is then applied to this univariate function to obtain σ_{i1} for which g is a minimum. In this program we choose the starting value of λ as 0.01 and the above procedure is repeated for successive values with step size +0.01. The corresponding value pair (σ_{i1}, λ_i) for which g was a global minimum is chosen. Since we are interested only up to first 2 decimal places of lambda the choice of the step size for lambda is well justified. Moreover the iterations were performed only for $\lambda_i \in [0, 0.1]$ as it is very unlikely that $\lambda_i > 0.1$.

σ_{i2} is found by substituting $\sigma_{i1} \ and \ \lambda_i$ into

$$\sigma_{i2} = \frac{\sigma_i - \lambda_i \sigma_{i1}}{(1 - \lambda_i)}$$

We now have the calibrated values of *sigma11, sigma12, sigma21, sigma22, lambda1 and lambda2*. We are left with the calibrated correlation values before we can start pricing American spread options using the 3-D binomial tree approach. In order to calibrate the correlation we apply the same procedure as earlier but by assuming that *rhoct* equals *rhotc*. In this case we fix the value of *rhott* and try to find *rhocc* for which f is a minimum. This is repeated for successive values of *rhott* with step size 0.05 with starting value as + or - 1.

The calibrated volatilities, lambda and correlation values are then fed into the binomial tree module to find the price of American spread option.

5.1.4 Binomial tree module

This module calculates the American spread option price using the 3-D binomial tree discussed in 4.1. There is no interaction between this module and the rest. The calibrated values of correlation and volatilities, initial stock prices, time-step size and other usual data serve as the input to this module.

This module has an array implementation of a 3-D binomial tree where an array is logically manipulated as a tree with no physical links similar to a tree. In a 3-D tree as described in Option theory (James, 2002) the number of nodes in a particular level k equals k^2 allows for easy traversal within the tree. Hence the total number of nodes is given by $n(n+1)(2n+1)/6$. Since a 1-D array is implemented as a tree the index $'I'_k$ of the starting node of a particular level is given by $I_k = (k-1)k(2k+3)/6$. The nodes at a higher level are called parent nodes while their successors (in the successive level) are called daughter nodes. The terminal nodes are called leaf nodes.

The module is divided into three parts - building the tree, calculating the terminal option price and back-tracking the tree from leaf nodes to calculate the required price. Every node has three elements - stock price 1, stock price 2 and the spread option price. Every node, excepting the leaf nodes, is linked to four other nodes in the successive level. At any node the stock price S_1 and S_2 can be computed from one of its parent nodes. While back-tracking the numbering pattern is altered as shown in the figure. In this case the price at a particular node is given as a weighted sum of the prices of the four daughter nodes. The nodes are numbered as shown in the diagram 4.1.

The value of stocks at every node is given by the set of equations 4.1. The probability that a stock price could move up or down was chosen to

be 0.25. The maximum profit condition for American options is taken into consideration by including a conditional statement in the backtracking part. This statement compares the calculated price and the payoff function at that node and stores the greatest among them.

5.2   Working of code

Having explained the different modules we shall see how these are linked and executed as a whole. Figure 5.1 shows the flow of control from one module to the other. Each arrow represents a function call, with the arrow directed to the called function. The program was written in C++ and MATLAB was used for plotting the results. All the modules mentioned below function as discussed in the previous section.

The various functions defined are given in 5.3 along with their input parameters and functionalities. One would notice that there are few functions bearing the same name but have different associated functionalities. This is called as *function overloading* in C++. Function overloading allows for defining multiple functions under the same name, but with different set of input parameters. When a function is called, C++ automatically chooses the appropriate function by comparing the parameters. In the program the functions that are overloaded are *g(), dg(), and newton(). g()* calculates the value of function g as in (5.1), *dg()* calculates the first order differential and *newton()* performs the Newton's method for both volatility and correlation. This enhances the clarity of the code as name is associated more with the *function* of the function and not with a block of code!

Going back to figure 5.1 we can see that the function *main()* forms the
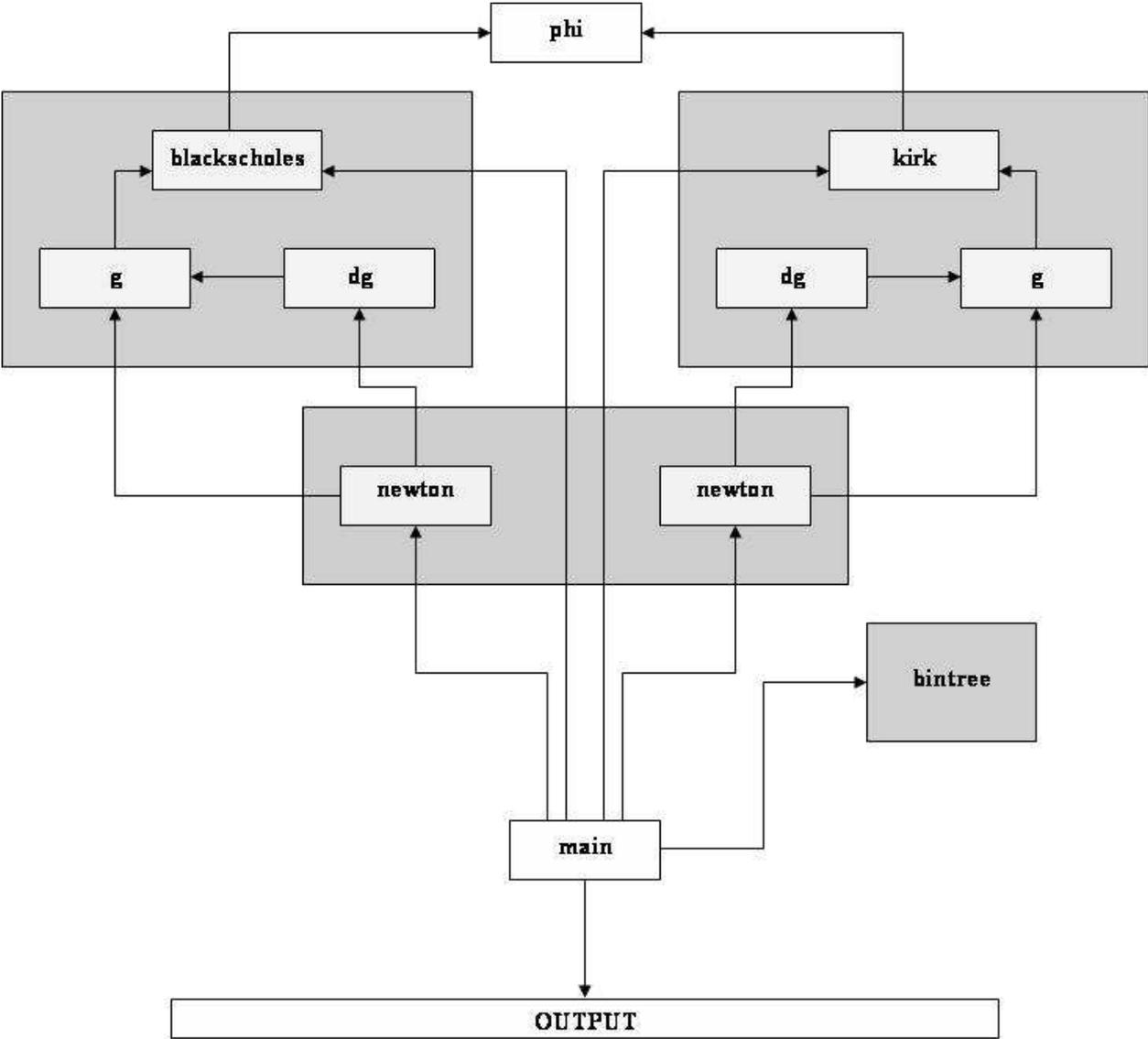
Figure 5.1: Control flow diagram

