The University of Reading
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Modelling Glacier Flow

by

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August 2007

Acknowledgements

I would firstly like to thank NERC for their generous funding, without their help I would not have been able complete this course. Secondly, I would like to thank my supervisor Mike Baines for his much appreciated help and support throughout this dissertation. Special thanks must also go to my family and friends, especially Stuart, who have given me the love and support I needed to get me through the more difficult times in the year, Thank you.

Declaration

I confirm that this is my own work and the use of all materials from other sources have been properly and fully acknowledged.

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Abstract

In this dissertation a moving mesh method is used to produce a numerical approximation of a simple glacier model, which is a second order nonlinear diffusion equation, for the purpose of investigating how a glacier moves over time. The same model is also solved on a fixed grid for qualitive comparison with the moving mesh method.

Contents

Chapter 1

Introduction

This dissertation will provide a moving mesh method of the simple glacier equation, which is also the highly nonlinear second order PDE

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial H}{\partial x}\right) + M \tag{1.1}$$

with the snow term M and D defined as

$$D = cH^5 \left|\frac{\partial H}{\partial x}\right|^2 \tag{1.2}$$

with boundary conditions $H = 0$ at the boundaries, and initial condition, when $t = 0$, $H = 0$ at all points.

In chapter 2 a brief introduction to glaciers will be given, with aspects such how glaciers are formed and the environmental impact of glaciers. In chapter 3 the simple model defined above shall be derived from the continutity equation, and conservation of mass principle. Chapter 4 will see the self-similar solution calculated, and the model shown to be scale invariant under the mappings defined. The model will be approximated on a fixed grid in chapter 5, and instabilities found will be smoothed

in order to get an approximation of the model. The moving mesh method is then introduced in chaper 6, with a derivation of the ice thickness equation and velocity equation from the simple glacial model, and also the new mesh point equation found using time integration. Again stability issues will be tackled and plots given of the moving glacier representation. Then, in chapter 7, an analysis is made of the results gained in the previous three chapters. Finally, chapter 8 contains conclusions about the dissertation and any further work that may be possible from this work.

Chapter 2

Glaciers and Ice Sheets

A glacier can form in any climate zone where the input of snow exceeds the rate at which it melts. The amount of time required for a glacier to form depends on the rate at which the snow acculumates and turns to ice. Once a glacier has formed, its survival depends on the balance between acculumation and ablation (melting), this balance is known as mass balance and is largely dependant on climate.

The study of glacier mass balance is concerned with inputs to and outputs from the glacial system. Snow, hail, frost and avalanched snow are all inputs, output is generally from melting of snow, and calving of ice into the sea. If these inputs survive the summer ablation period the transformation into glacier ice will begin, the snow that has begun this transformation is called firn. The transformation involves three steps: compaction of the snow layers; the expulsion of the trapped air; and the growth of ice crystals. Again, the rate at which this transformation takes place depends on climate. If the rate of acculumation is high and significant melting occurs then the process can be very rapid since the older snow is buried by fresh snow which then compacts the firn; while alternate freezing and melting encourages the growth of new ice crystals. In contrast, if precipitation is low and little melting occurs, the process can be very slow.

During the period between 1550 and 1850 known as the Little Ice Age, glaciers increased. After this period, up unitl about 1940, glaciers worldwide retreated in response to an increase in climate temperature. This recession slowed during the period 1950 to 1980, in some cases even reversing, due to small global cooling. However, widespread retreat of glaciers has increased rapidly since 1980, increasing even more since 1995. Excluding the ice caps and sheets of the arctic and antarctic, the total surface area of glaciers worldwide has decreased by 50% since the end of the 19th century.

There are many different reasons for studying glaciers, the environmental impacts being at the forefront in recent years. Climatologists argue that increased temperature will lead to increased melting of glacial ice which can contribute to rising sea levels. The total ice mass on the Earth covers almost 14,000,000 km^2 of area, that is around 30,000,000 km^3 of ice. If all this ice melted the sea level would rise by approximately 70m. The third IPCC report [2] stated that an increase of $1.5°C$ to $4.5°C$ is estimated to lead to an increase of 15 to 95cm in sea level.

Beside rising sea level,increased melting of glaciers has other impacts. The loss of glaciers can cause landslides, flash floods and glacial lake overflow, and also increases annual variation in water flows in rivers, from [2]. For instance, the Hindu Kush and Himalayan glacial melts are the principal water source during the dry season of the major rivers in the South, East and Southeast Asian mainland. Increased melting would cause greater flow for several decades, after which "some areas of the most populated regions on Earth are likely to run out of water" as source glaciers are depleted, from [3].

Another reason for interest in glaciers is glacial surge. A glacial surge is when a glacier can move up to velocities 100 times faster then normal, during this time the glacier can advance significantly. These events don't last very long but they take place regularly and periodically. The period inbetween surges is called the

quiescent phase; during this period the glacier retreats significantly due to much lower velocities. Glacial surges can cause problems in shipping lanes where large amounts of ice discharge into the sea.

Chapter 3

The Simple Glacier Model

A glacier flows in the direction of decreasing surface elevation due to driving stress, resistance of this force can come from the glacier bed and at lateral margins, or it may be associated with gradients in longitudinal stresses. Most numerical models of ice-flow are based on the lamellar flow where the driving stress is taken to be opposed entirely by basal drag, so the longitudinal stresses and lateral shear are neglected.

3.1   The Continuity Equation

Numerical models of glaciers require that no ice may be created or lost. Changes in ice thickness at any point must be due to the flow of the ice and local snowfall or loss due to melting or calving. When integrated over the entire glacier the average rate of change of ice thickness must equal the total amount of ice added at the surface through snowfall minus the loss due to melting and calving. This conservation of mass is expressed by the continuty equation.

