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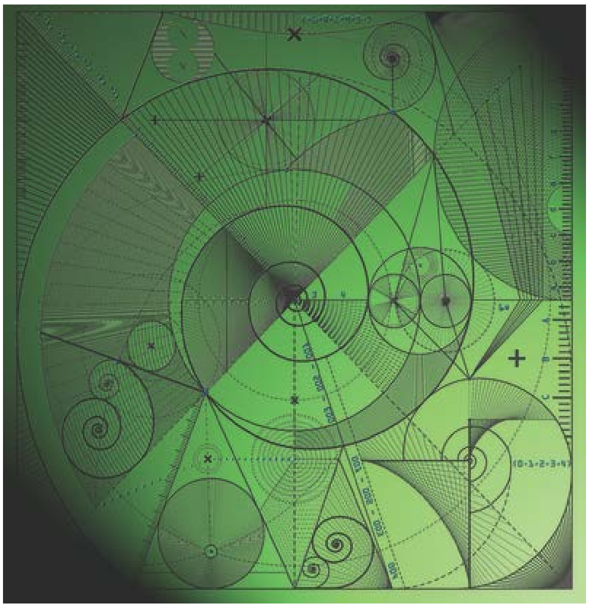

EXISTENCE, UNIQUENESS AND STRUCTURE OF SECOND ORDER ABSOLUTE MINIMISERS

NIKOS KATZOURAKIS AND ROGER MOSER

Abstract. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open $C^{1,1}$ set. In this paper we prove the existence of a unique second order absolute minimiser u_∞ of the functional

$$\mathrm{E}_\infty(u,\mathcal{O}) := \|\mathrm{F}(\cdot,\Delta u)\|_{L^\infty(\mathcal{O})}, \quad \mathcal{O} \subseteq \Omega \text{ measurable},$$

with prescribed boundary conditions for u and $\mathrm{D}u$ on $\partial\Omega$ and under natural assumptions on F. We also show that u_∞ is partially smooth and there exists a harmonic function $f_\infty \in L^1(\Omega)$ such that

$$\mathrm{F}(x,\Delta u_\infty(x)) = e_\infty \operatorname{sgn}\big(f_\infty(x)\big)$$

for all $x \in \{f_\infty \neq 0\}$, where e_∞ is the infimum of the global energy.

1. Introduction

For $n \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let also $\mathrm{F} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a real function that is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R})$-measurable, namely, measurable with respect to the product σ-algebra of the Lebesgue subsets of Ω with the Borel subsets of $\mathbb{R}$. In this paper we consider variational problems for second order supremal functionals of the form

$$\mathrm{E}_\infty(u,\mathcal{O}) := \big\|\mathrm{F}(\cdot,\Delta u)\big\|_{L^\infty(\mathcal{O})}, \quad \mathcal{O} \subseteq \Omega \text{ measurable}, \tag{1.1}$$

where the admissible functions u range over the (Fréchet) Sobolev space

$$\mathcal{W}^{2,\infty}(\Omega) := \bigcap_{1<p<\infty} \Big\{ u \in W^{2,p}(\Omega) \ : \ \Delta u \in L^\infty(\Omega) \Big\}. \tag{1.2}$$

The following is a natural notion of minimiser for variational problems of this type.

Definition 1 (Second order absolute minimisers)**.** A function $u \in \mathcal{W}^{2,\infty}(\Omega)$ is called a second order absolute minimiser of (1.1) on Ω if

$$\mathrm{E}_\infty(u,\mathcal{O}) \leq \mathrm{E}_\infty(u+\phi,\mathcal{O})$$

for all open sets $\mathcal{O} \subseteq \Omega$ and all $\phi \in \mathcal{W}^{2,\infty}_0(\mathcal{O})$.

Here we have used the obvious notation $\mathcal{W}^{2,\infty}_0(\mathcal{O}) := \mathcal{W}^{2,\infty}(\mathcal{O}) \cap \mathcal{W}^{2,2}_0(\mathcal{O})$. Given $u_0 \in \mathcal{W}^{2,\infty}(\Omega)$, we will also write $\mathcal{W}^{2,\infty}_{u_0}(\Omega) = u_0 + \mathcal{W}^{2,\infty}_0(\Omega)$.

The concept of an absolute minimiser would be unnecessary for variational problems given in terms of integrals rather than the essential supremum, where it would suffice to consider global minimisers on Ω for fixed boundary data. But here, since the functional (1.1) is not a measure (not additive) in the domain argument, a global minimiser of $\mathrm{E}_\infty(\cdot,\Omega)$ will not necessarily be a second order absolute minimiser.

Key words and phrases. ∞-Laplacian; ∞-Bilaplacian; Second Order absolute minimisers; Calculus of Variations in L^∞.

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Example. *For $\Omega = (-1,0)\cup(0,1)$, consider the functional $\mathrm{E}_\infty(u,\mathcal{O}) = \|u''\|_{L^\infty(\mathcal{O})}$ and let $Q(x) := x\chi_{(-1,0)}(x) - x(x-1)\,\chi_{(0,1)}(x)$, $x \in \Omega$. Then Q is a global minimiser of $\mathrm{E}_\infty(\cdot,\Omega)$ in $\mathcal{W}^{2,\infty}_Q(\Omega)$ with $\mathrm{E}_\infty(Q,\Omega) = 2$ and is also a second order absolute minimiser. However, for any function $\zeta \in C^\infty_c(-1,0)$ with $0 < \|\zeta''\|_{L^\infty(-1,0)} < 1$, the perturbation $Q+\zeta$ still satisfies $\mathrm{E}_\infty(Q+\zeta,\Omega) = 2$ and lies in $\mathcal{W}^{2,\infty}_Q(\Omega)$, but does not minimise $\mathrm{E}_\infty(\cdot,(-1,0))$ over $W^{2,\infty}_Q((-1,0))$ because the only minimiser on $(-1,0)$ with boundary data Q is the identity.*

On the other hand, if $u \in \mathcal{W}^{2,\infty}(\Omega)$ minimises $\mathrm{E}_\infty(\cdot,\Omega)$ *uniquely* in $\mathcal{W}^{2,\infty}(\Omega)$ with respect to its own boundary conditions, then we will show that u is actually the unique second order absolute minimiser. This is the situation that we will find in the main results of this paper. We first give a condition that guarantees that u is the unique minimiser for its boundary values on any subdomain. While it is not obvious that this condition can be met, we will subsequently prove that it is satisfied by exactly one function under given boundary conditions and very mild additional assumptions.

In the following, we will use the symbolisation "sgn" for the sign function, with the convention that $\mathrm{sgn}(0) = 0$. We will assume also that

$$\Omega \text{ is a bounded connected open subset of } \mathbb{R}^n,\ n \geq 1, \tag{1.3}$$

and

$$\begin{cases} \mathrm{F} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R} \text{ is } \mathcal{L}(\Omega)\otimes\mathcal{B}(\mathbb{R})\text{-measurable and for a.e.} \\ x \in \Omega,\ \xi \mapsto \mathrm{F}(x,\xi) \text{ is strictly increasing with } \mathrm{F}(x,0) = 0. \end{cases} \tag{1.4}$$

Our first main result therefore is:

Theorem 2 (Criterion for unique minimisers)**.** *Suppose (1.3)-(1.4) hold and consider (1.1) and a function $u_* \in \mathcal{W}^{2,\infty}(\Omega)$. If there exist a number $e_* \geq 0$ and a function $f_* \in L^1(\Omega)$ satisfying*

$$\Delta f_* \,=\, 0, \quad \text{on } \Omega, \tag{1.5}$$

such that

$$\mathrm{F}(\cdot,\Delta u_*) \,=\, e_*\,\mathrm{sgn}(f_*), \quad \text{a.e. on } \Omega, \tag{1.6}$$

then

$$\mathrm{E}_\infty(u_*,\mathcal{O}) \,<\, \mathrm{E}_\infty\big(u_*+\phi,\mathcal{O}\big),$$

for any open $\mathcal{O} \subseteq \Omega$ and any $\phi \in \mathcal{W}^{2,\infty}_0(\mathcal{O}) \setminus \{0\}$.

Namely, for any open subset $\mathcal{O} \subseteq \Omega$, the function u_* is the unique global minimiser of $\mathrm{E}_\infty(\cdot,\mathcal{O})$ in $\mathcal{W}^{2,\infty}_{u_*}(\mathcal{O})$.

Consequently, any such function u_* as in Theorem 2 is the unique second order absolute minimiser as well as the unique minimiser in $\mathcal{W}^{2,\infty}(\Omega)$ with respect to its own boundary conditions on $\partial\Omega$.