core, calling all the other modules. Since our aim is to obtain American spread option price it is imperative to obtain calibrated values of volatility and correlation at first. Moreover we need to calibrate volatility before calibrating correlation.

In order to calibrate the volatility we run the *newton()* routine to find optimum values of volatility. This inturn calls the UNM module to obtain the value of function g as in (5.1) and *dg*. The UNM module is called for every iteration of calibration for reasons discussed earlier.

The calibrated volatilities are then fed into the overloaded *newton()* routine, this time to calibrate correlation. The BNM module is called in this case for calculating the values of g *and* dg as in [equation].

The whole process of calibration involves running the *newtons()* routine for different values of λ_i/ρ_{TT}, incremented in successive iterations, to obtain the final values of volatility and correlation.

These values are then fed into the *bintree()* function that gives the required American spread option price. The binomial tree module runs the *bintree()* function for different time steps δt ranging between (0,T/2), where T is the final time.

We shall look at the results and plots in the following chapter.

5.3 List of functions

phi( )	
x	Calculates the probability density Φ. Uses Simpson's rule for evaluating the line integral from 0 to x.

blackscholes( )	
r, q, T, K, sigma, s	Gives the Black-Scholes price for a European option.

kirk( )	
r, q, T, K, sigma, s	Gives the Black-Scholes price for a European option.

g( )	
r, q, T, K, sigma, x, s, lambda	Gives the square of the difference between the Univariate normal mixture price and Black-Scholes price with a specified tolerance value that is assumed to be 0.05 % times the Black-Scholes price.

g( )	
r, q1, q2, T, K, sigma1, sigma2, sigma11, sigma21, s1, s2, lambda1, lambda2, rho, rhocc, rhott	Gives the square of the difference between the Bivariate normal mixture price and Kirk's price with zero tolerance value.

dg( )	
r, q, T, K, sigma, x, s, lambda	Finds the first order derivative of the function g with respect to sigma

dg( )	
r, q1, q2, T, K, sigma1, sigma2, sigma11, sigma21, s1, s2, lambda1, lambda2, rho, rhocc, rhott	Finds the first order derivative of the function g with respect to rhocc

newton( )	
r, q, T, K, sigma, x, s, lambda	Finds the value of sigma for which g is a minimum using Newton's method

newton( )	
r, q1, q2, T, K, sigma1, sigma2, sigma11, sigma21, s1, s2, lambda1, lambda2, rho, rho, rhott	Finds the value of rho for which g is a minimum using Newton's method

bintree( )	
r, q1, q2, T, K, sigma1, sigma2, s1, s2, rho, dt	Calculates the value of American Spread Option price using a 3D Binomial tree model for a specified time step dt.

Chapter 6

Analysis

In this section we shall look at the results of the program discussed in 5.2 and discuss its performance. We compare the output of the calibrated Univariate normal mixture(UNM) model with Black-Scholes' and that of Bivariate normal mixture (BNM) model with Kirk's. The behaviour of the prices obtained from each of these models are shown in the figures that follow. Unless specified the stock prices of assets 1 and 2 are taken to be 100. The volatility of stock 1 is 25% and stock 2 is 40%. The correlation between the stocks is -0.5.

Fig. 6.1 shows the Black-Scholes price (p_{BS}) as a function of strike and maturity. The Black-Scholes price increases linearly with strike as shown and tries to imitate the actual pay off function.

When $K < S$, the Black-Scholes price is comparatively low and when $K \geq S$, the price increases linearly with strike as shown in fig. 6.1. The change of price with respect to time to maturity (T) is lesser. Fig. 6.2 shows how the price curve shifts away from the actual payoff as T increases.