By definition the ice flux through any vertical section of ice is HU where H is the ice thickness and U is the discharge velocity. HU is then defined as the amount
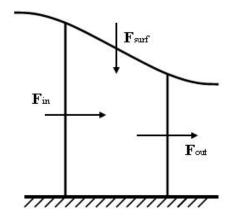
Figure 3.1: Diagram of ice fluxes into and out of a vertical column of ice extending from the bed to the surface

of ice flowing through the section per unit time per unit width in the cross-section direction. So the flux into the column is defined to be

$$F_{in} = HU(x) \tag{3.1}$$

and the ice flowing out of the column is defined as

$$F_{out} = HU(x + \Delta x) \tag{3.2}$$

If M is the accumulation rate of the ice, that is the snowfall and melting or calving, then the ice flux into the surface is defined to be

$$F_{surf} = M\Delta x \tag{3.3}$$

The ice becomes thicker or thinner when these three terms are not in balance. The density of the ice is taken to be zero so that the densification in the upper firn

layers can be neglected, therefore the conservation of mass corresponds to conservation of volume. By definition, the rate of change of the ice thickness is $\frac{\partial H}{\partial x}$ and so the rate of change of volume of the column is defined as $\frac{\partial H}{\partial t}\Delta x$. Therefore conservation of mass (or volume) is

$$
\begin{aligned}
\frac{\partial H}{\partial t}\Delta x &= F_{in} - F_{out} + F_{surf} \\
&= HU(x) - HU(x + \Delta x) + M\Delta x
\end{aligned}
\tag{3.4}
$$

dividing both sides by Δx and taking the limit as $\Delta x \to 0$ gives the continuity equation

$$
\frac{\partial H}{\partial t} = -\frac{\partial(HU)}{\partial x} + M
\tag{3.5}
$$

3.2 Derivation of Simple Model

From [1] the vertical-mean velocity is given by

$$
U = \frac{2AH}{n+2}\tau_{dx}^{n} + U_s
\tag{3.6}
$$

where τ_{dx} is the driving stress, U_s is the sliding velocity, and A and n are the flow parameters from Glen's Flow Law. Again, from [1] the driving stress is given by

$$
\tau_{dx} = -\rho g H \frac{\partial h}{\partial x}
\tag{3.7}
$$

where ρ is the ice density, g is the acceleration due to gravity, and h is the surface elevation of the ice.

If the basal sliding is set to zero, i.e. $U_s = 0$, and (3.7) is substituted into (3.6) then U is given by

$$U = -\frac{2AH}{n+2}(\rho g H)^n \left|\frac{\partial h}{\partial x}\right|^{n-1}\frac{\partial h}{\partial x} \tag{3.8}$$

The model used in this dissertation is solved on a flat bed so $H = h$, so (3.8) can be rewritten as

$$U = -\frac{2AH}{n+2}(\rho g H)^n \left|\frac{\partial H}{\partial x}\right|^{n-1}\frac{\partial H}{\partial x} \tag{3.9}$$

Substituting this equation for U into (3.5) gives

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(cH^{n+2}\left|\frac{\partial H}{\partial x}\right|^{n-1}\frac{\partial H}{\partial x}\right) + M \tag{3.10}$$

where

$$c = \frac{2A}{n+2}(\rho g)^n \tag{3.11}$$

Equation (3.10) can be written as the diffusion equation

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial H}{\partial x}\right) + M \tag{3.12}$$

with diffusivity

$$D = \frac{2A}{n+2}(\rho g)^n H^{n+2}\left|\frac{\partial H}{\partial x}\right|^{n-1} \tag{3.13}$$

The snow term is a function of distance only and decreases as x increases, for this dissertation M is defined as $M = b(x_b - ax)$ where b, a are arbitrary constants and

x_b is the right hand boundary. A is set to be a constant for this model, a common value used is 0.8×10^{-16}, and n shall be set as $n = 3$ from Glen's Flow Law. Glen's Flow Law gives a relationship between effective stress rate and strain rate, further information can be found in chapter 2 of [1]. The density of ice is set as 910kgm^{-3}, so the constant $c = 0.000022765$.

This makes the simple glacier model

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(\frac{2A}{5}(\rho g)^3 H^5 \left(\frac{\partial H}{\partial x}\right)^3\right) + b(x_b - ax) \tag{3.14}$$

with conditions $H = 0$ at the boundaries, and initial condition $H = 0$ for all x. This is the model that is used in this dissertation. In the next chapter the scale invariance of the simple model shall be looked at.

Chapter 4

Scale Invariance

An equation is said to be scale invariant if, for the partial differential equation

$$u_t = f(x, u, u_x, u_{xx} \ldots) \tag{4.1}$$

a transformation can be defined mapping the system (u, x, t) to a new system $(\hat{u}, \hat{x}, \hat{t})$ such that the PDE remains unchanged. That is that the physical quantities of the PDE are not dependant on the the system in which they are observed. For any nonlinear partial differential equation that is satisfied by the system (u, x, t), the transformation mapping to the new system $(\hat{u}, \hat{x}, \hat{t})$ is defined as

$$u = \lambda^\gamma \hat{u}, \quad x = \lambda^\beta \hat{x}, \quad t = \lambda^\alpha \hat{t}. \tag{4.2}$$

for some arbitrary positive parameter, λ. The solution is self-similar if it is invariant under the mappings defined above.

Consider the second order simple glacer model, defined on the system (H, x, t)

$$\frac{\partial H}{\partial t} = c \frac{\partial}{\partial x} \left(H^5 \left(\frac{\partial H}{\partial x} \right)^3 \right) + b(x_b - ax) \tag{4.3}$$

with boundary conditions $H = 0$ at the boundaries and initial condition $H = 0$ everwhere; where a, b are constants, x_b is the right hand boundary and $c = \frac{2A}{5}(\rho g)^3$ is constant. Then, using the transformations (4.2) with H substituted for u gives

$$\frac{\partial H}{\partial x} = \frac{\partial(\lambda^\gamma \hat{H})}{\partial(\lambda^\alpha \hat{t})} = \lambda^{\gamma-\alpha}\frac{\partial \hat{H}}{\partial \hat{t}} \tag{4.4}$$