Clearly, (1.6) implies that $e_* = \mathrm{E}_\infty(u_*,\Omega)$ unless $f_* \equiv 0$. If we fix boundary data by choosing $u_0 \in \mathcal{W}^{2,\infty}(\Omega)$ and insist that $u_* \in \mathcal{W}^{2,\infty}_{u_0}(\Omega)$, then there is at most one minimiser $u_* \in \mathcal{W}^{2,\infty}_{u_0}(\Omega)$ that satisfies the condition and e_* is uniquely determined as the infimum of $\mathrm{E}_\infty(\cdot,\Omega)$ over the space. On the other hand, *the function f_* is not determined uniquely by the boundary conditions.* Note further that since $\mathrm{F}(x,\cdot)$

is by assumption strictly increasing, (1.6) is equivalent to the next *representation formula* for u_*:

$$\Delta u_*(x) \,=\, \mathrm{F}(x,\cdot)^{-1}\Big(e_*\,\mathrm{sgn}\,\big(f_*(x)\big)\Big), \qquad \text{a.e. } x\in\Omega.$$

Moreover, by using standard argument involving Green functions (see e.g. [GT, Ch. 2]), we could represent u_* in terms of $\mathrm{F}, f_*, e_*, u_*|_{\partial\Omega}, \mathrm{D}_\nu u_*|_{\partial\Omega}$.

Theorem 2 gives a connection between the variational problem and a PDE system of second order equations with a parameter consisting of (1.5) and (1.6). We will see later that under certain assumptions on Ω, F and the boundary data, the system has in fact a solution (u_*, f_*, e_*) with $f_*\not\equiv 0$ if $e_*>0$. It then follows that (u_*, e_*) is unique and the system is equivalent to unique global minimality under prescribed boundary data. *We may think of* (1.5) *and* (1.6) *as a PDE formulation of the L^∞ variational problem.* There does exist, however, a more conventional analogue of the "Euler-Lagrange equation" for (1.1). This is the fully nonlinear PDE of third order

$$\mathrm{F}(\cdot,\Delta u)\mathrm{F}_\xi(\cdot,\Delta u)\big|\mathrm{D}\big(|\mathrm{F}(\cdot,\Delta u)|^2\big)\big|^2 \,=\, 0, \quad \text{on } \Omega. \tag{1.7}$$

A particular model case of (1.1)-(1.7) is what we call the "∞-Bilaplacian" and arises from the choice $\mathrm{F}(x,\xi)=\xi$. Then, equation (1.6) becomes $\Delta u_* = e_*\,\mathrm{sgn}(f_*)$ and (1.7) becomes $\Delta u|\mathrm{D}(|\Delta u|^2)|^2=0$. Due to the particular structure of the functional (1.1), however, in this case (1.7) becomes redundant since all the structural information of second order absolute minimisers can be obtained directly from the L^∞ variational problem.

For our existence result, we will assume that $\mathrm{F}:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfies

$$\left\{\begin{array}{l} \mathrm{F}\in C^2(\Omega\times\mathbb{R}),\\ \mathrm{F}(x,0)\,=\,0, \quad x\in\Omega,\\ \exists\, c>0\ :\ \left\{\begin{array}{ll} c\,\le\,\mathrm{F}_\xi(x,\xi)\,\le\,\dfrac{1}{c}, & (x,\xi)\in\Omega\times\mathbb{R},\\ \mathrm{F}(x,\xi)\,\mathrm{F}_{\xi\xi}(x,\xi)\,\ge\,-\dfrac{1}{c}, & (x,\xi)\in\Omega\times\mathbb{R}, \end{array}\right. \end{array}\right. \tag{1.8}$$

where subscripts of F denote partial derivatives. The conditions of (1.8) imply that for any fixed $x\in\Omega$ the partial function $|\mathrm{F}(x,\cdot)|$ is level-convex on $\mathbb{R}$ (i.e. has convex sublevel sets) but in general they do not imply convexity.

In addition to existence, we will prove that the absolute minimisers of the L^∞ problem can be approximated by solutions of corresponding minimisers of L^p problems as $p\to\infty$. For $p\in(1,\infty)$, we therefore define

$$\mathrm{E}_p(u)\,:=\,\bigg(\fint_\Omega \big|\mathrm{F}(\cdot,\Delta u)\big|^p\bigg)^{1/p}, \quad u\in W^{2,p}(\Omega)$$

where the slashed integral sign denotes the average over Ω.

Theorem 3 (Existence, structure and approximation)**.** *Let Ω satisfy* (1.3) *and also have $C^{1,1}$ boundary. Let also* F *satisfy* (1.8) *and fix a function u_0 in $\mathcal{W}^{2,\infty}(\Omega)$ with $\Delta u_0\in C(\overline{\Omega})$.*

(I) *There exist a global minimiser $u_\infty\in\mathcal{W}^{2,\infty}_{u_0}(\Omega)$ of $\mathrm{E}_\infty(\cdot,\Omega)$ and a harmonic function $f_\infty\in L^1(\Omega)$ such that*

$$\mathrm{F}(\cdot,\Delta u_\infty)\,=\,e_\infty\,\mathrm{sgn}(f_\infty), \quad \textit{a.e. on } \Omega,$$

where $e_\infty = \mathrm{E}_\infty(u_\infty, \Omega)$. *Further,* $f_\infty \not\equiv 0$ *if* $e_\infty > 0$.

(II) *Let*

$$\Gamma_\infty \,:=\, f_\infty^{-1}(\{0\}).$$

If $e_\infty > 0$*, then* u_∞ *belongs to* $C^{3,\alpha}(\Omega \setminus \Gamma_\infty)$ *for any* $\alpha \in (0,1)$ *and* Γ_∞ *is a Lebesgue nullset. If* $e_\infty = 0$*, then* u_∞ *is a harmonic function.*

(III) *For* $p \in \mathbb{N}$*, let*

$$e_p \,:=\, \inf\big\{ \mathrm{E}_p(u) \,:\, u \in W^{2,p}_{u_0}(\Omega) \big\}.$$

Then, for any p *large enough there exists a global minimiser* $u_p \in W^{2,p}_{u_0}(\Omega)$ *of* E_p *satisfying* $\mathrm{E}_p(u_p) = e_p$*. Moreover,* $e_p \longrightarrow e_\infty$ *as* $p \to \infty$ *and there exists a subsequence* $(p_\ell)_{\ell=1}^\infty$ *such that* $u_{p_\ell} \relbar\!\rightharpoonup u_\infty$ *in the weak topology of* $\mathcal{W}^{2,\infty}(\Omega)$ *as* $\ell \to \infty$*. In addition,*

$$\left\{ \begin{array}{ll} u_{p_\ell} \longrightarrow u_\infty, & \text{in } C^1(\overline{\Omega}), \\ \mathrm{D}^2 u_{p_\ell} \relbar\!\rightharpoonup \mathrm{D}^2 u_\infty, & \text{in } L^q(\Omega, \mathbb{R}^{n\times n}) \text{ for all } q \in (1,\infty), \\ \Delta u_{p_\ell} \longrightarrow \Delta u_\infty, & \text{a.e. on } \Omega \text{ and in } L^q(\Omega) \text{ for all } q \in (1,\infty), \end{array} \right.$$

as $\ell \to \infty$*. Furthermore,* $\Delta u_{p_\ell} \longrightarrow \Delta u_\infty$ *locally uniformly on* $\Omega \setminus \Gamma_\infty$ *if* $e_\infty > 0$ *and locally uniformly on* Ω *if* $e_\infty = 0$*.*

(IV) *Let*

$$f_p \,:=\, \frac{1}{e_p^{p-1}} \big| \mathrm{F}(\cdot, \Delta u_p) \big|^{p-2} \, \mathrm{F}(\cdot, \Delta u_p) \, \mathrm{F}_\xi(\cdot, \Delta u_p) \tag{1.9}$$

if $e_p \neq 0$ *and* $f_p \equiv 0$ *if* $e_p = 0$*. Then, the harmonic function* f_∞ *in* (I) *may be chosen such that* $f_{p_\ell} \longrightarrow f_\infty$ *as* $\ell \to \infty$ *in the strong local topology of* $C^\infty(\Omega)$*.*

By invoking Theorem 2, an immediate consequence is that the modes of convergence in Theorem 3(III) as $p \to \infty$ are actually *full and not just subsequential*. Also, known results on the regularity of nodal sets of solutions to elliptic equations [HS] imply that Γ_∞ *is countably rectifiable, being equal to the union of countably many smooth submanifolds of* Ω *and a set of vanishing* $(n-2)$*-dimensional Hausdorff measure*. This, however, uses only the fact that f_∞ is harmonic and it seems plausible that the full statement in (I) could give further information.