Fig. 6.3 shows how the UNM price function behaves with respect to strike and maturity for the calibrated values of volatility. As discussed in sec. 4.2, since the UNM model was calibrated to the BS model (with different volatil-
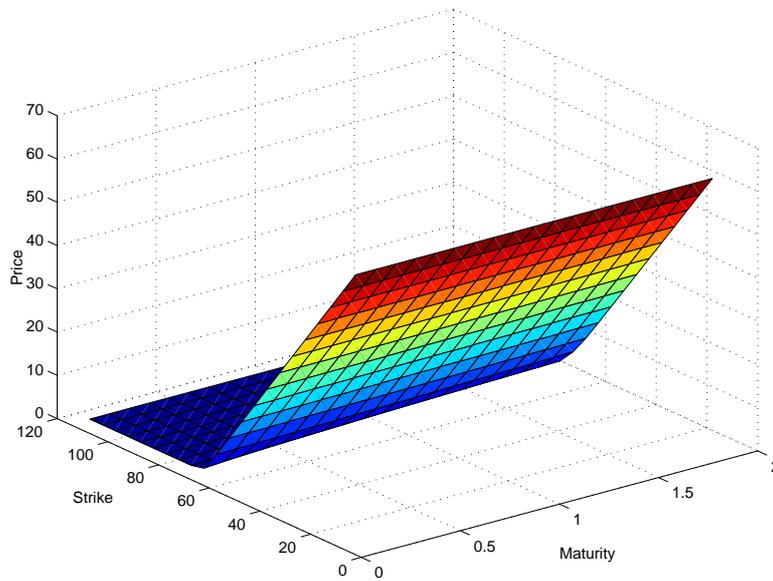
Figure 6.1: BS price versus strike and maturity
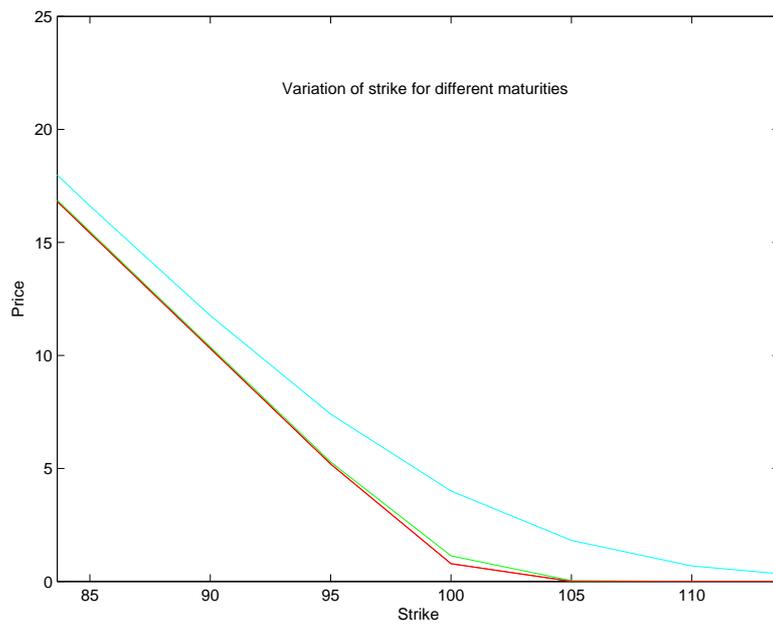
Figure 6.2: BS price versus strike - different maturities
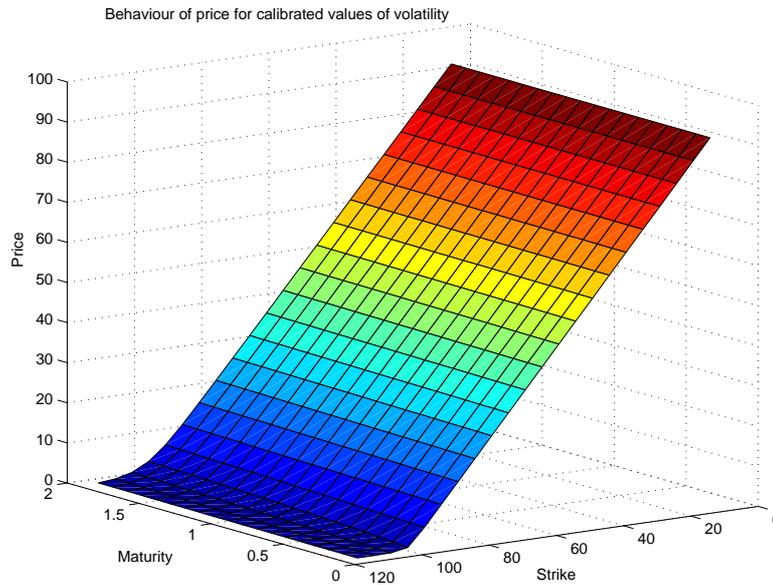
Figure 6.3: Calibrated UNM price variation with strike and maturity (stock1)

ities) it can be expected to behave in the same way as BS model. This can be verified in fig. 6.3. The only difference is that the curves slope upward for different strikes (100 in the former case and 70 in the latter) as the stock prices were taken to be different.

The higher volatility component has a lower effect on the price while the lower volatility component has a higher effect

A plot of the 'square of the error' (g) in the UNM price for stocks 1 and 2 is shown in figs. 6.6 and 6.7. They were plotted for $1 > \sigma_1 > \sigma$ and $0 < \lambda < 0.1$ (sigma2 varies accordingly).

When lambda is fixed, g decreases first and then increases as sigma1 decreases.

The graph is convex and hence the Newton-Raphson method can be im-
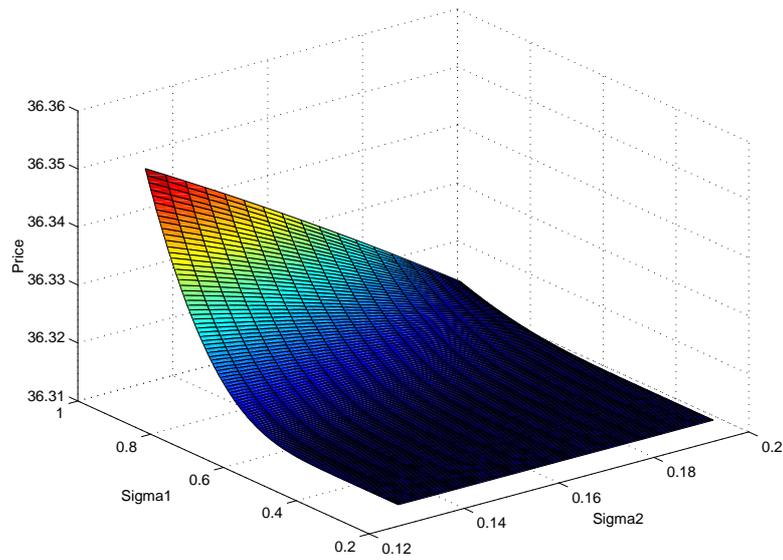
Figure 6.4: UNM price of stock 1 as a function of sigma1 and sigma2
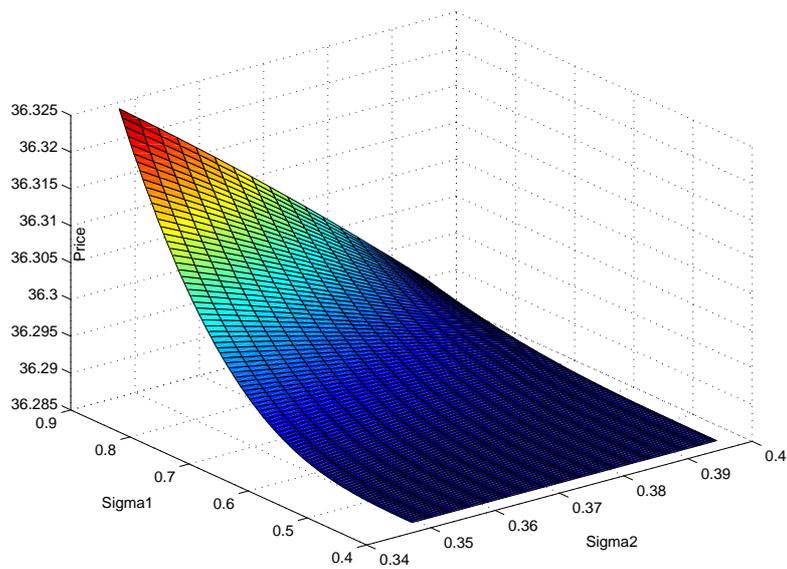
Figure 6.5: UNM price of stock 2 as a function of sigma1 and sigma2

