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# Mathematical Modelling for the Digital Society

by

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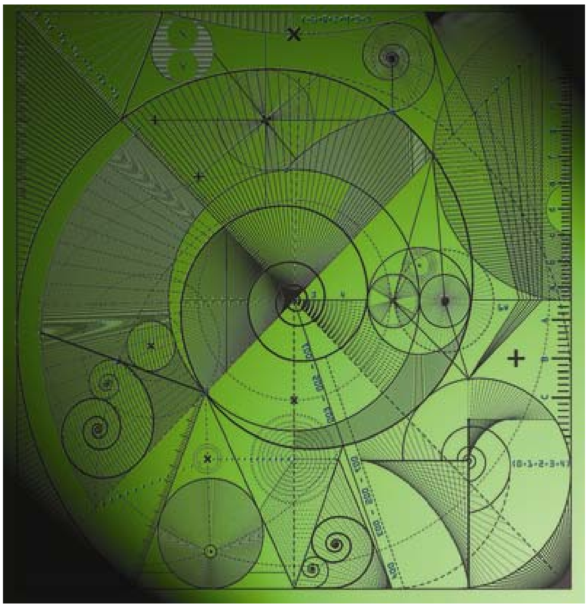

# Mathematical Modelling for the Digital Society

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## 1 The information economy and the digital society

We live in a digital world. The upcoming generations within our society are increasingly technology savvy. Individuals simply accept all of the rapidly evolving, disruptive, innovative feature-functionality of the digital, information economy in the same way that their forbears simply accepted the gifts of modern electricity supply, automobiles, TVs, plastics, and pharmaceuticals (within the old manufacturing economy). They are vaguely aware of how new technologies might or might not work, yet they are much more prepared to invest their time and energy into exploiting their advantages rather than in understanding their nuts and bolts.

Across all aspects of our society, modern advances in IT, fast communications, and the convergence of useable platforms are producing new spaces in which individuals can work, rest and play. Services and applications often go viral and are often adaptive. Users educate each other, and that is part of the thrill. Moreover, where the innovations resonate with the users, in both intended and unintended ways, we see an explosive take-up which radically disrupts and changes entire commercial sectors, our social norms, and our own aspirations. Forever.

Of course it isn't just the growth of communications in Digital Britain [1], the Digital Economy[2], or across the world; there are other forces that make such changes irresistible. The empowerment of individuals, the democracy (or tyranny) of the internet, and the rise of the virtual self, through blogs, social networking, citizen journalism, broadcast messaging (tweets), all mean that our opinions, perceptions and ideas can now be heard. The changes in the demographics for the nation, including the rise of a tech-savvy generation of over fifty tear olds, mean that issues of health, well being and aging are to the fore. Moreover the public concerns over the environment, energy, and climate issues, and the ethics of global sourcing are also empowering drivers. This digital

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society gives individuals, public institutes, and businesses a new voice, a presence, and a connectivity that is both empowering and staggering.

Many of the aspects and activities within the digital society have a trust basis, and many are implicitly altruistic. At its inception many people thought e-bay [3] could not work: why would somebody send money to sellers for goods that may be nonexistent, faulty or never arrive? Yet it worked. The provision of many services such as search engines for free has produced a whole generation of users who just expect things should be free. As an unintended consequences file sharing and similar activities, seem obvious and natural, yet fly in the face of property rights. Attempts to fight this trend can appear draconian: yet the ethics of fairness and respect for property ownership must be set against the urge for access and empowerment.

In some sectors of the Digital Society modern IT and communications has indeed already been extremely transformative. The music industry for example with music downloads reducing cost to the consumer and MySpace pages performing the A&R function, instead of having these within recording companies. Conversely the value has been returned to live music. For many business to business, business to customer, and peer to peer interactions the internet alone has provided access, competition, audiences, and theatre. As digital voice, recording, and cameras became commonplace, on palms and phones, so skype, youtube, and face-book have made us massively more connected, more aware, and more public. Indeed we all manage a single offline persona and multiple online personas. All of this is upside. One can watch or amuse one's friends, avoid UK censorship, Google one's prescriptions, buy pharmaceutical and designer drugs, sell one's old stuff, buy books and music, protest on the No 10 petitions page. One can be creative, political, caring and in touch with whoever, whenever.