$$c\frac{\partial}{\partial x}\left(H^5\left(\frac{\partial H}{\partial x}\right)^3\right) = c\frac{\partial}{\partial(\lambda^\beta \hat{x})}\left((\lambda^\gamma \hat{H})^5\left(\frac{\partial(\lambda^\gamma \hat{H})}{\partial(\lambda^\beta \hat{x})}\right)^3\right) \tag{4.5}$$

$$b(x_b - ax) = \lambda^\beta b(\hat{x}_b - a\hat{x}) \tag{4.6}$$

and so,

$$\lambda^{\gamma-\alpha}\frac{\partial \hat{H}}{\partial \hat{t}} = c\lambda^{8\gamma-4\beta}\frac{\partial}{\partial \hat{x}}\left(\hat{H}^5\left(\frac{\partial \hat{H}}{\partial \hat{x}}\right)^3\right) + \lambda^\beta(\hat{x}_b - a\hat{x}) \tag{4.7}$$

In order for (4.3) to be scale invariant $\lambda^{\gamma-\alpha}$, $\lambda^{8\gamma-4\beta}$, and λ^β must cancel, i.e.

$$\gamma - \alpha = 8\gamma - 4\beta = \beta$$

so,

$$8\gamma = 5\beta \Rightarrow \gamma = \frac{5}{8}\beta$$

Now, let $\alpha = 1$, then

$$\gamma - 1 = \beta \Rightarrow \beta = -\frac{8}{3}$$

and,

$$\gamma = -\frac{5}{3}$$

So, the PDE is scale invariant under the transformations

$$H = \lambda^{-\frac{5}{3}}\hat{H}, \quad x = \lambda^{-\frac{8}{3}}\hat{x}, \quad t = \lambda\hat{t}$$

4.1   Self-Similar Solutions

Using these transformations a self similar solution can be found which can be used to test the numerical methods used to solve the model. Firstly the similarity variables are defined as $\xi = \frac{x}{t^{\beta}} = \frac{x}{t^{-8/3}}$ and $\eta = \frac{H}{t^{\gamma}} = \frac{H}{t^{-5/3}}$. Substituting these variables into the left hand side of (5.3) gives

$$
\begin{aligned}
\frac{\partial H}{\partial t} &= \frac{\partial}{\partial t}(t^{-5/3}\eta) \\
&= t^{-5/3}\frac{\partial \eta}{\partial t} - \frac{5}{3}t^{-8/3}\eta \\
&= t^{-5/3}\frac{\partial \xi}{\partial t}.\frac{d\eta}{d\xi} - \frac{5}{3}t^{-8/3}\eta
\end{aligned}
\tag{4.8}
$$

since

$$\frac{\partial \xi}{\partial t} = \frac{8}{3}t^{5/3}x \tag{4.9}$$

then

$$
\begin{aligned}
\frac{\partial H}{\partial t} &= \frac{8}{3}x\eta' - \frac{5}{3}t^{-8/3}\eta \\
&= \frac{8}{3}\xi t^{-8/3}\eta' - \frac{5}{3}t^{-8/3}\eta
\end{aligned}
\tag{4.10}
$$

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Now consider the right hand side of (5.3), since

$$\frac{\partial \xi}{\partial x} = t^{8/3} \tag{4.11}$$

then

$$
\begin{aligned}
c\frac{\partial}{\partial x}\left(H^5\left(\frac{\partial H}{\partial x}\right)^3\right) &= c\frac{\partial \xi}{\partial x}\frac{d}{d\xi}\left[(t^{-5/3}\eta)^5\left(\frac{\partial \xi}{\partial x}\frac{d}{d\xi}(t^{-5/3}\eta)\right)^3\right] \\
&= ct^{8/3}\frac{d}{d\xi}\left[t^{-25/3}\eta^5\left(t^{8/3}\frac{d}{d\xi}(t^{-5/3}\eta)\right)^3\right] \\
&= ct^{-8/3}\frac{d}{d\xi}\left[\eta^5\left(\frac{d\eta}{d\xi}\right)^3\right] \tag{4.12}
\end{aligned}
$$

The snow term becomes

$$b(x_b - ax) = t^{-8/3}b(\xi_b - a\xi) \tag{4.13}$$

and so,

$$\frac{8}{3}\xi t^{-8/3}\eta' - \frac{5}{3}t^{-8/3}\eta = ct^{-8/3}(\eta^5\eta'^3)' + t^{-8/3}b(\xi_b - a\xi) \tag{4.14}$$

cancelling the $t^{-8/3}$ terms gives

$$\frac{8}{3}\xi\eta' - \frac{5}{3}\eta = c\left(\eta^5\eta'^3\right)' + b(\xi_b - a\xi) \tag{4.15}$$

So the second order PDE in (H, x, t) becomes a second order ordinary differential equation in (ξ, η). This ODE can be solved numerically using finite differences to approximate the equation.

4.2 Numerical Approximation of the Self-Similar Solution

The self-similar solution is approximated over a normalised fixed grid with $\xi \in [0, 1]$, and $\eta = 0$ at both boundaries, corresponding to $H = 0$ at the boundaries; initially $\eta = 0$ at all points, again to correspond with $H = 0$ initially, i.e. there is no ice to start with. Using a central difference approximation, the following is obtained

$$\frac{8}{3}ih\left(\frac{\eta_{i+1} - \eta_{i-1}}{2h}\right) - \frac{5}{3}\eta_i = c\left[\frac{(\eta^5\eta'^3)_{i+\frac{1}{2}} - (\eta^5\eta'^3)_{i-\frac{1}{2}}}{h}\right] + b(1 - aih) \qquad (4.16)$$

expanding the right hand side of the equation

$$\begin{aligned}
&= \frac{c}{h}\left[\left(\frac{\eta_{i+1} + \eta_i}{2}\right)^5\left(\frac{\eta_{i+1} - \eta_i}{h}\right)^3 - \left(\frac{\eta_i + \eta_{i-1}}{2}\right)^5\left(\frac{\eta_i - \eta_{i-1}}{h}\right)^3\right] \\
&\quad + b(1 - aih) \qquad (4.17)
\end{aligned}$$

where ξ has been approximated by $\xi = ih$ for all i.