plemented to find the minimum. For fixed lambda values the curves spread
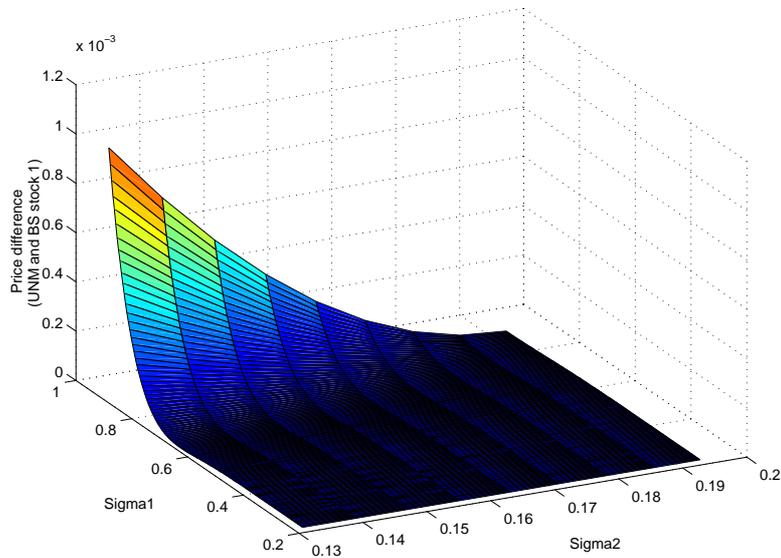
Figure 6.6: Plot of the square of difference between UNM and BS prices versus sigma1 and sigma1 - Stock 1

out with increasing price as sigma1 increases and sigma2 decreases. Figs. 6.8 and 6.9 shows the contours of 6.6 and 6.7 for a particular lambda value.

The observed pattern of the difference between the calibrated UNM price and BS price in fig. 6.11 suggests that the UNM prices are always greater than or equal the Black-Scholes prices. When $K = S$ the difference is a minimum and as the strike moves away from the stock price it increases and then drops. Since every parameter except for the volatility is the same in both the models the disparity in the prices can be explained by the BS model only by increasing the volatility. That is, the BS implied volatility smiles suggesting that the calibrated UNM model is *smile consistent*!