The optimal regularity of Γ_∞ is an open question which we do not attempt to answer here. Certainly, full regularity of the set Γ_∞ cannot be expected as there are limitations: *in* [KP2] *it was noted that* Γ_∞ *in general may not be a smooth submanifold, as for certain data* u_0 *the intersection of transversal lines in* Ω *was observed in numerical experiments*. In most cases the set Γ_∞ is necessarily non-empty and divides Ω into two distinct parts, whilst the equations $\mathrm{F}(\cdot, \Delta u_\infty) = \pm e_\infty$ do not permit any solutions in $\mathcal{W}^{2,\infty}_{u_0}(\Omega)$ for most boundary data, even if $n = 1$. Therefore, the Laplacian of u_∞ will have a jump on Γ_∞ and, in terms of u_∞, no more regularity than $\mathcal{W}^{2,\infty}(\Omega)$ can be expected (see the numerical and explicit solutions in [KP2]).

Let us also note further that (I)–(IV) above have been obtained in [KP2] for $n = 1$ and in some other special cases (although were not stated in this explicit fashion), whilst the qualitative behaviour emerging here was observed numerically for $n = 2$ and $\mathrm{F}(x, \xi) = \xi$.

When combined, Theorem 2 and Theorem 3 imply in particular the following.

Corollary 4. *Under the hypotheses of Theorem 3, there exists a unique global minimiser u_∞ of $\mathrm{E}_\infty(\cdot,\Omega)$ in $\mathcal{W}^{2,\infty}_{u_0}(\Omega)$, which is a second order absolute minimiser and a strong solution to the Dirichlet problem for (1.7):*

$$\left\{\begin{aligned} \mathrm{F}(\cdot,\Delta u)\mathrm{F}_\xi(\cdot,\Delta u)\big|\mathrm{D}\big(|\mathrm{F}(\cdot,\Delta u)|^2\big)\big|^2 \,&=\, 0, && \text{in } \Omega,\\ u \,&=\, u_0, && \text{on } \partial\Omega,\\ \mathrm{D}u \,&=\, \mathrm{D}u_0, && \text{on } \partial\Omega, \end{aligned}\right.$$

More precisely, u_∞ is thrice differentiable a.e. on Ω and satisfies the PDE in the pointwise sense.

The study of supremal functionals and of their associated equations is known as Calculus of Variations in the space L^∞. Second order variational problems in L^∞ have only relatively recently been studied and are still poorly understood. It is remarkable that for our specific problem, we obtain not just unique absolute minimisers, but also a fair amount of detailed information about their structures, with relatively simple means. On the other hand, our methods take advantage of the special structure of the problem and are unlikely to work in general, although they allow the following modest generalisation: all of the preceding results hold for the seemingly more general case where the Laplacian is replaced by the projection $\mathbf{A}:\mathrm{D}^2u = \sum_{i,j}\mathbf{A}_{ij}\mathrm{D}^2_{ij}u$ on a fixed positive symmetric matrix $\mathbf{A}\in\mathbb{R}^{n\times n}_+$. This gives rise to the following functional:

$$\mathrm{E}_\infty(u,\mathcal{O}) \,=\, \big\|\mathrm{F}(\cdot,\mathbf{A}\!:\!\mathrm{D}^2u)\big\|_{L^\infty(\mathcal{O})},\quad \mathcal{O}\subseteq\Omega \text{ measurable.}$$

However, this case can easily be reduced to the case we study herein via the change of variables $x\mapsto \Lambda O^\top x$ for a diagonal $n\times n$ matrix Λ and an orthogonal matrix $O\in\mathrm{O}(n)$ arising from the spectral representation $\mathbf{A}=O\Lambda^2O^\top$.

Some of the techniques that underpin Theorem 3 have been successfully deployed to problems somewhat different to (1.1) (with dependence on u in addition to Δu) [MS, S1], which suggests that further generalisation might be possible. In order to keep the presentation simple, however, we do not explore this possibility any further in this work.

We conclude this introduction by placing the L^∞ problem we study herein into the wider context of Calculus of Variations. Variational problems for first order functionals of the form

$$(u,\mathcal{O})\longmapsto \underset{x\in\mathcal{O}}{\mathrm{ess\,sup}}\,\mathrm{H}\big(x,u(x),\mathrm{D}u(x)\big),\quad u\in W^{1,\infty}(\Omega),\ \mathcal{O}\subseteq\Omega, \tag{1.10}$$

together with the associated PDEs, first arose in the work of Aronsson in the 1960s [A1]–[A3]. The first order case is very well developed and the relevant bibliography is very extensive. For a pedagogical introduction to the theme which is accessible to non-experts, we refer to the monograph [K8] (see also [C]). The vectorial case of (1.10) for maps $u:\mathbb{R}^n\supseteq\Omega\longrightarrow\mathbb{R}^N$ is a much more modern and rapidly developing topic which first arose in recent work of the first author in the early 2010s (see [K1]–[K7], [K9]–[K13] as well as the joint works with Abugirda, Ayanbayev, Croce, Kristensen, Manfredi, Pisante and Pryer [AK, KM, CKP, KP, KK, AyK]).

In a very recent paper, the first author, jointly with Pryer (see [KP2] and also [KP3]), initiated the study of higher order variational problems and of their associated PDEs. As a first step they considered functionals of the form

$$(u,\mathcal{O})\longmapsto \underset{x\in\mathcal{O}}{\mathrm{ess\,sup}}\,\mathrm{H}\big(\mathrm{D}^2u(x)\big),\quad u\in W^{2,\infty}(\Omega),\ \mathcal{O}\subseteq\Omega,$$

with dependence on pure second derivatives only. Some preliminary investigations (relevant to the second order case of energy density $\mathrm{H}(\cdot, u, u', u'')$ when $n = 1$) had previously been performed via different methods by Aronsson and Aronsson-Barron in [A4, AB].

Apart from the intrinsic mathematical interest, the motivation to study higher order L^∞ minimisation problems comes from several diverse areas. In applied disciplines like Data Assimilation in the geosciences, PDE-constrained optimisation in aeronautics, etc. (see e.g. the model problem in [K9] and references therein, as well as the classical monograph [L]), a prevalent underlying problem is the construction of approximate solutions to second order ill-posed PDE problems. For instance, in the modelling of aquifers, one needs to solve a Poisson equation $\Delta u = f$ coupled with a pointwise constraint of the form $K(u) = k$ for given functions f, K, k. By minimising the error function $|\Delta u - f|^2 + |K(u) - k|^2$ in L^∞, one can obtain uniformly (absolutely) best approximations.

Minimisation problems in L^∞ similar to the above have also been studied in the context of differential geometry and in questions related to the Yamabe problem. In particular, the second author, together with Schwetlick [MS], and subsequently Sakellaris [S1] considered the problem of minimising the scalar curvature of a Riemannian metric on a given manifold and in a given conformal class. When formulated in terms of differential operators, this gives rise to a functional similar in structure to (1.1). This work uses different boundary conditions, however, and no attempt is made to prove uniqueness or find second order absolute minimisers. Nevertheless, some of the tools in the proofs of the above results originate in the above quoted papers.

We close with some remarks about generalised solutions to the equations governing the "extremals" of Calculus of Variations in L^∞. In the scalar first order case, the theory of viscosity solutions of Crandall-Ishii-Lions (see [CIL, C, K8]) proved to be an apt framework within which the generally non-smooth solutions to the so-called Aronsson equation, which is a second order PDE, can be studied rigorously. However, viscosity solutions are of purely scalar nature and fail to work in either the vectorial or the higher order case (where we have either a second order system with discontinuous coefficients, or a fully nonlinear third order PDE). In the recent papers [K9, K10] a new theory of generalised solutions has been introduced which is based on a probabilistic representation of derivatives which do not exist classically and in the papers [AK, AyK, CKP, K11, K12, K13, KP, KP2, KP3] several results have been obtained in this framework. However, in the setting of the present paper, the particular structure of the problem at hand allows to prove directly existence of strong solutions to the fully nonlinear PDE (1.7).

2. Proofs

In this section we establish the proofs of Theorems 2 and 3 and of Corollary 4.

Proof of Theorem 2. Fix $\phi \in \mathcal{W}^{2,\infty}_0(\Omega)$ with $\phi \not\equiv 0$ on Ω. Since f_* is a harmonic function in $L^1(\Omega)$, it follows that

$$\int_\Omega f_* \, \Delta\phi \,=\, 0. \tag{2.1}$$

We set

$$\Gamma_* \,:=\, f_*^{-1}(\{0\}).$$

By standard results on the nodal set of solutions to elliptic equations [HS] and the connectedness of Ω, it follows that if $f_* \not\equiv 0$ then Γ_* is a Lebesgue nullset and if $f_* \equiv 0$ then $\Gamma_* = \Omega$.