To approximate, (4.17) is made linear by splitting $\left(\frac{\eta_{i+1} - \eta_i}{h}\right)^3$ and $\left(\frac{\eta_i - \eta_{i-1}}{h}\right)^3$ into two components, one at level k and one at level $k + 1$, such that

$$\left(\frac{8}{6}i + \frac{c}{h^2}M\right)\eta_{i+1}^{(k+1)} + \left(\frac{c}{h^2}L + \frac{c}{h^2}M - \frac{5}{3}\right)\eta_i^{(k+1)} + \left(-\frac{8}{6}i - \frac{c}{h^2}M\right)\eta_{i-1}^{(k+1)} = b(aih - 1) \qquad (4.18)$$

where

$$L = \left(\frac{\eta_{i+1}^{(k)} + \eta_i^{(k)}}{2}\right)^5\left(\frac{\eta_{i+1}^{(k)} - \eta_i^{(k)}}{h}\right)^2 \qquad (4.19)$$

$$M = \left(\frac{\eta_i^{(k)} + \eta_{i-1}^{(k)}}{2}\right)^5\left(\frac{\eta_i^{(k)} - \eta_{i-1}^{(k)}}{h}\right)^2 \qquad (4.20)$$

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This gives a tridiagonal matrix that is inverted in the program using the Gauss-Seidel method.

4.3 Numerical Results of Self-Similar Solution

Figure 4.1 shows the self-similar solution of the simple glacier model with $h = 0.005$.
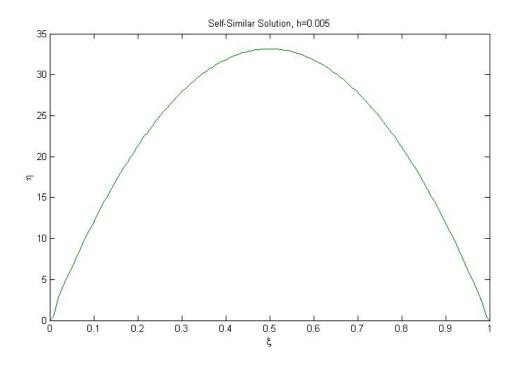

Figure 4.1: Self-Similar Solution of Model, $h = 0.005$

The self-similar solution is a particular solution of the simple glacier model that can be used to test the numerical approximations in chapters 5 and 6. The graph shows the solution after convergence of the solution has been reached.

The graph is quite symmetrical, it is not expected that the numerical approximations in the next chapters will reflect this property as the snow term in this approximation is of a slightly different form. In the following the chapters the snow

term is unable to be negative, instead at these points it is set to zero; this is not reflected in the ODE solved here. It would be expected that there would be a bias of mass to the left if this was the case.

Chapter 5

A Fixed Grid Approximation

Basic numerical approximations are done on grids with a fixed number of nodes, and a fixed grid spacing, where a node is calculated using the values of the nodes that surround it, or the boundary conditions if the node is close to or on a boundary. There are many different methods for solving such approximations, most of which involve inverting matrices. These methods can be explicit, where the solution is calculated from previous time steps, or implicit; where the solution is calculated using information from both previous time steps and the current time step.

5.1 A Fixed Grid Approximation

In this chapter the simple glacier model is solved fully implicitly by the θ-method with $\theta = 1$, using the Thomas Algorithm to invert the matrix on a fixed grid. Recalling (3.12)

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial H}{\partial x}\right) + M \tag{5.1}$$

with D defined as

$$D = cH^5 \left| \frac{\partial H}{\partial x} \right|^2 \tag{5.2}$$

as usual, a, b, c, x_b are constants. Using a finite difference approximation to approximate the spacial derivative of H at the time level $t = n$ gives the following approximation of the diffusivity

$$D_i^{(n)} = cH_i^{5(n)} \left[\frac{\left| H_{i+1}^{(n)} - H_{i-1}^{(n)} \right|}{2\Delta x} \right]^2 \tag{5.3}$$

The ice flux $F_{i\pm1/2}$ is approximated using an average of the diffusivity and a forward or backward difference approximation of $\frac{\partial H}{\partial x}$

$$F_{i+1/2} = \frac{1}{2} \left(D_{i+1}^{(n)} + D_i^{(n)} \right) \left[\frac{H_{i+1}^{(n+1)} - H_i^{(n+1)}}{\Delta x} \right]^2 \tag{5.4}$$

The flux term in (5.1) is approximated by $\frac{F_{i+1/2} - F_{i-1/2}}{\Delta x}$, expanding gives

$$\frac{c}{2\Delta x} \left[H_i^{5(n)} \left[\frac{H_{i+1}^{(n)} - H_{i-1}^{(n)}}{2\Delta x} \right]^2 + H_{i+1}^{5(n)} \left[\frac{H_{i+1}^{(n)} - H_i^{(n)}}{\Delta x} \right]^2 \right] \left[\frac{H_{i+1}^{(n+1)} - H_i^{(n+1)}}{\Delta x} \right]$$

$$- \frac{c}{2\Delta x} \left[H_i^{5(n)} \left[\frac{H_{i+1}^{(n)} - H_{i-1}^{(n)}}{2\Delta x} \right]^2 + H_{i-1}^{5(n)} \left[\frac{H_i^{(n)} - H_{i-1}^{(n)}}{\Delta x} \right]^2 \right] \left[\frac{H_i^{(n+1)} - H_{i-1}^{(n+1)}}{\Delta x} \right]$$

The time derivative can be approximated using Euler's Explicit Scheme, i.e.

$$\frac{\partial H}{\partial t} = \frac{H_i^{(n+1)} - H_i^{(n)}}{\Delta t} \tag{5.5}$$

The ice thickness, H, is approximated using

$$\frac{H_i^{(n+1)} - H_i^{(n)}}{\Delta t} = \frac{F_{i+1/2} - F_{1-1/2}}{\Delta x} + M \tag{5.6}$$