Fig. 6.12 shows the implementation of Newtons method for two different lambda values. The sigma1 for which g is a global minimum is the required value. Below is a plot showing the Kirks price as a function of strike and
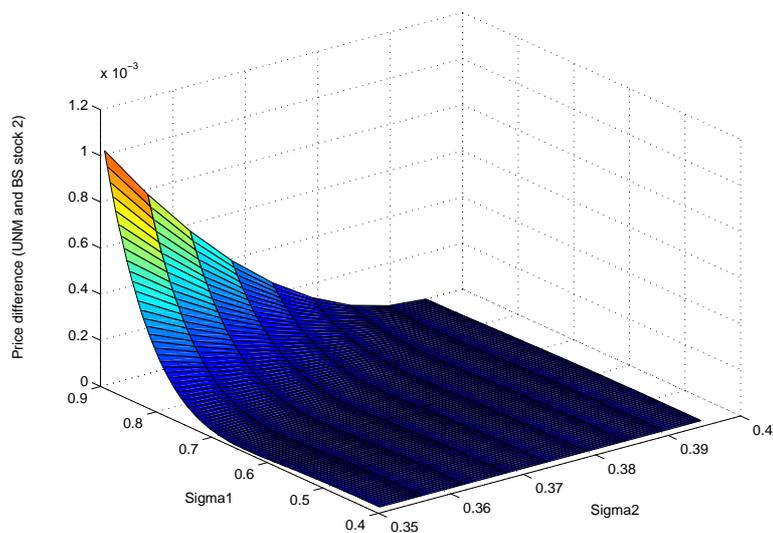
Figure 6.7: Plot of the square of difference between UNM and BS prices versus sigma1 and sigma2 - stock 2
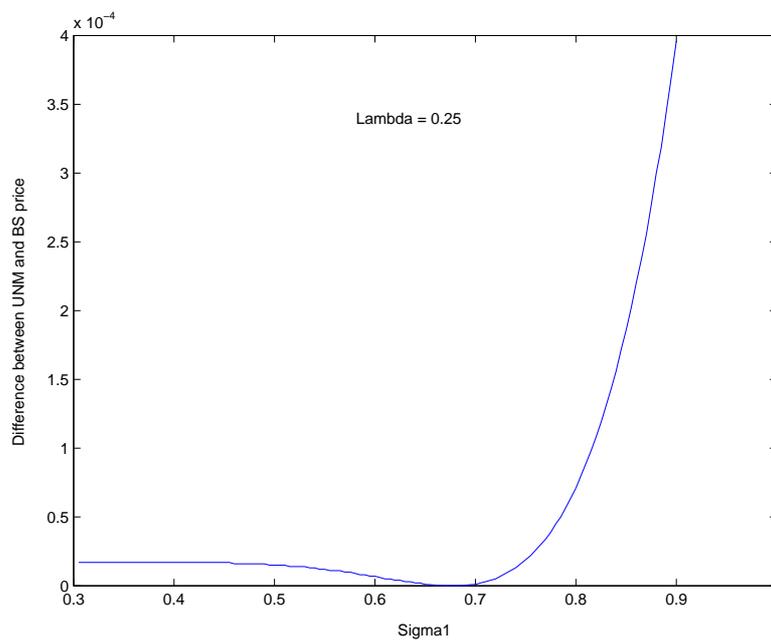
Figure 6.8: Square of difference between UNM and BS prices (lambda = 0.07, stock 1)
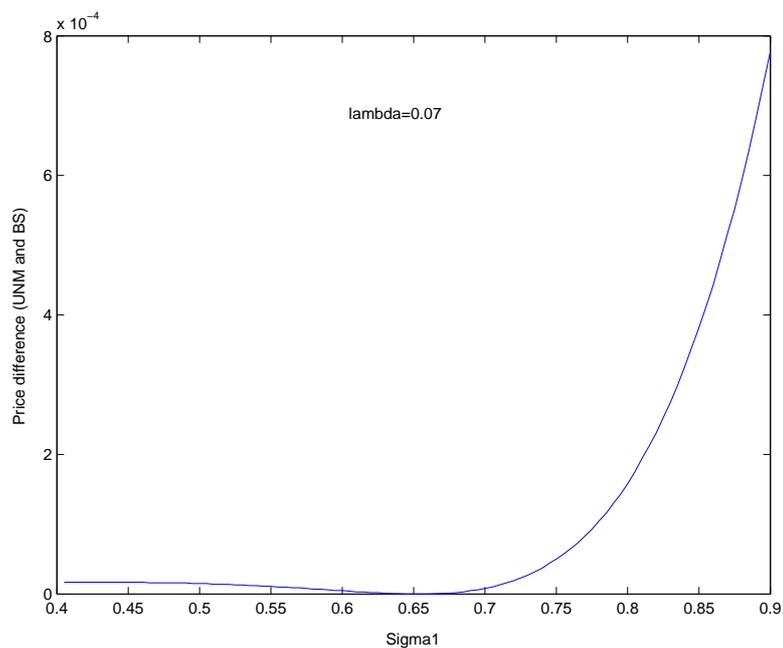
Figure 6.9: Square of difference between UNM and BS prices (lambda = 0.05, stock 2)
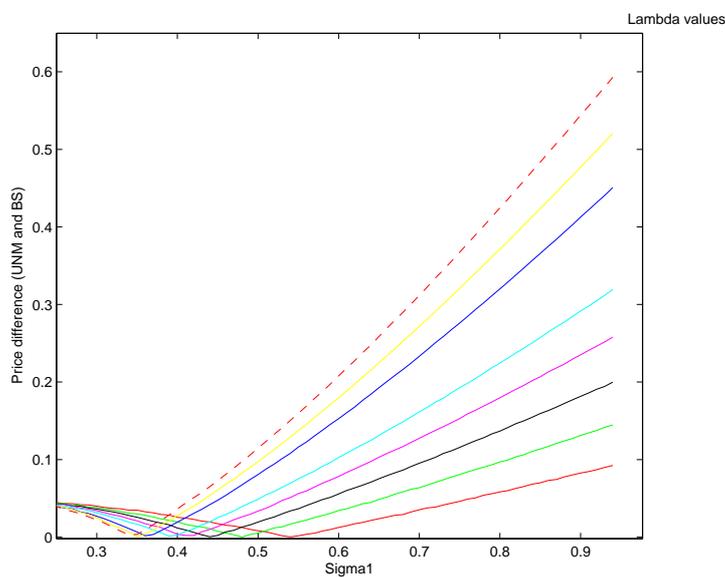
Figure 6.10: Square of difference between UNM and BS prices as a function of sigma1 (for different lambda values)
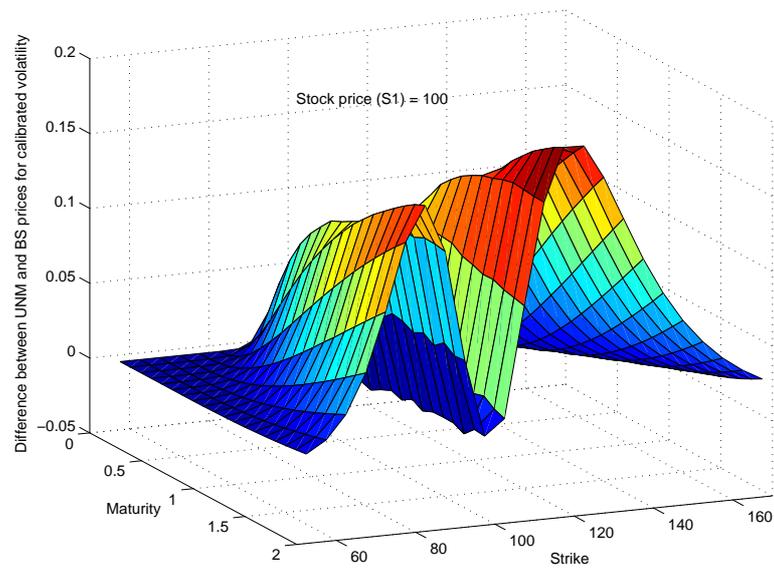
Figure 6.11: Difference between calibrated UNM price and BS price as a function of strike and maturity
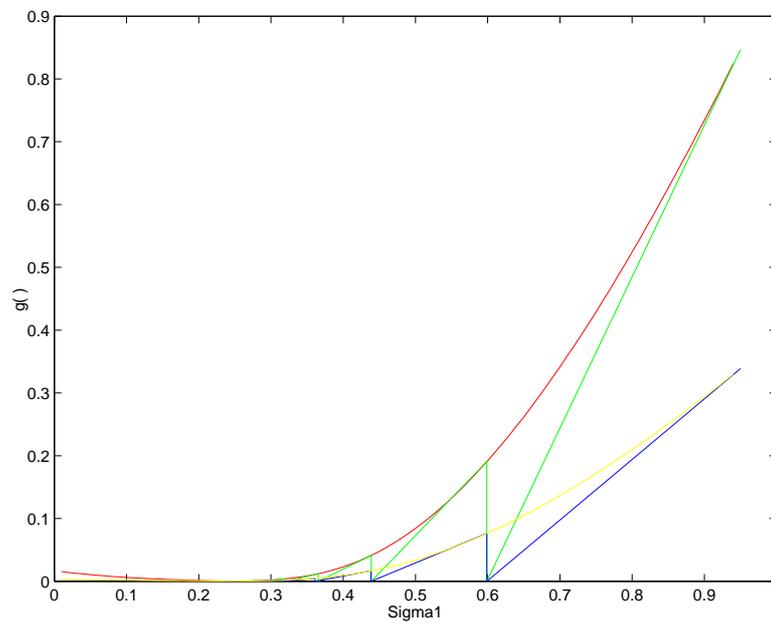
Figure 6.12: Newtons method

maturity.