Let us first consider the case $f_* \not\equiv 0$. Note that $\Delta\phi$ cannot vanish almost everywhere on Ω (as this would imply that $\phi \equiv 0$ by uniqueness of solutions of the Dirichlet problem for the Laplace equation). Therefore, we deduce that $f\Delta\phi \neq 0$ on a subset of positive Lebesgue measure in Ω. Hence, (2.1) implies that there exist measurable sets $\Omega^\pm \subseteq \Omega$ with $\mathcal{L}^n(\Omega^\pm) > 0$ such that

$$\pm f_*\, \Delta\phi \, > \, 0, \quad \text{a.e. on } \Omega^\pm,$$

where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure. If (1.6) holds true, then we have $|\mathrm{F}(\cdot, \Delta u_*)| = e_*$ a.e. on Ω and

$$\mathrm{sgn}(\Delta u_*) \, = \, \mathrm{sgn}(f_*) \, = \, \mathrm{sgn}(\Delta\phi), \qquad \text{a.e. on } \Omega^+.$$

As $\mathrm{F}(x, \cdot)$ is strictly increasing for a.e. $x \in \Omega$, this gives that

$$|\mathrm{F}(\cdot, \Delta u_* + \Delta\phi)| \, > \, |\mathrm{F}(\cdot, \Delta u_*)| \, = \, e_*, \quad \text{a.e. in } \Omega^+.$$

Therefore,

$$\mathrm{E}_\infty(u_*, \Omega) \, = \, e_* \, < \, \mathrm{E}_\infty(u_* + \phi, \Omega).$$

It remains to consider the case $f_* \equiv 0$. Then, the hypothesis (1.6) implies that $\Delta u_* = 0$ almost everywhere and so $\mathrm{E}_\infty(u_*, \Omega) = 0$. On the other hand, by arguing as above it follows that $\mathrm{E}_\infty(u_* + \phi, \Omega) > 0$ for any $\phi \in \mathcal{W}^{2,\infty}_0(\Omega) \setminus \{0\}$. Hence, we arrive at the same conclusion.

Finally, if $\mathcal{O} \subseteq \Omega$ is a non-empty open set and $\phi \in \mathcal{W}^{2,\infty}_0(\mathcal{O}) \setminus \{0\}$, by repeating the previous reasoning with $\mathcal{O}$ is the place of Ω and $\Gamma_* \cap \mathcal{O}$ in the place of Γ_*, we obtain once again the strict inequality $\mathrm{E}_\infty(u_*, \mathcal{O}) < \mathrm{E}_\infty(u_* + \phi, \mathcal{O})$. The theorem ensues. ☐

The proof of Theorem 3 is more involved and requires some preparation. We begin with an elementary preliminary result.

Lemma 5. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\mathrm{F} \in C^2(\Omega \times \mathbb{R})$ a function satisfying (1.8). Then*

$$\mathrm{sgn}\big(\mathrm{F}(x,\xi)\big) \, = \, \mathrm{sgn}(\xi), \tag{2.2}$$

$$\frac{1}{c}|\xi| \, \geq \, \big|\mathrm{F}(x,\xi)\big| \, \geq \, c|\xi|, \tag{2.3}$$

$$\big(|\mathrm{F}(x,\cdot)|^p\big)_{\xi\xi}(\xi) \, \geq \, 0, \qquad \textit{if } \; p \geq \frac{1}{c^3} + 1, \tag{2.4}$$

$$\mathrm{F}_\xi(x,\xi)\, |\xi| \, \geq \, c^2 |\mathrm{F}(x,\xi)|, \tag{2.5}$$

for all $(x,\xi) \in \Omega \times \mathbb{R}$, where $c > 0$ is the same constant as in (1.8).

Proof of Lemma 5. We first note that (2.2) is obvious, while (2.3) follows, by integration, from $F(x,0) = 0$ and the bounds $c \leq \mathrm{F}(x,\cdot) \leq 1/c$ of (1.8). For (2.4), we differentiate in ξ and use (1.8). Then

$$\begin{aligned} \big(|\mathrm{F}|^p\big)_{\xi\xi} \, &= \, p|\mathrm{F}|^{p-2}\Big(\mathrm{F}\,\mathrm{F}_{\xi\xi} \, + \, (p-1)(\mathrm{F}_\xi)^2\Big) \\ &\geq \, p|\mathrm{F}|^{p-2}\Big(-\,\frac{1}{c} \, + \, (p-1)c^2\Big), \end{aligned}$$

which establishes the desired bound. Finally, (2.5) is a consequence of (1.8), which gives $|\xi|\mathrm{F}_\xi(x,\xi)\geq c|\xi|$, and of (2.3), which gives $c|\xi|\geq c^2|\mathrm{F}(x,\xi)|$. $\square$

We will construct the solutions to our problem by approximation with minimisers of L^p functionals. Therefore, we need to understand the behaviour of the latter.

Proposition 6. *Suppose that* $\mathrm{F}\in C^2(\Omega\times\mathbb{R})$ *satisfies* (1.8)*. Then for any* $p> c^{-3}+1$ *there exists a minimiser* u_p *of* E_p *over the space* $W^{2,p}_{u_0}(\Omega)$*. Moreover,* u_p *is a weak solution to the Dirichlet problem for the Euler-Lagrange equation associated with the functional* E_p*:*

$$\left\{\begin{array}{rcll} \Delta\Big(\big|\mathrm{F}(\cdot,\Delta u)\big|^{p-2}\,\mathrm{F}(\cdot,\Delta u)\,\mathrm{F}_\xi(\cdot,\Delta u)\Big) &=& 0, & \text{in } \Omega,\\ u &=& u_0, & \text{on } \partial\Omega,\\ \mathrm{D}u &=& \mathrm{D}u_0, & \text{on } \partial\Omega. \end{array}\right.$$

Furthermore, there exist a (global) minimiser u_∞ *of the functional* $\mathrm{E}_\infty(\cdot,\Omega)$ *over the space* $\mathcal{W}^{2,\infty}_{u_0}(\Omega)$ *such that* $\mathrm{E}_p(u_p)\longrightarrow\mathrm{E}_\infty(u_\infty,\Omega)$ *as* $p\to\infty$*. Also, there exists a subsequence* $(p_\ell)_1^\infty$ *such that*

$$\left\{\begin{array}{ll} u_{p_\ell}\longrightarrow u_\infty, & \text{in } C^1(\overline{\Omega}),\\ \mathrm{D}^2u_{p_\ell}\rightharpoonup \mathrm{D}^2u_\infty, & \text{in } L^q(\Omega,\mathbb{R}^{n\times n}),\ \text{for all } q\in(1,\infty), \end{array}\right.$$

as $\ell\to\infty$*.*

Proof of Proposition 6. By (2.3)–(2.4) of Lemma 5, for $p>c^{-3}+1$ the functional E_p is convex in $W^{2,p}_{u_0}(\Omega)$ and

$$\mathrm{E}_p(u)\,\geq\,c(\mathcal{L}^n(\Omega))^{-1/p}\|\Delta u\|_{L^p(\Omega)},$$

for any $u\in W^{2,p}_{u_0}(\Omega)$. Since $u-u_0\in W^{2,p}_0(\Omega)$, by the Calderon-Zygmund L^p estimates (e.g. [GT, GM]) and the Poincaré inequality we have a positive constant $c_0=c_0(p,\Omega)$ that

$$\|\Delta u\|_{L^p(\Omega)}\,\geq\,c_0\,\|u\|_{W^{2,p}(\Omega)}\,-\,(c_0+1)\|u_0\|_{W^{2,p}(\Omega)}. \tag{2.6}$$

Hence, E_p is coercive on $W^{2,p}_{u_0}(\Omega)$ and by setting

$$e_p\,:=\,\inf\big\{\mathrm{E}_p(u)\,:\,u\in W^{2,p}_{u_0}(\Omega)\big\}$$

we also have the bound

$$0\,\leq\,e_p\,\leq\,\mathrm{E}_p(u_0)\,<\,\infty$$

because $u_0\in W^{2,p}(\Omega)$. By applying the direct method of the Calculus of Variations (e.g. [D]), we deduce the existence of a global minimiser $u_p\in W^{2,p}_{u_0}(\Omega)$. Further, E_p is Gateaux differentiable at the minimiser as a result of the bound

$$\big||\mathrm{F}(\cdot,\xi)|^{p-1}\mathrm{F}_\xi(\cdot,\xi)\big|\,\leq\,C|\xi|^{p-1}$$

and well-known results (see e.g. [D, GM]).