Applications on mobile phones are transforming lives: they can exploit pervasive and location aware technology. The convergence of phones, palms, and notebooks, with intuitive interfaces means that functionality is now available to developers and users, as never before. This was evident ten years ago: some commentators thought that sms would never catch on - but look at it now. The rise of application for mobiles, messaging, of mobile banking, and digital advertising, media players (iplayer, ipod) dominate the digital traffic (the iplayer alone provides a significant challenge to the UK infrastructure)

There is a downside too. In the very early days of the internet (dial up access at home!), before spam filters became popular, the web porn industry made opening emails a daily education for all. There are chat rooms and social network opportunities for grooming, stalking, and cyber threats to personal safety and security. These are made worse by the particular exposure of young and trusting (naive) members of society. If we think of the digital society as a town, then there are no nice neighborhoods. Everybody is up close to everybody else. So there is an obvious need to take care. Yet this contradicts the underlying users' desires for openness, trust and altruism that is so liberating for them, and despite obvious and repeated warnings the latter is the more powerful driver.

But there is worse. The digital society is wide open to abuse from those who would attack our society, or seek to undermine it. Cyber enabled threats are a huge concern. More so than threats to the cyber infrastructure itself. Indeed would-be terrorists benefit from the existence of the internet, and the intelligence and connectivity it provides. There are also cyber threats that are organized, or self organized, hacking attacks (from patriotic or motivated individuals) aimed at public and company infrastructures. These are highly competitive and trophy driven, and are observed daily. Some of these are state sponsored or state driven, designed to attack infrastructure in services, supply or even health and wellbeing. Advances in technology, and online or application functionality (Google Earth, Street View, Youtube, Facebook, Twitter...) are a huge help to terrorist operations, from their recruitment, and propaganda production , through to their operations (reconnaissance, planning, to action), and their ability to amplify the impact and publish the public's concerns and perceptions.

Pervasive communications and computing provide further opportunities for us all to be connected and aware. As individuals we leave a digital footprint wherever we go. Now the rise of cloud computing and software as a service will change for ever the small medium and large corporate IT spends - once, of course, the security situation is sorted out. Public institutional use is less likely to change as rapidly. This is because it is less cost driven and institutionally slow to change and less adventurous.

In the UK these changes have brought opportunities to entrepreneurs and investors. It was a national concern that the research base was relatively slow to act. For data analysis and modelling this may be doubly true. Indeed the analysis of data from the digital society is much more keenly discussed within companies and business schools than within science faculties [4, 5, 6, 7, 8, 9]. Now however we have a UK programme: the RCUK Digital Economy programme [2]. It is truly multidisciplinary with creative arts, social sciences, business, marketing, computer science, and maths input; and a range of applied research (prototypes and demonstrators), research in the wild (doing things with real people), training (DTCs), outreach, and industry engagement, as well as public, ethics, and security considerations.

This Digital Economy challenge includes mathematical modelling too. In this paper we will describe some of the problems and ideas that this represents. We will argue that there will be a genuine two-way dividend. Real benefits for the digital economy players; and some real benefits for mathematics. This will force us to respond to both the conceptual and modelling challenges.

This points to a fundamental weakness in the (research base's) complexity modelling cannon. There we are modelling, predicting, inferencing, and simulating multilevel systems. In the digital society, at the microscopic level we have people not atoms. There are inconsistencies, and imperfect information. Whereas in physical or chemical complex systems, at the microscopic level we have conservation laws (energy, mass, momenta). On the other hand, in physical and chemical complex systems at the macroscopic level we can have field equations (density functions, continuum mechanics and so on). Yet within macro economics, for example, forecasting and modelling is relatively unreliable and ineffective. Even in well defined markets, such as financial instrument modelling, assumptions of perfect information and arbitrage are problematic and may fly in the face of phenomena observed.

Thus, implicit in this challenge from the digital society is the development of new concepts and models that will describe social behaviour, economic decision making, and peer to peer interactions (voice, messaging, proximity, location, transactions). The ace up our sleeve is the availability of the data: massive data sets from the individuals' digital footprints. The complete lack of such mass data in the past has held back progress in many areas in traditional socioeconomic modelling. But where hard data has been available, for example with the very popular retailer loyalty cards or online transactions, the progress in the past ten to fifteen years has been extraordinary (we will return to this topic later).

So the capture of, and access to, the digital data is what sets the *digital society* apart from the old, nondigital, society. And this is also the very resource that is needed to challenge we modellers to come up with fresh ideas and new mathematics.

# 2 The birth of new mathematical applications

Some have been through this type of mathematical phase change before. In the late 1970s and early 1980s, there was a new mishmash of ideas and applications that were together loosely termed *mathematical biology* . Some of these components were not new, going back to the work of Darcy Thompson and Alan Turing: other were very radical. Indeed many of these concepts and models did much of challenge and exploit theory of nonlinear dynamical systems (in iterations, ODES and PDEs). It was a zoo: there were all sorts of ideas. By 1985 most universities in the UK had at least one mathematician, if not a few, prepared to call their work mathematical biology. Order was achieved through great leadership and great scholarship.