Since the Kirks formula was derived based on the Black-Scholes model by expressing the option price as an expectation of the payoff function, it is natural to expect Kirks spread option price to behave on the lines of the BS price. As in fig. 6.13 we can see that the price increases smoothly as the strike increases. The plot of the surface of the BNM price against sigma1
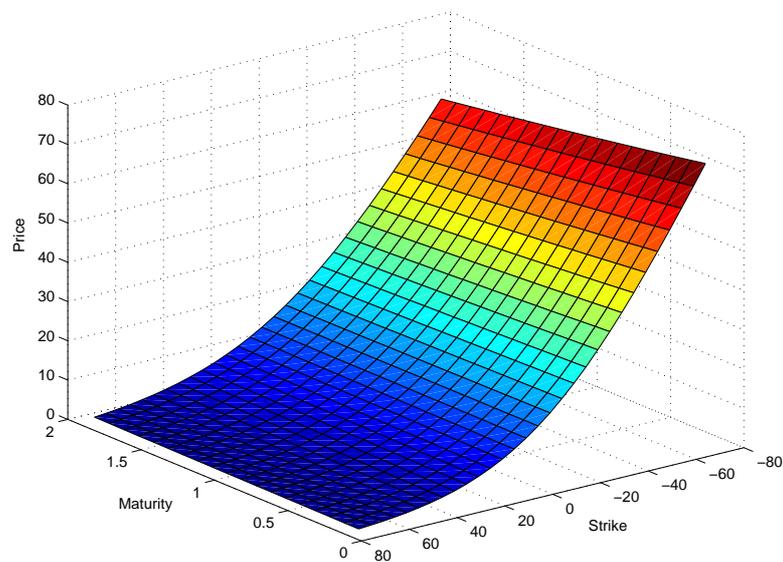
Figure 6.13: Kirks price versus strike and maturity

and sigma2 , although similar to its UNM counterpart exhibits a different behaviour. In this case, the price of the option increases with increase in sigma1 and sigma2 as against the decrease of sigma1 and increase of sigma2 in the UNM case.

A two dimensional plot of the square of the error versus the core correlation is shown in fig.6.15. The minimum was found using the same approach as Fig. 6.15 is similar in all respects to figs. 6.8 and 6.9.   The difference
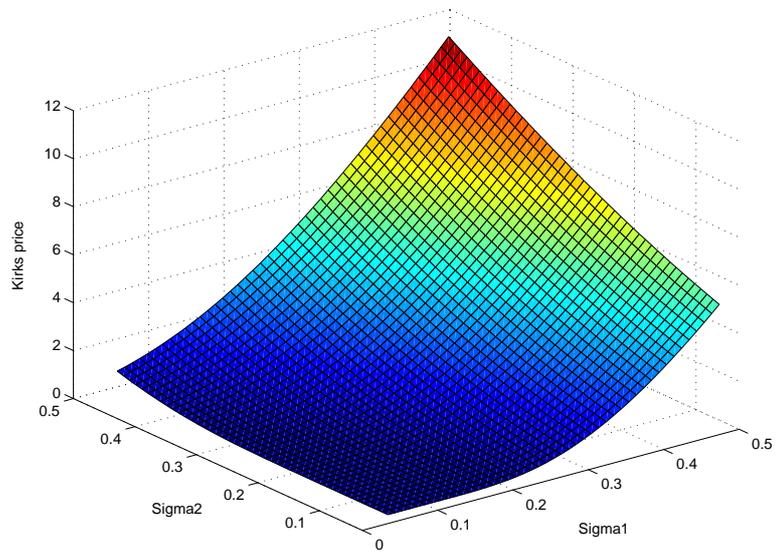
Figure 6.14: Kirks price versus sigma1 and sigma2
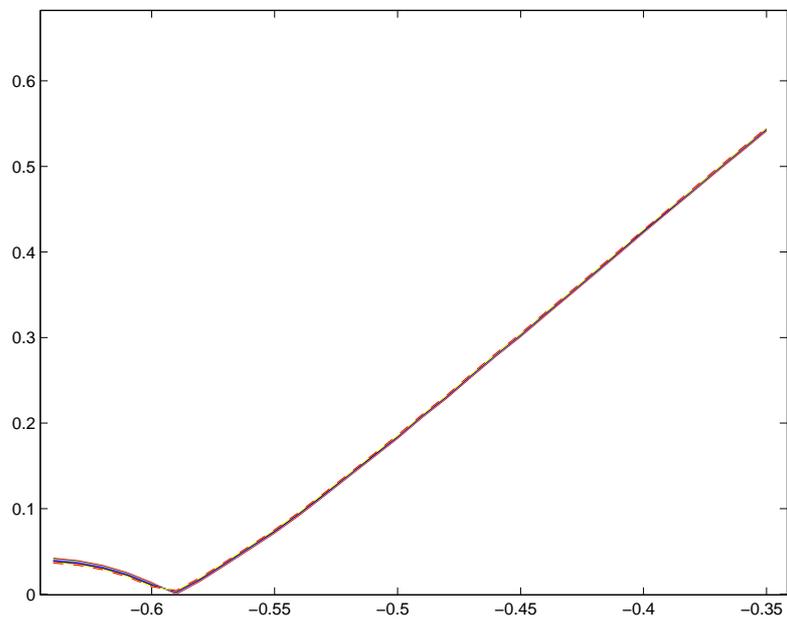
Figure 6.15: Square of difference between BNM and Kirk's prices versus rhoCC (calibrated volatility values)
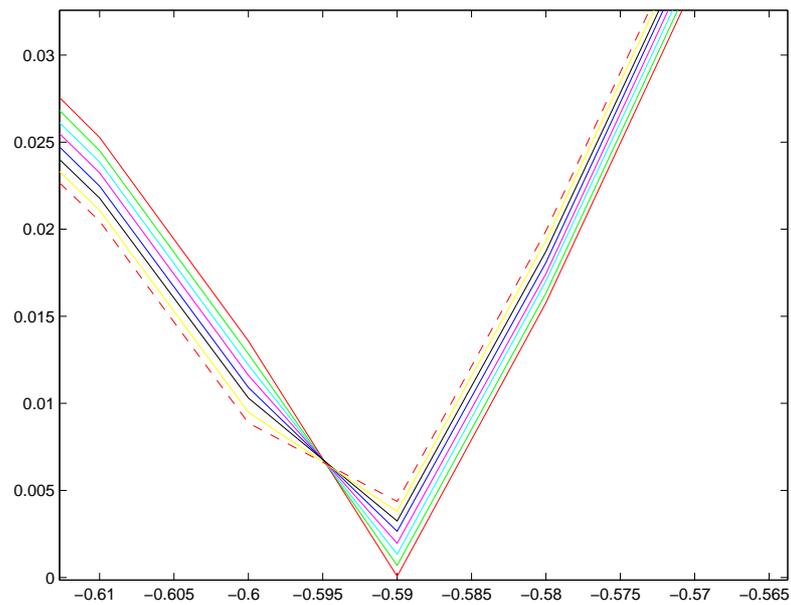
Figure 6.16: Square of difference between BNM and Kirk's prices versus rhoCC - (zoomed near minimum)