Consider a family of minimisers $(u_p)_{p\geq p_0}$ where

$$p_0\,:=\,\big\{\text{integer part of }\max\{n,c^{-3}\}+1\big\}$$

and fix $k\in\mathbb{N}$. For any $p\geq k$, by (2.3), Hölder's inequality and the minimality we have

$$c\,\|\Delta u_p\|_{L^k(\Omega)}\big(\mathcal{L}^n(\Omega)\big)^{-1/k}\,\leq\,\mathrm{E}_k(u_p)\,\leq\,\mathrm{E}_p(u_p)\,\leq\,\mathrm{E}_p(u_0)\,\leq\,\mathrm{E}_\infty(u_0,\Omega) \tag{2.7}$$

and hence $(\Delta u_p)_{p\geq p_0}$ is bounded in $L^k(\Omega)$. By the previous arguments and (2.6), we conclude that $(u_p)_{p\geq p_0}$ is bounded in $W^{2,k}_{u_0}(\Omega)$ for any $k\in\mathbb{N}$. By a standard diagonal argument, weak compactness and the Morrey theorem, there exists

$$u_\infty \in \bigcap_{1<k<\infty} W^{2,\infty}_{u_0}(\Omega)$$

such that the desired convergences hold true along a subsequence as $p_\ell\to\infty$. When we pass to the limit as $\ell\to\infty$ in (2.7), the weak lower semicontinuity of the L^k norm implies

$$\|\Delta u_\infty\|_{L^k(\Omega)} \,\leq\, \frac{(\mathcal{L}^n(\Omega))^{1/k}}{c}\,\mathrm{E}_\infty(u_0,\Omega).$$

Letting $k\to\infty$ we obtain $\Delta u_\infty\in L^\infty(\Omega)$. Thus, $u_\infty\in\mathcal{W}^{2,\infty}_{u_0}(\Omega)$, as desired. It remains to show the convergence of $\mathrm{E}_p(u_p)$ and minimality of u_∞.

Hölder's inequality and minimality show that

$$\mathrm{E}_p(u_p)\,\leq\,\mathrm{E}_p(u_q)\,\leq\,\mathrm{E}_q(u_q),\quad\text{whenever } p\leq q.$$

Therefore, the limit $\lim\limits_{p\to\infty}\mathrm{E}_p(u_p)$ exists. Since $u_p-u_\infty\in W^{2,p}_0(\Omega)$, for any $\phi\in\mathcal{W}^{2,\infty}_0(\Omega)$ the minimality and Hölder's inequality imply

$$\begin{aligned}
\mathrm{E}_\infty(u_\infty,\Omega)\,&=\,\lim_{k\to\infty}\mathrm{E}_k(u_\infty)\\
&\leq\,\liminf_{k\to\infty}\Big(\liminf_{\ell\to\infty}\mathrm{E}_k(u_{p_\ell})\Big)\\
&\leq\,\lim_{p\to\infty}\mathrm{E}_p(u_p) \qquad\qquad (2.8)\\
&\leq\,\limsup_{p\to\infty}\mathrm{E}_p(u_\infty+\phi)\\
&\leq \mathrm{E}_\infty(u_\infty+\phi,\Omega).
\end{aligned}$$

Inequality (2.8) implies that u_∞ is indeed a global minimiser of $\mathrm{E}_\infty(\cdot,\Omega)$ over $\mathcal{W}^{2,\infty}_{u_0}(\Omega)$. In addition, the choice $\phi=0$ in (2.8) gives

$$\mathrm{E}_\infty(u_\infty,\Omega)\,\leq\,\lim_{p\to\infty}\mathrm{E}_p(u_p)\,\leq\,\mathrm{E}_\infty(u_\infty,\Omega)$$

and hence $\mathrm{E}_p(u_p)\longrightarrow\mathrm{E}_\infty(u_\infty,\Omega)$, as claimed. $\square$

The next result is an essential part of our constructions and this is the only point at which we make use of the $C^{1,1}$ boundary regularity of $\partial\Omega$ and the slightly higher regularity of the boundary condition $u_0\in\mathcal{W}^{2,\infty}(\Omega)$ (that is, that the Laplacian Δu_0 is continuous on $\overline{\Omega}$).

Subsequently, we will be using the following symbolisation for the r-neighbourhood of the boundary $\partial\Omega$ in Ω:

$$\Omega_r\,:=\,\{x\in\Omega\,:\,\mathrm{dist}(x,\partial\Omega)<r\},$$

for $r>0$.

Lemma 7 (Improving the the boundary data). *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with $C^{1,1}$ boundary and consider a function $u_0\in\mathcal{W}^{2,\infty}(\Omega)$ with $\Delta u_0\in C(\overline{\Omega})$. Then, for any $\epsilon>0$ there exist a number $r=r(\varepsilon)>0$ and a function $w=w(\varepsilon)\in\mathcal{W}^{2,\infty}_{u_0}(\Omega)$ such that*

$$\|\Delta w\|_{L^\infty(\Omega_r)}\,\leq\,\varepsilon.$$

In other words, given any boundary condition $u_0 \in \mathcal{W}^{2,\infty}(\Omega)$ such that Δu_0 is continuous up to the boundary, we can find another function w in the same space with the same boundary data, the Laplacian of which is as small as desired in a neighbourhood of the boundary.

Proof of Lemma 7. Since $\partial\Omega$ is $C^{1,1}$-regular, the result of the appendix establishes that the distance function $\mathrm{dist}(\cdot,\partial\Omega)$ belongs to $C^{1,1}(\overline{\Omega_{2r_0}})$ for some $r_0 > 0$ small enough.

Let d be an extension of $\mathrm{dist}(\cdot,\partial\Omega)$ from Ω_{r_0} to $\overline{\Omega}$ which is in the space $W^{2,\infty}(\Omega)$. Extend Δu_0 and d by zero on $\mathbb{R}^n \setminus \Omega$. Let $(\eta^\delta)_{\delta>0} \subseteq C_c^\infty(\mathbb{R}^n)$ be a standard mollifying family (as e.g. in [E]). We set

$$v_\delta \,:=\, u_0 \,-\, \frac{d^2}{2}\big(\eta^\delta * \Delta u_0\big). \tag{2.9}$$

Then $v_\delta - u_0 \in \mathcal{W}^{2,\infty}_0(\Omega)$ since $d = 0$ on $\partial\Omega$ and

$$\begin{aligned}
\mathrm{D}v_\delta \,&=\, \mathrm{D}u_0 \,-\, d\left\{\frac{d}{2}\mathrm{D}\big(\eta^\delta * \Delta u_0\big) \,+\, \big(\eta^\delta * \Delta u_0\big)\mathrm{D}d\right\},\\
\mathrm{D}^2 v_\delta \,&=\, \mathrm{D}^2 u_0 \,-\, \big(\mathrm{D}d\otimes \mathrm{D}d\big)\big(\eta^\delta * \Delta u_0\big) \,-\, d\bigg\{\frac{d}{2}\mathrm{D}^2\big(\eta^\delta * \Delta u_0\big)\\
&\qquad + \mathrm{D}\big(\eta^\delta * \Delta u_0\big)\otimes \mathrm{D}d \,+\, \mathrm{D}d\otimes \mathrm{D}\big(\eta^\delta * \Delta u_0\big) \,+\, \big(\eta^\delta * \Delta u_0\big)\mathrm{D}^2 d\bigg\}.
\end{aligned}$$

By using that

$$\mathrm{tr}(\mathrm{D}d\otimes \mathrm{D}d) \,=\, |\mathrm{D}d|^2 \,=\, 1 \quad \text{on } \Omega_{r_0},$$

for $0 < r < r_0$ we deduce

$$\begin{aligned}
\|\Delta v_\delta\|_{L^\infty(\Omega_r)} \,\leq\, \big\|\Delta u_0 - \eta^\delta * \Delta u_0\big\|_{L^\infty(\Omega_{r_0})} \,+\, C_1 r\bigg(&\big\|\Delta u_0 * \mathrm{D}^2\eta^\delta\big\|_{L^\infty(\mathbb{R}^n)}\\
+\, \big\|\Delta u_0 * \mathrm{D}\eta^\delta\big\|_{L^\infty(\mathbb{R}^n)} \,+\, &\big\|\Delta u_0\big\|_{L^\infty(\mathbb{R}^n)}\bigg)
\end{aligned} \tag{2.10}$$

for some constant $C_1 > 0$. Since

$$\eta^\delta(x) \,=\, \delta^{-n}\eta(|x|/\delta)$$

for some fixed function $\eta \in C_c^\infty(\mathbb{B}_1(0))$, by Young's inequality for convolutions we have the estimate

$$\big\|\Delta u_0 * \mathrm{D}^k\eta^\delta\big\|_{L^\infty(\mathbb{R}^n)} \,\leq\, \frac{1}{\delta^k}\,\big\|\Delta u_0\big\|_{L^\infty(\mathbb{R}^n)}\,\|\mathrm{D}^k\eta\|_{L^1(\mathbb{R}^n)}, \qquad k = 1, 2. \tag{2.11}$$

Hence, by invoking (2.11) we see that (2.10) gives

$$\|\Delta v_\delta\|_{L^\infty(\Omega_r)} \,\leq\, \big\|\Delta u_0 - \eta^\delta * \Delta u_0\big\|_{L^\infty(\Omega_{r_0})} + C_2 r\left(\frac{1}{\delta^2} \,+\, \frac{1}{\delta} \,+\, 1\right)\big\|\Delta u_0\big\|_{L^\infty(\Omega_{r_0})},$$

where $C_2 = C_1\|\eta\|_{W^{2,1}(\mathbb{R}^n)}$. By choosing

$$\delta \,:=\, r^{1/4}$$

and also choosing r_0 sufficiently small, we obtain the desired statement as a consequence of the continuity of Δu_0. $\square$

Now we can show that the minimiser u_∞ obtained in Proposition 6 satisfies the desired formula of part (I) in Theorem 3. The rest of the proof is then not difficult.