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Rearranging gives the equation for approximating the ice thickness $H_i^{(n+1)}$ in matrix form, this is given as

$$A\mathbf{H} = \mathbf{f} \tag{5.7}$$

where A is the tridiagonal matrix

$$
\mathbf{A} =
\begin{pmatrix}
b_0 & c_0 & 0 & 0 & \dots \\
a_1 & b_1 & c_1 & 0 & \dots \\
0 & a_2 & b_2 & c_2 & \dots \\
\vdots & \ddots & \ddots & \ddots & \dots \\
0 & a_i & b_i & c_i & \dots \\
\vdots & \ddots & \ddots & \ddots & \dots
\end{pmatrix}
$$

with

$$
\begin{aligned}
a_i &= \frac{k}{4} H_i^{5(n)} \left[H_{i+1}^{(n)} - H_{i-1}^{(n)} \right]^2 + k H_{i+1}^{5(n)} \left[H_{i+1}^{(n)} - H_i^{(n)} \right]^2 \\
b_i &= -\frac{k}{2} H_i^{5(n)} \left[H_{i+1}^{(n)} - H_{i-1}^{(n)} \right]^2 - k H_{i+1}^{5(n)} \left[H_{i+1}^{(n)} - H_i^{(n)} \right]^2 - k H_{i-1}^{5(n)} \left[H_i^{(n)} - H_{i-1}^{(n)} \right]^2 - 1 \\
c_i &= \frac{k}{4} H_i^{5(n)} \left[H_{i+1}^{(n)} - H_{i-1}^{(n)} \right]^2 + k H_{i-1}^{(n)} \left[H_i^{(n)} - H_{i-1}^{(n)} \right]^2
\end{aligned}
$$

where $k = \frac{c\Delta t}{2(\Delta x)^4}$,

$$\mathbf{H} = (H_0^{(n+1)}, H_1^{(n+1)} \dots, H_i^{(n+1)} \dots, H_N^{(n+1)})^T \tag{5.8}$$

and

$$\mathbf{f} = -H_i^{(n)} - \Delta t b(x_b - ax) \tag{5.9}$$

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The matrix A will be inverted using the Thomas Algorithm, this algorithm only acts on the interior points and so first the matrix needs to be rearranged into

$$
\begin{bmatrix}
b_1 & c_1 & 0 & 0 & \ldots \\
a_2 & b_2 & c_2 & 0 & \ldots \\
\ddots & \ddots & \ddots & \ldots & c_{N-2} \\
0 & 0 & \ldots & a_{N-1} & b_{N-1}
\end{bmatrix}
\begin{bmatrix}
H_1^{(n+1)} \\
H_2^{(n+1)} \\
\vdots \\
H_{N-1}^{(n+1)}
\end{bmatrix}
=
\begin{bmatrix}
f_1 - a_1 H_0^{(n+1)} \\
f_2 \\
\vdots \\
f_{N-1} - c_{n-1} H_N^{(n+1)}
\end{bmatrix}
$$

The Thomas Algorithm then follows two distinct steps. Firstly, a forward sweep of the matrix "removes" the diagonal of a_i's to obtain a matrix with two diagonals. The new system is defined by

$$
\begin{bmatrix}
b_1' & c_1' & 0 & 0 & \ldots \\
0 & b_2' & c_2' & 0 & \ldots \\
\ddots & \ddots & \ddots & \ldots & c_{N-2}' \\
0 & 0 & \ldots & 0 & b_{N-1}'
\end{bmatrix}
\begin{bmatrix}
H_1^{(n+1)} \\
H_2^{(n+1)} \\
\vdots \\
H_{N-1}^{(n+1)}
\end{bmatrix}
=
\begin{bmatrix}
f_1' - a_1 H_0^{(n+1)} \\
f_2' \\
\vdots \\
f_{N-1}' - c_{n-1} H_N^{(n+1)}
\end{bmatrix}
$$

where the new coeffients are defined as

$$
\begin{aligned}
b_i' &= b_i - c_{i-1}' \frac{a_i}{b_{i-1}'} \\
c_i' &= c_i \\
f_i' &= f_i - f_{i-1}' \frac{a_i}{b_{i-1}'}
\end{aligned}
$$

for all $i = 2, 3 \ldots, N - 1$, and

$$
\begin{aligned}
b_1' &= b_1 \\
f_1' &= f_1
\end{aligned}
$$

The second step involves a backward sweep of the new matrix in order the calculate the solution. Initially

$$H_{N-1}^{(n+1)} = \frac{f'_{N-1}}{b'_{N-1}} \qquad (5.10)$$

then $H_{N-1}^{(n+1)}$ is used to calculate the next solution, and the loop continues with

$$H_i^{(n+1)} = \frac{f'_i - c'_i H_{i+1}^{(n+1)}}{b'_i} \qquad (5.11)$$

for all $i = N - 2, N - 3 \dots, 1$.

Figure 5.1 shows the approximation of the simple glacier model on a fixed grid with $\Delta t = 0.0001$, $\Delta x = 0.1$, and $n = 200000$.
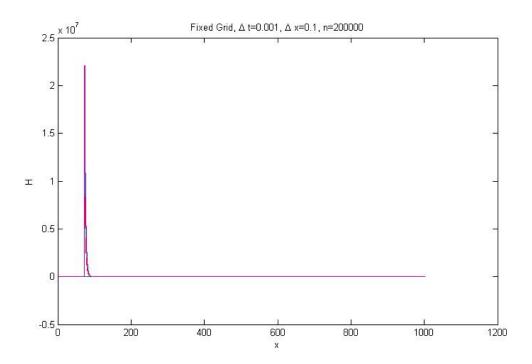

Figure 5.1: Unstable Fixed Grid Method, $\Delta t = 0.001$, $\Delta x = 0.1$, $n = 200000$

It can clearly be seen this is not an accurate representation of a glacier. The model becomes unstable and shoots up at one point. To make the approximation stable an extra diffusion term is added to the model. So the model is

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(D\left(\frac{\partial H}{\partial x}\right)\right) + \epsilon\frac{\partial^2 H}{\partial x^2} + M \tag{5.12}$$

where ϵ is a constant and D is as before.