The key point is that not everything in the original zoo made it all the way out of the 1980s into what we now call mathematical biology. Some ideas were red herrings, some were dead ends, some were ill-defined at best, or just ultimately wrong. The boiling up and distillation of theory and applications is the essential right of passage for new mathematical fields. It is natural.

We could recount a similar history of the rise of *mathematical finance*, from the work of Black and Scholes onwards. Such theory and applications are now commonly part of mainstream mathematical courses and mathematical research. Indeed they are such a part of the furniture they may be blamed for the ills of others, as in the origins of the credit crunch.

Of course in the birth of any new field of mathematical application there is also risk to the mathematicians. The provision of leadership and scholarship itself requires people to commit to these new fields: fields that others simply may not value, understand, nor accept, at best; or at worst may deride or despise, as competitors (for funding) to their own fields of research. This should not happen, but we know it does. A particular sore point for some mathematicians is the multidisciplinary nature of some new fields: with funding for mathematical research being so tight (especially in the UK) how much can the community carry? So cross disciplinary, thematic programmes, such as the Digital Economy programme, represent an additional opportunity and do not require leverage or investment from existing subject specific programmes.

An adherence to a new discipline of *mathematics for the digital society* may be validated every time one speaks to the potential exploiters, especially within service sector companies: in retail, consumer goods, telecoms, online businesses, energy, finance and IT and communications (including software an sevices). All of these companies' operational and research groups will recognize the value of analysis that rises to these digital challenges. They are intrinsic to the future success of our companies, our economy and our international compettiveness. So we can feel comfortable in having this very strong pull from potential users and exploiters. These are companies which work in hugely competitive sectors, that are internationally excellent, and where every decision for investment and activity is tensioned. Of course if members of the mathematical community do not choose to engage in such conversations then they will not realize the huge potential for impact that these sectors offer. They are not interested, and have their own roads to hoe. For some it is an obvious opportunity though: we should want our mathematical ideas to be points of differentiation, providing an insightful advantage for their users. We need them also to challenge the existing mathematical cannon: to tell us what we can not do presently.

The contention argued here is that this *boiling up* phase, prior to distillation, is what is happening now. In ten year's time the mathematical modelling and research of digital societies, businesses and economies will be common place. We can expect every mathematics department in the UK will be doing some of it. Industry and commerce needs it, the government and public regulators will require it, and our students will be attracted towards it: jobs, careers, entrepreneurial opportunities, and research challenges with be the drivers.

# 3 Mathematical challenges

In this section we highlight some particular problems that themselves suggest new areas of mathematical modelling and analysis may be required. These are examples - part of the current zoo. We cannot offer any solutions, but rather directions; and we may discuss the deficiencies of some current approaches.

They point to some wider issues too: the need for a mixed community of public and private research. As mathematicians we need to become much more comfortable with this: we may trade openness for publication, in the short term, with *impact*. Indeed some of these subjects cannot proceed rapidly, in my view, without the route to commercial exploitation and the sector *pull*. In the longer term of course (10-20 years) we may expect the community will settle on a balanced symbiosis. We have seen similar tensions in biosciences (genome analysis, informatics and so on), and closer to home within mathematical finance, where investment houses may be commercially secretive: yet that field has become more mainstream over twenty years. This mixture of public and private research is not a new issue for mathematics, and we should not be precious about it.

## 3.1 Evolving networks

There has been a huge amount written about social networks and communication networks in recent years, based on the growth of mobile telephone, online messaging and social networks themselves. For the most part, when quantitative, this has encompassed attempts to (a) describe snap shots of such networks (characterized as small world [10], or scale free [11], and so on), or (b) to monitor the evolution of macroscopic network properties (clustering coefficients, degrees distributions, connectivity and centrality) [12, 13]. Yet both of these approaches avoid a fundamental property: some edges are more permanent than others. Imagine taking successive daily, hourly or secondly sequential snapshots of a communication network. People stand at the vertices and the edges indicated some transient two way communication. As time evolves some of the edges appear and disappear repeatedly, some are constant, and some are almost always absent. We see an object that evolves, even without adding more vertices. An example is shown in Figure 1. The network constantly moves from one form to another: we have a sequence of graphs $\{G_k | k = 0, 1, 2, ...\}$ indexed with time. As modellers we should define some dynamical mathematical objects which evolve similarly. By comparing our theoretical models of evolving networks with those given by the observed data, we can rule in or out alternative modelling assumptions and make some key inferences.