Proof of Theorem 3. Let us begin by setting

$$e_\infty := \inf\big\{\mathrm{E}_\infty(u,\Omega) \,:\, u \in \mathcal{W}^{2,\infty}_{u_0}(\Omega)\big\}.$$

If $e_\infty = 0$, then everything in Theorem 3 follows from Proposition 6 or is trivial. (Note that in this case $e_p = 0$ for every p and every minimiser of the corresponding functionals is a harmonic function.) Therefore, we may assume that $e_\infty > 0$.

Since $e_p \longrightarrow e_\infty$ as $p \to \infty$ by Proposition 6, it follows that $e_p > 0$ for large p, say for $p \geq p_0$. Then for all $p \geq p_0$, the formula in (IV) gives rise to a measurable function $f_p : \Omega \longrightarrow \mathbb{R}$. Then the Euler-Lagrange equation in Proposition 6 can be expressed in the form

$$\Delta f_p \,=\, 0, \quad \text{on } \Omega. \tag{2.12}$$

That is, f_p is harmonic on Ω and hence belongs to $C^\infty(\Omega)$.

Let $p' = p/(p-1)$ be the conjugate exponent of $p \in (1,\infty)$. Then (1.8) implies

$$\begin{aligned}
\bigg(\fint_\Omega |f_p|^{p'}\bigg)^{1/p'} \,&=\, \frac{1}{e_p^{p-1}}\bigg(\fint_\Omega \Big|\mathrm{F}^{p-1}(\cdot,\Delta u_p)\mathrm{F}_\xi(\cdot,\Delta u_p)\Big|^{p'}\bigg)^{1/p'}\\
&\leq\, \frac{1}{c\, e_p^{p-1}}\bigg(\fint_\Omega \Big|\mathrm{F}^{p-1}(\cdot,\Delta u_p)\Big|^{p/(p-1)}\bigg)^{(p-1)/p}\\
&=\, \frac{1}{c},
\end{aligned}$$

which gives the following uniform L^1 bound of $(f_p)_{p\geq p_0} \subseteq C^\infty(\Omega)$:

$$\|f_p\|_{L^1(\Omega)} \,\leq\, \mathcal{L}^n(\Omega)\bigg(\fint_\Omega |f_p|^{p'}\bigg)^{1/p'} \,\leq\, \frac{\mathcal{L}^n(\Omega)}{c}. \tag{2.13}$$

By the mean value theorem for harmonic functions and by the standard interior derivative estimates (e.g. [GT]) we have, for any $k \in \mathbb{N}\cup\{0\}$ and for any compactly contained $\mathcal{O} \Subset \Omega$, a constant $C = C(k,\mathcal{O},\Omega) > 0$ such that

$$\|\mathrm{D}^k f_p\|_{L^\infty(\mathcal{O})} \,\leq\, C\|f_p\|_{L^1(\Omega)}.$$

Hence, the family $(f_p)_{p\geq p_0}$ is bounded (in the locally convex sense) in the topology of $C^\infty(\Omega)$ and as a consequence there exist $f_\infty \in C^\infty(\Omega)$ and a sequence $p_\ell \to \infty$ such that $f_{p_\ell} \longrightarrow f_\infty$ as $\ell \to \infty$. From (2.12) it follows that f_∞ is harmonic:

$$\Delta f_\infty \,=\, 0, \quad \text{on } \Omega. \tag{2.14}$$

Fix $r > 0$ and consider the inner r-neighbourhood Ω_r of $\partial\Omega$. Since $f_{p_\ell} \longrightarrow f_\infty$ in $C\big(\overline{\Omega\setminus\Omega_r}\big)$, inequality (2.13) implies that

$$\|f_\infty\|_{L^1(\Omega\setminus\Omega_r)} \,=\, \lim_{\ell\to\infty}\|f_{p_\ell}\|_{L^1(\Omega\setminus\Omega_r)} \,\leq\, \frac{\mathcal{L}^n(\Omega)}{c}.$$

Letting $r \to 0$ we conclude that $f_\infty \in L^1(\Omega)$ and

$$\|f_\infty\|_{L^1(\Omega)} \,\leq\, \frac{\mathcal{L}^n(\Omega)}{c}. \tag{2.15}$$

We now show that $f_\infty \not\equiv 0$ using Lemma 7. Fix $\epsilon > 0$ small and let $r > 0$ and $w \in \mathcal{W}^{2,\infty}_{u_0}(\Omega)$ be as constructed in Lemma 7 with $|\Delta w| \leq \varepsilon$ on the inner zone

$\Omega_r \subseteq \Omega$ of the boundary $\partial\Omega$. Since $u_p - w \in W_0^{2,p}(\Omega)$, it is an admissible test function and by (2.12), integration by parts gives

$$\int_\Omega f_p \, \Delta(u_p - w) \, = \, 0.$$

Hence, by the above together with (1.9), (1.8), (2.2) and (2.5), we obtain

$$\begin{aligned}
\int_\Omega f_p \, \Delta w \, &= \, \int_\Omega f_p \, \Delta u_p \\
&= \, \frac{1}{e_p^{p-1}} \int_\Omega \big|\mathrm{F}(\cdot, \Delta u_p)\big|^{p-2} \, \mathrm{F}(\cdot, \Delta u_p) \, \mathrm{F}_\xi(\cdot, \Delta u_p) \Delta u_p \\
&= \, \frac{1}{e_p^{p-1}} \int_\Omega \big|\mathrm{F}(\cdot, \Delta u_p)\big|^{p-1} \, \mathrm{F}_\xi(\cdot, \Delta u_p) \, |\Delta u_p| \\
&\geq \, \frac{c^2}{e_p^{p-1}} \int_\Omega \big|\mathrm{F}(\cdot, \Delta u_p)\big|^{p}.
\end{aligned}$$

Thus,

$$\int_\Omega f_p \, \Delta w \, \geq \, c^2 \, \mathcal{L}^n(\Omega) \, e_p. \tag{2.16}$$

Now, we use (2.16), (2.13) and Lemma 7 to estimate

$$\begin{aligned}
c^2 \, \mathcal{L}^n(\Omega) \, e_p \, &\leq \, \int_{\Omega_r} f_p \, \Delta w \, + \, \int_{\Omega \setminus \Omega_r} f_p \, \Delta w \\
&\leq \, \varepsilon \|f_p\|_{L^1(\Omega)} \, + \, \int_{\Omega \setminus \Omega_r} f_p \, \Delta w \\
&\leq \, \frac{\varepsilon \mathcal{L}^n(\Omega)}{c} \, + \, \int_{\Omega \setminus \Omega_r} f_p \, \Delta w.
\end{aligned}$$

Recalling that $e_{p_\ell} \longrightarrow e_\infty$ and also $f_{p_\ell} \longrightarrow f_\infty$ in $C\big(\overline{\Omega \setminus \Omega_r}\big)$, we pass to the limit as $\ell \to \infty$ to find that

$$\int_{\Omega \setminus \Omega_r} f_\infty \, \Delta w \, \geq \, \mathcal{L}^n(\Omega) \Big(c^2 \, e_\infty \, - \, \frac{\varepsilon}{c} \Big).$$

By choosing $\varepsilon > 0$ small enough, we deduce that

$$\int_{\Omega \setminus \Omega_r} f_\infty \, \Delta w \, > \, 0,$$

which implies $f_\infty \not\equiv 0$, as claimed.

Let us now define the map

$$\Phi \, : \, \Omega \times \mathbb{R} \longrightarrow \Omega \times \mathbb{R}, \qquad \Phi(x, \xi) \, := \, (x, \mathrm{F}(x, \xi)).$$

Under the assumption (1.8), this is a C^2 diffeomorphism.