So the new coefficients of the matrix A are defined to be

$$
\begin{aligned}
a_i &= \frac{k}{4}H_i^{5(n)}\left[H_{i+1}^{(n)} - H_{i-1}^{(n)}\right]^2 + kH_{i+1}^{5(n)}\left[H_{i+1}^{(n)} - H_i^{(n)}\right]^2 + \frac{\epsilon}{(\Delta x)^2} \\
b_i &= -\frac{k}{2}H_i^{5(n)}\left[H_{i+1}^{(n)} - H_{i-1}^{(n)}\right]^2 - kH_{i+1}^{5(n)}\left[H_{i+1}^{(n)} - H_i^{(n)}\right]^2 - kH_{i-1}^{5(n)}\left[H_i^{(n)} - H_{i-1}^{(n)}\right]^2 \\
&\quad -1 - \frac{2\epsilon}{(\Delta x)^2} \\
c_i &= \frac{k}{4}H_i^{5(n)}\left[H_{i+1}^{(n)} - H_{i-1}^{(n)}\right]^2 + kH_{i-1}^{(n)}\left[H_i^{(n)} - H_{i-1}^{(n)}\right]^2 + \frac{\epsilon}{(\Delta x)^2}
\end{aligned}
$$

This has the effect of smoothing the unstable point and is therefore a much better approximation.

5.2 Numerical Results of Fixed Grid Approxima-
tion

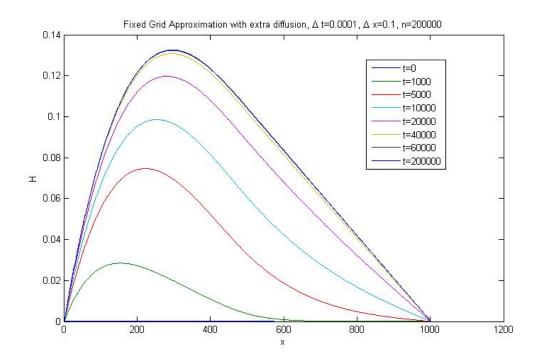Figure 5.2 shows the approximation with the smoothing term added.

Figure 5.2: Fixed Grid Approximation with Extra Diffusion, $\Delta t = 0.0001$, $\Delta x = 0.1$, $n = 200000$

The model gains mass from the snow term before $x_i = 50$ then once a sufficient amount of ice mas has built up the glacier is forced to move to the right. The model cannot move to the left as the boundary condition $H_0 = 0$ prevents this, this conditions represents a deep ocean into which the glacier cannot extend.

Since this is a fixed grid approximation the right hand boundary is also fixed, so the model cannot extend beyond this boundary. In the next chapter a moving mesh method is used to approximate the glacier. Using this method, the right hand boundary is able to move.

Chapter 6

A Moving Mesh Method

There are a number of techniques used to improve the accuracy of the basic numerical meshes such as the fixed grid used in chapter 5, the main three being h-refinement, p-refinement and, the method to be adopted here, r-refinement. In h-refinement extra nodes are added to small areas of the grid to improve local resolution; p-refinement involves the use of higher order numerical approximations to improve local accuracy. For r-refinement, the number of nodes in the grid remains fixed but they are strategically relocated at each time step to where they can provide the most information.

Most techniques for generating adaptive moving meshes can be categorised into one of two groups: location based methods and velocity based methods. Location based methods, as the name suggests, are concerned with directly controlling the location of the mesh points. Velocity methods, such as the the method applied in this dissertation, compute the velocity of the mesh, $\frac{dx}{dt} = \dot{x} = v$. The new mesh points can then be found from this equation using time integration.

6.1 A Moving Mesh Method

Consider the second order simple glacier model derived in chapter 3

$$\frac{\partial H}{\partial t} = c\frac{\partial}{\partial x}\left(cH^5\left(\frac{\partial H}{\partial x}\right)^3\right) + b(x_b - ax) \tag{6.1}$$

with $H = 0$ at the boundaries $(x_0(t), x_N(t))$ of the domain, $c = \frac{2A}{5}(\rho g)^3$ is constant, a, b, x_b are constants.

Integrating (6.1) over an arbitrary domain, with respect to x gives

$$\int_{a(t)}^{b(t)} \frac{\partial H}{\partial t}dx = c\int_{a(t)}^{b(t)} \frac{\partial}{\partial x}\left(cH^5\left(\frac{\partial H}{\partial x}\right)^3\right)dx + \int_{a(t)}^{b(t)} b(x_b - ax)dx \tag{6.2}$$

Leibniz's rule states

$$\frac{\partial}{\partial t}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}dx + f(b(t),t)\frac{\partial b}{\partial t} - f(a(t),t)\frac{\partial a}{\partial t} \tag{6.3}$$

so applying this to (6.2) gives

$$\frac{d}{dt}\int_{a(t)}^{b(t)} Hdx - \left[H\frac{dx}{dt}\right]_{a(t)}^{b(t)} = \left[cH^5\left(\frac{\partial H}{\partial x}\right)^3\right]_{a(t)}^{b(t)} + \int_{a(t)}^{b(t)} b(x_b - ax)dx \tag{6.4}$$

Matching the flux terms, the velocity is then defined by

$$-\left[H\frac{dx}{dt}\right]_{a(t)}^{b(t)} = \left[cH^5\left(\frac{\partial H}{\partial x}\right)^3\right]_{a(t)}^{b(t)} \tag{6.5}$$

which in turn implies that

$$\frac{d}{dt} \int_{a(t)}^{b(t)} H \, dx = \int_{a(t)}^{b(t)} b(x_b - ax) \tag{6.6}$$