between the calibrated BNM price and the Kirks price is plotted as a function of strike and maturity in 6.17. Unlike fig. 6.11 which dips when the strike is near the stock price, we see here that the dip occurs at a different point. This is explained by the complex relation between the stock prices of the two assets and the strike of the spread option. One might observe a different pattern if calibration was done using a different strike convention as explained in Alexander and Scourse(2004).

Alexander and Scourse show that the BNM price is lesser than or equal to 2GBM/Kirk's price due to uncertainty over correlation and greater than or less than 2GBM/Kirk's price due to uncertainty over volatility. Fig. 6.17 also shows that the BNM prices can be greater than or lesser than the Kirk's. Since our calibrated BNM model conforms to this it is *frown consistent*!

The graph in fig. 6.18 shows the 3-D tree price of an American spread
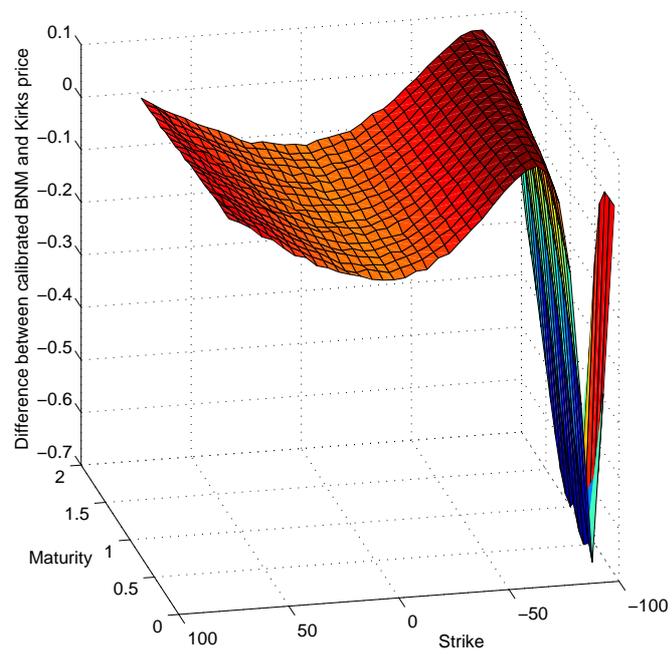
Figure 6.17: Difference between calibrated BNM model price and Kirk's price as a function of strike and maturity
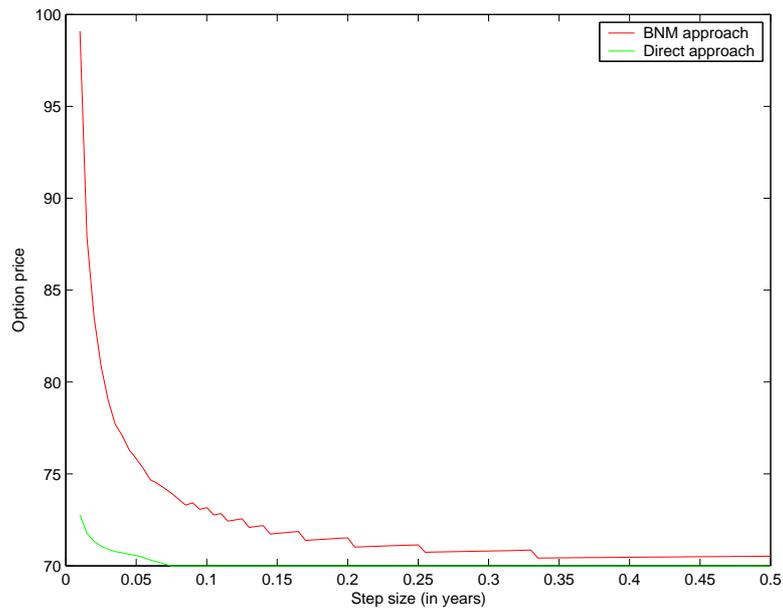
Figure 6.18: American Spread option price using 3-D tree model

option as a function of step size. With decreasing step size the resulting price increases. The price obtained using the BNM approach is found to be greater than that of a direct implementation (by substituting σ_1, σ_2, and ρ) of 3-D tree approach. One can also see that as the step size increases the gap between the former and latter approaches widen. If the calibrated BNM approach were to be correct then a direct 3-D tree approach can be said to underestimate the spread option price.
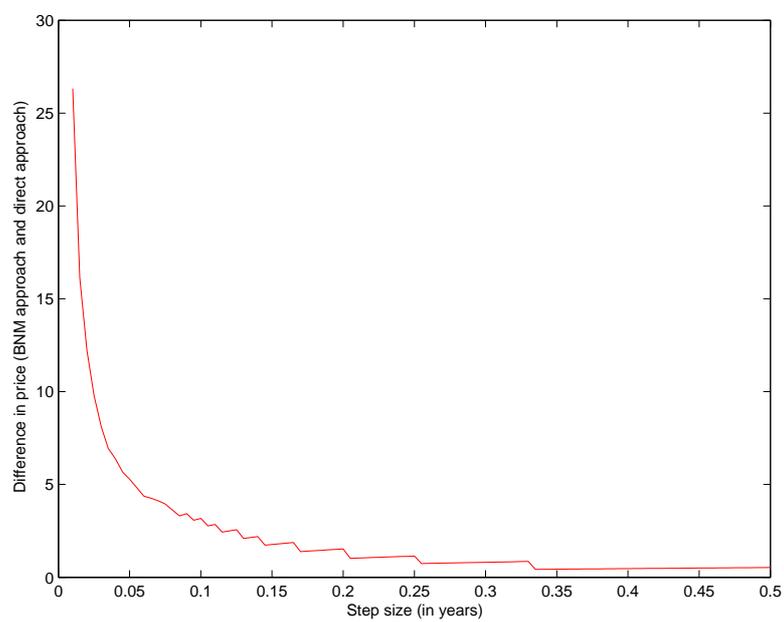
Figure 6.19: Difference between BNM approach and direct approach - 3Dtree model

Chapter 7
Summary and Conclusion

This project aims to price American spread options by extending the BNM approach using a 3-D binomial tree approach. The BNM model is an extension of the UNM model which assumes that the marginal densities of the asset returns are a mixture of log-normal densities. In the BNM model case this assumption is extended to two correlated underlying assets. Calibration was the foremost where the UNM and BNM models were calibrated to the market model. This was done for the models to be consistent with the volatility smiles and correlation frowns. The calibrated values were later used in the 3-D tree approach to numerically find the price of the American spread option.