Since the inverse function of $t \mapsto |t|^{p-2} t$ is given by $s \mapsto \mathrm{sgn}(s)|s|^{1/(p-1)}$, we may rewrite the formula (1.9) defining the harmonic function f_p, as

$$\mathrm{F}(\cdot, \Delta u_p) \, = \, e_p |f_p|^{\frac{1}{p-1}} \big[\mathrm{F}_\xi(\cdot, \Delta u_p)\big]^{-\frac{1}{p-1}} \, \mathrm{sgn}(f_p)$$

or as

$$\big(x, \Delta u_p(x)\big) = \Phi^{-1} \Big(\cdot \, , \, e_p |f_p|^{\frac{1}{p-1}} \big[\mathrm{F}_\xi(\cdot, \Delta u_p)\big]^{-\frac{1}{p-1}} \, \mathrm{sgn}(f_p) \Big) \, (x), \tag{2.17}$$

for $x \in \Omega \setminus \Gamma_\infty$. On any compact set $K \subseteq \Omega \setminus \Gamma_\infty$, we have the uniform convergence $f_{p_\ell} \longrightarrow f_\infty$ as $\ell \to \infty$, whereas F_ξ is uniformly bounded from above and below by (1.8). Hence, by restricting ourselves along the subsequence p_ℓ and letting $\ell \to \infty$ we obtain uniform convergence of the right-hand side of (2.17) to $\Phi^{-1}(\cdot, e_\infty \operatorname{sgn}(f_\infty))$ on K. But since we already know that $\Delta u_p \rightharpoonup \Delta u_\infty$ weakly in $L^2(\Omega)$, it follows that

$$(x, \Delta u_\infty(x)) = \Phi^{-1}\Big(x, e_\infty \operatorname{sgn}\big(f_\infty(x)\big)\Big), \quad x \in K. \tag{2.18}$$

As a consequence,

$$\mathrm{F}\big(x, \Delta u_\infty(x)\big) \,=\, e_\infty \operatorname{sgn}\big(f_\infty(x)\big), \quad x \in K.$$

Now let us recall that $\mathcal{L}^n(\Gamma_\infty) = 0$. This is a consequence of general regularity results for nodal sets of solution to elliptic equations [HS]. The statement of item (I) then follows.

In order to prove item (II), we note that (2.18) implies that $\Delta u_\infty \in C^2(\Omega \setminus \Gamma_\infty)$. The desired statement then follows from standard Schauder theory [GT].

For item (III), first recall the subsequential convergence of the E_p-minimisers $(u_p)_1^\infty$ of Proposition 6 along $(p_\ell)_1^\infty$ as $\ell \to \infty$. We also have the desired respective convergence of the global infima $(e_p)_1^\infty$ of the energies.

The a.e. convergence of the Laplacians $(\Delta u_{p_\ell})_{\ell=1}^\infty$ follows from the fact that $\Omega \setminus \Gamma_\infty$ has full Lebesgue measure and that the sequence converges locally uniformly thereon.

The strong convergence of the Laplacians $(\Delta u_{p_\ell})_{\ell=1}^\infty$ in $L^q(\Omega)$ for all $q \in (1, \infty)$ is a consequence of the Vitaly convergence theorem (see e.g. [FL]) and of the following facts:

i) the weak convergence of the Laplacians over the same spaces,
ii) the a.e. convergence of the Laplacians on Ω,
iii) the boundedness of Ω,
iv) the L^q equi-integrability estimate

$$\|\Delta u_{p_\ell}\|_{L^q(E)} \,\leq\, \left(\sup_{\ell \in \mathbb{N}} \|\Delta u_{p_\ell}\|_{L^{q+1}(\Omega)}\right) \big(\mathcal{L}^n(E)\big)^{\frac{1}{q(q+1)}}$$

which holds true for any measurable subset $E \subseteq \Omega$.

Finally, the statement in part (IV) has already been proven. □

We conclude this section by noting that Corollary 4 is an immediate consequence of Theorem 2, Theorem 3, and the observation that $\mathcal{L}^n(\Gamma_\infty) = 0$ if $e_\infty > 0$. On the other hand, u_∞ is a harmonic function if $e_\infty = 0$ and the result follows trivially.

Appendix: $C^{1,1}$ regularity of the distance function for $C^{1,1}$ domains

Suppose that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with $C^{1,1}$ boundary $\partial\Omega$. Let

$$d \,\equiv\, \operatorname{dist}(\cdot, \partial\Omega) \;:\; \Omega \longrightarrow \mathbb{R}$$

symbolise the distance function to the boundary. For $r > 0$, let Ω_r denote again the inner r-neighbourhood of $\partial\Omega$ in Ω:

$$\Omega_r \,=\, \{x \in \Omega \,:\, d(x) < r\}.$$

In this appendix we establish that $d \in C^{1,1}(\overline{\Omega_r})$ when r is sufficiently small. This fact is probably known but we couldn't locate a precise reference in the literature.

The C^2 regularity of the distance function for a C^2 boundary $\partial\Omega$ is a classical result, see e.g. [GT, Appendix 14.6]. On the other hand, the case of C^1 regularity of the distance function when the boundary $\partial\Omega$ is C^1 holds under the extra hypothesis that the distance is realised at *one* point; see e.g. [F].

In order to prove the desired $C^{1,1}$ regularity of the distance function near $\partial\Omega$ when the boundary itself is a $C^{1,1}$ manifold (which we utilised in Lemma 7), we first note the following fact: suppose that $r > 0$ is such that $1/r$ is larger than the essential supremum of the curvature of $\partial\Omega$. If $x \in \Omega$ and $y \in \partial\Omega$ with $|x-y| = s \leq r$ and such that the tangent hyperplanes of $\partial\Omega$ and $\partial\mathbb{B}_s(x)$ coincide at y, then it follows that $\mathbb{B}_s(x) \subseteq \Omega$ and $\partial\mathbb{B}_s(x) \cap \partial\Omega = \{y\}$. (This is easy to see when $\partial\Omega$ is C^2 regular and follows by an approximation of $\partial\Omega$ with C^2 manifolds otherwise.) Therefore, in the above situation, it follows that $d(x) = s$. Moreover, for $x \in \Omega$ with $d(x) \leq r$, it follows that there exists a *unique* point $y \in \partial\Omega$ such that $|x-y| = d(x)$. Moreover, if ν denotes the outer normal vector on $\partial\Omega$, then $x = y - d(x)\nu(y)$ and $\mathrm{D}d(x) = -\nu(y)$.

Now fix $x_0 \in \partial\Omega$. Our aim is to prove $C^{1,1}$ regularity of d near x_0. To this end, we may assume without loss of generality that there exist open sets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^{n-1}$ such that $x_0 \in U$ and $0 \in V$ and there exists a function $f \in C^{1,1}(V)$ such that

$$\Omega \cap U \;=\; \Big\{(x', x_n) \in V \times \mathbb{R} \,:\, x_n > f(x')\Big\} \cap U$$

and such that $x_0 = (0, f(0))$ and $\mathrm{D}f(0) = 0$. Define

$$N(x') \;:=\; \frac{(-\mathrm{D}f(x'), 1)}{\sqrt{1 + |\mathrm{D}f(x')|^2}}, \quad x' \in V,$$

so that $N(x') = -\nu(x', f(x'))$ for $x' \in V$. Note that this map is Lipschitz continuous. We now define $\Psi : V \times \mathbb{R} \longrightarrow \mathbb{R}^n$ by

$$\Psi(x', t) \;:=\; (x', f(x')) \;+\; t\, N(x').$$

Then Ψ is $C^{0,1}$ near $(0, 0)$. Note also that Ψ is injective in a sufficiently small neighbourhood of $(0, 0)$: if we had $\Psi(x', s) = \Psi(y', t) =: z$ with $0 < s \leq t \leq r$, then it would follow that $\partial\mathbb{B}_t(z)$ and $\partial\Omega$ have the same tangent hyperplanes at $(y', f(y'))$. By the above observations, this would imply that $\mathbb{B}_t(z) \subseteq \Omega$ and $\partial\mathbb{B}_t(z) \cap \partial\Omega = \{(y', f(y'))\}$, and therefore $x' = y'$ and $s = t$. So if U and V are chosen appropriately, then Ψ is a bijection between $V \times [0, r)$ and $U \cap \overline{\Omega}$. Moreover, we compute

$$\mathrm{D}_j \Psi_i(x', t) \;=\; \delta_{ij} \;+\; t\, \mathrm{D}_j N_i(x'), \qquad i, j = 1, \ldots, n-1,$$

and

$$\mathrm{D}_j \Psi_n(x', t) \;=\; \mathrm{D}_j f(x') + t\, \mathrm{D}_j N_n(x'), \qquad j = 1, \ldots, n-1,$$

while

$$\mathrm{D}_t \Psi(x', t) \;=\; N(x').$$

Since $\mathrm{D}f(0) = 0$ and $N(0) = (0, \ldots, 0, 1)$, it follows that $\mathrm{D}\Psi$ is of full rank in some neighbourhood of $(0, 0)$ and moreover, the inverse $(\mathrm{D}\Psi)^{-1}$ is essentially bounded in this neighbourhood. That is, by making V and r smaller if necessary, without loss of generality we may assume that

$$\Psi^{-1} \;\in\; C^{0,1}\big(U \cap \overline{\Omega}; V \times [0, r)\big).$$