6.2 Numerical Approximations of The Moving Mesh Method

Equation (6.6) will be used to calculate the ice thickness in each interval so setting $b(t) = x_{i+1}$ and $a(t) = x_{i-1}$ gives

$$\frac{d}{dt} \int_{x_{i-1}}^{x_{i+1}} H \, dx = \int_{x_{i-1}}^{x_{i+1}} b(x_b - ax) dx \tag{6.7}$$

Using Euler's Explicit Scheme to approximate the time derivative gives

$$\int_{x_{i-1}}^{x_{i+1}} H \, dx|_{t=n+1} = \int_{x_{i-1}}^{x_{i+1}} H \, dx|_{t=n} + \Delta t \int_{x_{i-1}}^{x_{i+1}} b(x_b - ax) dx \tag{6.8}$$

then, applying the Midpoint Rule to approximate the H integrals gives

$$\frac{H_i^{(n+1)}}{2}(x_{i+1}^{(n+1)} - x_{i-1}^{(n+1)}) = \frac{H_i^{(n)}}{2}(x_{i+1}^{(n)} - x_{i-1}^{(n)}) + b\Delta t \left[x_b x_i^{(n)} - a\frac{x_i^{2(n)}}{2} \right]_{x_{i-1}}^{x_{i+1}} \tag{6.9}$$

Rearranging,

$$H_i^{(n+1)} = \frac{H_i^{(n)}(x_{i+1}^{(n)} - x_{i-1}^{(n)}) + 2b\Delta t \left[x_b(x_{i+1}^{(n)} - x_{i-1}^{(n)}) - \frac{a}{2}(x_{i+1}^{2(n)} - x_{i-1}^{2(n)}) \right]}{(x_{i+1}^{(n+1)} - x_{i-1}^{(n+1)})} \tag{6.10}$$

This is the equation used to calculate the ice thickness, H, with boundary conditions $H(x_0, t) = H(x_N, t) = 0$, and initial condition $H(x_i, 0) = 0$ for all $i = 0, 1, \ldots, N$.

For the velocity, recalling (6.6) and integrating from x_0 to x_i gives

$$- H_i \left(\frac{\partial x}{\partial t} \right)_i + H_0 \left(\frac{\partial x}{\partial t} \right)_0 = cH_i^5 \left(\frac{\partial H}{\partial x} \right)_i^3 - cH_0^5 \left(\frac{\partial H}{\partial x} \right)_0^3 \tag{6.11}$$

since $H_0 = 0$ and with $v = \frac{dx}{dt}$, then

$$- H_i v_i = cH_i^5 \left(\frac{\partial H}{\partial x} \right)_i^3 \tag{6.12}$$

and so

$$v_i = -cH_i^4 \left(\frac{\partial H}{\partial x} \right)_i^3 \tag{6.13}$$

This is the equation that is used to determine the velocity of each node.

To approximate the velocity (6.13) is first rearranged as

$$
\begin{aligned}
v_i &= - \left(H^{4/3} \frac{\partial H}{\partial x} \right)^3 \\
&= - \left(\frac{3}{7} \frac{\partial}{\partial x} H^{7/3} \right)^3
\end{aligned} \tag{6.14}
$$

Then a forward difference approximates the derivative, giving

$$v_i = - \left(\frac{3}{7} \right)^3 \left[\frac{\left(H^{7/3} \right)_{i+1} - \left(H^{7/3} \right)_i}{x_{i+1} - x_i} \right]^3 \tag{6.15}$$

This is the equation used to approximate the velocity. From the velocity, the new grid point can be obtained using time integration. By definition $v = \frac{dx}{dt}$, so applying Euler's Explicit Scheme again gives

$$\frac{x_i^{(n+1)} - x_i^{(n)}}{\Delta t} = v \qquad (6.16)$$

So the new position of x_i is obtained through

$$x_i^{(n+1)} = x_i^{(n)} + \Delta t v \qquad (6.17)$$

The initial grid is set up with gridpoints $x_i = ih$ for $i = 0, 1, \ldots, N$ with $h = 0.01$. Since $H_i = 0$ for all i initially, then $v_i = 0$, and the gridpoints remain unchanged, however they begin to move after this point. From [1] the snow term constant, b, is set at $b = 0.5$; Van der Veen also sets $a = 0.5 * 10^{-6}$, however the model used in this dissertation has been scaled and so a is set as 0.5; x_b is the right hand boundary of the initial grid and is set to $x_b = 1$. From [1] the constant in the model, c, is set to be $c = 0.000022765$. These constants make the snow term in the equation equal to $0.5(1 - 0.5x)$ meaning that once $x_i > 0.5$ there is no snow falling and so no positive input from this term into the model. It is possible to allow this snow term to become negative to allow for ablation, however in this model it has been chosen to not allow this, and has been set so that the snow term is equal to zero after $x_i = 0.5$. Physically this is inaccurate as it will allow the glacier to continue to move indefinitely when of course this is not possible. For this simple case, it is the movement of the glacier that is of more concern.

Just as an extra diffusion term was required for the fixed grid approximation, a smoothing term is required in the moving mesh approximation. Once the ice thickness, H_i, is calculated a new ice thickness is calculated by averaging H_i and its surrounding points, that is

$$H_i^{new} = \frac{H_{i+1}^{old} + 2H_i^{old} + H_{i-1}^{old}}{4} \qquad (6.18)$$

6.3 Numerical Results of The Moving Mesh Method

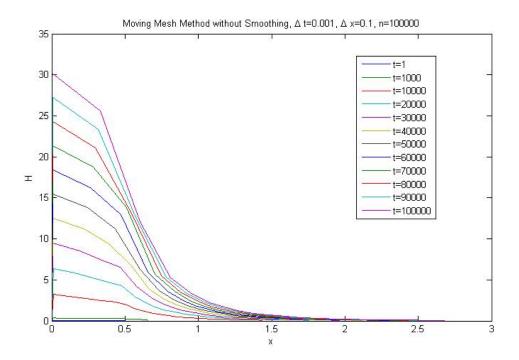Figure 6.1 shows the moving mesh approximation without the smoothing term included.