When calibrating the UNM and BNM models, their respective price functions were reduced to a univariate function and the Newtons method was applied to find the optimum volatility and correlation values for which the square of the error was a minimum. The 3-D tree model assumes that the stock prices of the two assets are governed by Geometric Brownian motion and that the effect of correlation was only on one asset but relative to the other. Each module of the code was tested for its correctness and the American spread option price was found.

The values of the American spread options were found to be greater than that of Europeans' as expected. But this is by no means an effective tool to validate the results obtained. A better conclusion can be arrived by comparing the results with the actual market data.

This project has contributed by using an amalgamation of analytic and numerical approaches to find the American spread option prices. The main advantage of this method, which has never been implemented so far, is its simplicity which is mainly borrowed from the BNM and 3-D binomial tree models. There is a greater scope for further research and one can find innumerable ways of pricing American spread options.

The 3-D binomial tree approach used was a basic approach and the results can be improved if we were to use the model described by Boyle(1988). On the numerical front, since we were interested in the lambda, sigma and rho values only up to two decimal places, the choice of the fixing the lambda and rho values and their step size (see sec. 5.1.3) is justified.

If one were to find more accurate results the univariate approach adopted would not prove a good choice. In that case we can adopt higher dimensional descent methods, like gradient methods, Krylow subspace method and others, for optimisation. Proposing the problem as a linear optimisation problem with a set of constraints would be a more efficient and elegant approach. In brief, by adopting the extended Kirk's formula, advanced 3-D tree approaches and efficient optimisation techniques this new approach can be refined further; the possibilities being infinite!

Appendix A

Newton-Raphson Method

Let f(x) be a continuous smooth monotonically increasing/decreasing or a convex function with only one zero. The Newton-Raphson method allows one to find the zero of the function iteratively considering the function, its derivative, and an arbitrary initial x-value. The value of the iterate depends on the value and derivative of the function at the previous point. It is given by:

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} \quad where \quad f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

where, x_n is the current known x-value, $f(x_n)$ represents the value of the function at x_n, and $f'(x_n)$ is the derivative (slope) at x_n. x_{n+1} represents the new x-value that we are trying to find. This method has a quadratic rate of convergence.

The first order derivative in the program was calculated using a Δx of 0.005 which produced a satisfactory approximation to the actual value. This makes a good choice as the values of the Kirk's and Black-Scholes formulas are not much altered for small changes in correlation and volatility, respectively.

Acknowledgment

I would like to thank Prof. Mike Baines for his supervision and useful suggestions on this project. Moreover, I would also like to express my gratitude to him for his continued support, encouragement and guidance all throughout the year. I would also like to thank Prof. Nancy Nichols for her valuable comments and advice.

Special thanks to Prof. Carol Alexander without whom this project would have been impossible. Her patience and concern was much appreciated. Thanks to her PhD student Mr. Leonardo Nogueira who has always been helpful in my understanding of the subject.

I express my sincere gratitude to the anonymous sponsor and the scholarship committee for awarding me the *Felix Scholarship* to enable me to study the M.Sc. course.

Lastly, I would like to thank my family and friends for their periodic reminders for the project and deadlines!

Bibliography

[1] Alexander C and A Scourse 2004 Bivariate normal mixture spread option valuation *Quantitative Finance* Vol **4** 637-648

[2] Black F. and M Scholes 1973 The Pricing of Options and Corporate Liabilities *Journal of Political Economy* **81** 634-54

[3] Boyle P 1988 A lattice framework for pricing with two state variables *J. Financial Quant. Anal* **23** 1-12

[4] Brigo D and F Mercurio 2001 Displaced and mixture diffusions for analytically-tractable smile models *Mathematical Finance*

[5] Broadie M and J Detemple 1997 The Valuation of American Options on Multiple Assets *Mathematical Finance* Vol. **7** 241-286

[6] Broadie M and P Glasserman 1997 Pricing American-style securities using simulation *J. Econom. Dynam. Control* **21**(8-9)1323-1352

[7] Broadie M and P Glasserman 2004 A stochastic mesh method for pricing high-dimensional American options *J. Comput. Finance* **7**(4) 35-72

[8] Carmona R and V Durrleman 2003 Pricing and hedging spread options *SIAM* Rev **45** 627-87

[9] Cox J, Ross S and M Rubinstein 1979 Option Pricing: A Simplified Approach *Journal of Financial Economics* **7** 229-64

[10] Eydeland A and K Wolyniec 2003 Energy and Power Risk Management: New developments in Modeling Pricing and Hedging *John Wiley and Sons* NY

[11] Hull J 2000 Options,Futures, and Other Derivatives, 5th ed. *Prentice-Hall Upper Saddle River NJ*

[12] James P 2003 Option theory *Wiley Finance* The Atrium Chichester England

[13] Jarrow, R. and A. Rudd 1983 Option Pricing*Irwin*

[14] Kirk E 1995 Correlation in energy market Managing Energy Price Risk 1st edn *Risk Publications* pp 71-8

[15] Longstaff F and E Schwartz 2001 Valuing American options by simulation: A simple least-squares approach *Rev. Financial Stud* **14** 113-148

[16] Margrabe W 1978 The value of an option to exchange one asset for another *J. Finance* **33** 177-86

[17] Ravindran K 1993 Low-fat spreads *JRisk Mag* **6** 66-7

[18] Shimko D 1994 Options on futures spreads: hedging, speculation and valuation *J. Futures Markets* **14** 182-213

[19] www.investopedia.com