Now note that

$$\mathrm{D}d(\Psi(x',t)) \,=\, N(x')$$

whenever $t>0$ is small enough. Hence if π denotes the projection onto $\mathbb{R}^{n-1}\times\{0\}$, then we obtain the formula

$$\mathrm{D}d \,=\, N\circ\pi\circ\Psi^{-1}$$

near x_0. The right-hand side is of class $C^{0,1}$, and thus d is of class $C^{1,1}$ near x_0. A compactness argument then proves the above statement.

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References

AK. H. Abugirda, N. Katzourakis, *Existence of 1D vectorial absolute minimisers in L^∞ under minimal assumptions*, Proceedings of the AMS, in press.

AyK. B. Ayanbayev, N. Katzourakis, *A Pointwise Characterisation of the PDE system of vectorial Calculus of variations in L^∞*, ArXiv preprint https://arxiv.org/pdf/1509.01811.pdf.

A1. G. Aronsson, *Minimization problems for the functional $sup_x\mathcal{F}(x,f(x),f'(x))$*, Arkiv für Mat. 6 (1965), 33 - 53.

A2. G. Aronsson, *Minimization problems for the functional $sup_x\mathcal{F}(x,f(x),f'(x))$ II*, Arkiv für Mat. 6 (1966), 409 - 431.

A3. G. Aronsson, *Extension of functions satisfying Lipschitz conditions*, Arkiv für Mat. 6 (1967), 551 - 561.

A4. G. Aronsson, *On Certain Minimax Problems and Pontryagin's Maximum Principle*, Calculus of Variations and PDE 37, 99 - 109 (2010).

AB. G. Aronsson, E.N. Barron, *L^∞ Variational Problems with Running Costs and Constraints*, Appl Math Optim 65, 53 - 90 (2012).

C. M. G. Crandall, *A visit with the ∞-Laplacian*, in *Calculus of Variations and Non-Linear Partial Differential Equations*, Springer Lecture notes in Mathematics 1927, CIME, Cetraro Italy 2005.

CIL. M. G. Crandall, H. Ishii, P.-L. Lions, *User's Guide to Viscosity Solutions of second Order Partial Differential Equations*, Bulletin of the AMS 27, 1-67 (1992).

CKP. G. Croce, N. Katzourakis, G. Pisante, *$\mathcal{D}$-solutions to the system of vectorial Calculus of Variations in L^∞ via the Baire Category method for the singular values*, ArXiv preprint, https://arxiv.org/pdf/1604.04385.pdf.

D. B. Dacorogna, *Direct Methods in the Calculus of Variations*, 2nd Edition, Volume 78, Applied Mathematical Sciences, Springer, 2008.

E. L.C. Evans, *Weak convergence methods for nonlinear partial differential equations*, Regional conference series in mathematics 74, AMS, 1990.

E2. L.C. Evans, *Partial Differential Equations*, AMS, Graduate Studies in Mathematics Vol. 19, 1998.

ES. L. C. Evans, O. Savin, *$C^{1,\alpha}$ Regularity for Infinity Harmonic Functions in Two Dimensions*, Calc. Var. 32, 325 - 347, (2008).

FL. I. Fonseca, G. Leoni, *Modern methods in the Calculus of Variations: L^p spaces*, Springer Monographs in Mathematics, 2007.

F. R.L. Foote, *Regularity of the distance function*, Proceedings of the AMS 92 (1), 153 - 155 (1984).

GM. M. Giaquinta, L. Martinazzi, *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*, Publications of the Scuola Normale Superiore 11, Springer, 2012.

GT. D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, reprint of the 1998 edition, Springer.

HS. R. Hardt, L. Simon, *Nodal sets for solutions of elliptic equations*, J. Differential Geometry 30, 505 - 522 (1989).

K1. N. Katzourakis, *L^∞-Variational Problems for Maps and the Aronsson PDE system*, J. Differential Equations, Volume 253, Issue 7 (2012), 2123 - 2139.

K2. N. Katzourakis, *Explicit 2D ∞-Harmonic Maps whose Interfaces have Junctions and Corners*, Comptes Rendus Acad. Sci. Paris, Ser.I, 351 (2013) 677 - 680.

K3. N. Katzourakis, *On the Structure of ∞-Harmonic Maps*, Communications in PDE, Volume 39, Issue 11 (2014), 2091 - 2124.

K4. N. Katzourakis, *∞-Minimal Submanifolds*, Proceedings of the Amer. Math. Soc., 142 (2014) 2797-2811.

K5. N. Katzourakis, *The Subelliptic ∞-Laplace System on Carnot-Carathèodory Spaces*, Advances in Nonlinear Analysis. Vol. 2, Issue 2 (2013), 213 - 233.

K6. N. Katzourakis, *Nonuniqueness in Vector-valued Calculus of Variations in L^∞ and some Linear Elliptic Systems*, Communications on Pure and Applied Analysis, Vol. 14, 1, 313 - 327 (2015).

K7. N. Katzourakis, *Optimal ∞-Quasiconformal Immersions*, ESAIM Control Optim. Calc. Var. 21 (2015), no. 2, 561 - 582.

K8. N. Katzourakis, *An Introduction to Viscosity Solutions for Fully Nonlinear PDE with Applications to Calculus of Variations in L^∞*, Springer Briefs in Mathematics, 2015, DOI 10.1007/978-3-319-12829-0.

K9. N. Katzourakis, *Absolutely minimising generalised solutions to the equations of vectorial Calculus of Variations in L^∞*, Calculus of Variations and PDE, to appear.

K10. N. Katzourakis, *Generalised solutions for fully nonlinear PDE systems and existence-uniqueness theorems*, ArXiv preprint, http://arxiv.org/pdf/1501.06164.pdf.

K11. N. Katzourakis, *Mollification of $\mathcal{D}$-solutions to fully nonlinear PDE systems*, ArXiv preprint, http://arxiv.org/pdf/1508.05519.pdf.

K12. N. Katzourakis, *A new characterisation of ∞-harmonic and p-harmonic maps via affine variations in L^∞*, ArXiv preprint, https://arxiv.org/pdf/1509.01811.pdf.

K13. N. Katzourakis, *Solutions of Vectorial Hamilton-Jacobi Equations are Rank-One absolute minimisers in L^∞*, ArXiv preprint https://arxiv.org/pdf/1509.01811.pdf.

KK. N. Katzourakis, J. Kristensen, *Strong convergence of derivatives of p-Polyharmonic mappings as $p \to \infty$*, manuscript in preparation.

KM. N. Katzourakis, J. Manfredi, *Remarks on the Validity of the Maximum Principle for the ∞-Laplacian*, Le Matematiche, Vol. LXXI (2016) Fasc. I, 63 - 74, DOI: 10.4418/2016.71.1.5.

KP. N. Katzourakis, T. Pryer, *On the numerical approximation of ∞-Harmonic mappings*, Nonlinear Differential Equations & Applications 23 (6), 1-23 (2016).

KP2. N. Katzourakis, T. Pryer, *2nd order L^∞ variational problems and the ∞-Polylaplacian*, ArXiv preprint, http://arxiv.org/pdf/1605.07880.pdf.

KP3. N. Katzourakis, T. Pryer, *On the numerical approximation of ∞-Biharmonic and p-Biharmonic functions*, preprint.

L. J.L. Lions, *Optimal control of systems governed by partial differential equaitons*, Springer-Verlag 1971.

MS. R. Moser, H. Schwetlick, *Minimizers of a weighted maximum of the Gauss curvature*, Annals of Global Analysis and Geometry, 41 (2), 199 - 207, 2012.

S1. Z. Sakellaris, *Minimization of Scalar Curvature in Conformal Geometry*, Annals of Global Analysis and Geometry, in press.

S2. O. Savin, *C^1 Regularity for Infinity Harmonic Functions in Two Dimensions*, Arch. Rational Mech. Anal. 176, 351 - 361, (2005).

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