Figure 6.1: Moving Mesh Method with No Smoothing, $\Delta t = 0.001$, $\Delta x = 0.1$, $n = 100000$

Figure 6.2 shows the evolution of the glacier from $t = 1$ to $t = 100000$, with a time step of $\Delta t = 0.001$ after the smoothing term has been included.

At $t = 1$ the glacier has had input from the snow term up to $x = 0.5$ and so begins to build. At $t = 1000$ a distinct build up of mass has occured, though not enough to force the glacier to start to move yet. It can be seen that at later times a sufficient amount of ice mass has built up to force the boundary to start to move, allowing the glacier to flatten and spread out. The left hand boundary condition has been set at $H(x_0) = 0$ to represent a deep ocean at this point, this means that the
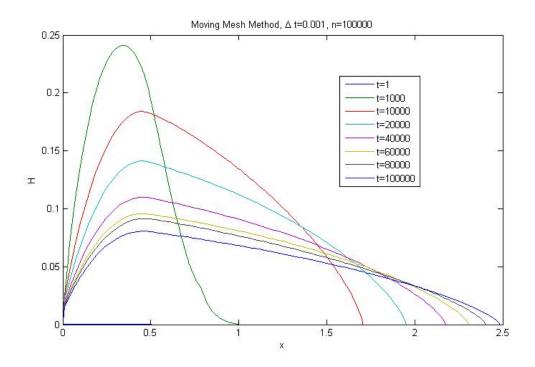
Figure 6.2: Moving Mesh Method, $\Delta t = 0.001$, $n = 100000$

ice cannot spread in this direction, however it is free to move as much as necessary at the right hand boundary.

Realistically, once the glacier passes the equilibrium line, where the accumulation zone and ablation zone meet, the ice would begin to melt and so would not extend past a certain point. It would eventually, assuming no change in climate, reach a steady state where the snow input equals the melting of the ice. It is also possible for ice to be lost at the sea boundary due to calving, which has also been left out of the simple model.

Figure 6.3 shows the moving mesh method after 1000000 time steps, it can be seen that the glacier continues to move and flatten in much the same way as it had in early timesteps, however always retaining the distinct bias of mass towards the left hand boundary due to the contribution of the snow term. In the next chapter,
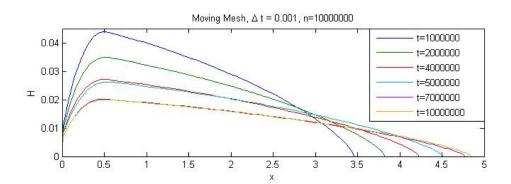
Figure 6.3: Moving Mesh Method, $\Delta t = 0.001$, $n = 10000000$

the results from the moving mesh method and the results from the scale invariance and fixed grids approximation shall be compared.

Chapter 7

An Analysis of Results

In chapter 4 transformations were defined that mapped the second order PDE in (H, x, t) to a second order ODE in (ξ, η). This gave a special solution of the model, a self-similar solution that was invariant under the defined transformations, it was found in order to test the numerical models of the later chapters. The graph was quite symmetrical and it was noted that it was not expected that this would be reflected in the later calculations due to the snow term not being able to be negative in chapters 5 and 6. It can be seen that this assumption was true, and there was a distinct bias of mass to the left in the numerical results of both the fixed grid approximation and the moving mesh approximation.

The numerical results from the fixed grid method and the moving mesh method have a similar shape, at least initially. The fixed grid is unable to move beyond the fixed right hand boundary and so settles into a steady state, whereas figure 6.2 has the ability to move the right hand boundary as much as necessary, this leads to the ice mass spreading out and flattening, though always with a bias of mass to the left.

Also, both the fixed grid approximation and moving mesh approximation required smoothing in order to control instablilty issues. The simple glacier model is highly nonlinear so small instabilites can take hold and grow extremely rapidly. For

the moving mesh, Δt could not be taken any larger than $\Delta t = 0.001$ as the solution blew up extremely quickly.

Chapter 8

Conclusions and Further Work

8.1 Summary

In this dissertation a moving mesh method has been used to solve numerically the simple glacier model to see how it models the movement of a glacier over time.

Chapter 2 saw an introduction to glaciers and ice sheets, including their formation process and their environmental impacts. In chapter 3 the simple glacier model was derived. In chapter 4 scale invariance was introduced, and a self-similar solution was found in order to test the numerical results of later chapters. Chapter 5 saw the approximation of the simple glacier model on a fixed grid using the fully implicit θ-method, and an extra diffusion term was added to the model in order to improve stability. The moving mesh method was introduced in chapter 6 in order to investigate how the glacier moves over time. In Chapter 7 the results found in chapters 5, 6, and 7 were compared and analysed the self-similar solution and results from the fixed grid approximations were compared with the moving mesh method to check for consistency.

8.2   Remarks and Further Work

The moving mesh method in chapter 6 will continue to grow and move indefinitely since there is no ablation term. The snow term in this model is unable to be negative and so provides only positive input or zero input to the model. As a small extension to this model a term could be added, or an alteration made of the snow term so that melting of the ice would be accounted for. Other possible extensions of the model include a non-horizontal bed, so that the movement down an arbitrary slope could be monitored; as well as the effect of outside climate change on the volume and velocity of the glacier.

For simplicity the inital grid in the moving mesh method was uniformly spaced. It is possible that a better approximation would be acheived with a non-uniform initial grid. Various initial grids could be used to investigate the effect they have on the growth and movement of the glacier model.

The flow parameter A was taken to be constant, however, this rate factor is affected by many aspects and so an investigation of the model when A is not constant could lead to more accurate results. Allowing the density of the ice ρ to alter is also a possible extension of the model.

Since glaciers flow through relatively narrow ridges most of their properties can be captured in a 1-D model, however, extending the model into 2-D could also yield some interesting results, and one could take into account the interaction of the ice with the valley wall. Also, taking the model into 2-D could then help describe the flow of ice sheets as well as glaciers. There are many extension to this simple lamellar flow model that are possible though, unfortunately, time restraints prevented them being investigated.

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