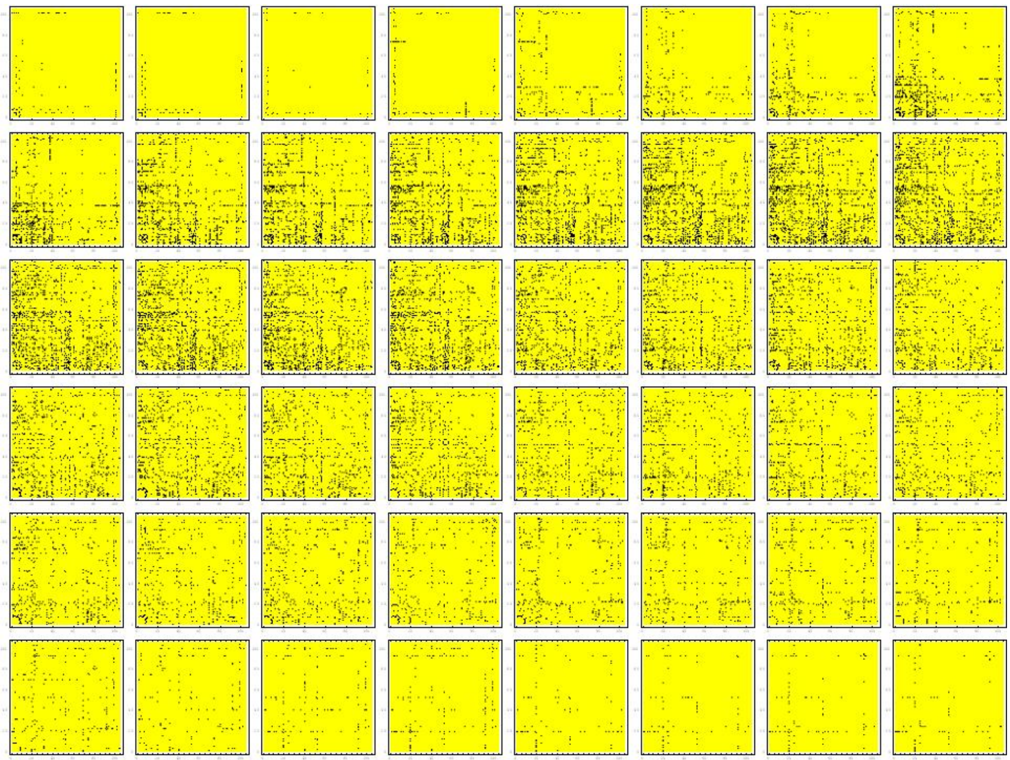

Figure 1. Reality mining data for weekly voice call interactions between 106 subjects, over 48 weeks: adjacency matrices, data from [17].

In recent work we have looked at evolving networks as Markov models, analyzed inverse problems (how to represent data within a class of such models), and information propagation problems (dynamical SIR epidemiological models defined on dynamical evolving networks) [14]. We have looked at evolving networks which have long term memory effects [15], and looked at the long term behaviour of networks subject to edge-dependent dynamics [16].

Models for evolving networks are simply stochastic processes defined over the set of all possible graphs. Our recent attempts to apply such models to social and communication networks are certainly naive but potentially game changing. At the very least models that yield the distribution of (conditional) probabilities $P(G_k|G_{k-1}, G_{k-2}, ...)$ are of real value to the network owners or to users, in estimating about what may happen next, and what is or is not aberrant behaviour.

What do we expect to see when we observe such networks? There are a number of applications in marketing, information management, and security that are of immediate interest. Let $A_k$ be the symmetric adjacency matrix corresponding to the undirected graph $G_k$, defines over $N$ vertices. An object of interest is the "communicability" matrix [19]. There are a number of ways to define this, but one such is

$$B_K = (I - \alpha A_0)^{-1}.(I - \alpha A_1)^{-1}.(I - \alpha A_2)^{-1}...(I - \alpha A_K)^{-1}.$$

This is non-symmetric owing to the ordering of the sequence. The $i, j$th term represents the number of walks, made up of (successive) edges, between vertex $i$ and $j$ as the network itself evolves, applying a geometrical discount to the walk lengths. Here we allow for any number (including zero) of edges from each successive time step $G_k$ to be used in a walk. The constant $\alpha > 0$ must be such that that $1/\alpha$ is greater than the spectral radii of all of the $A_k$.

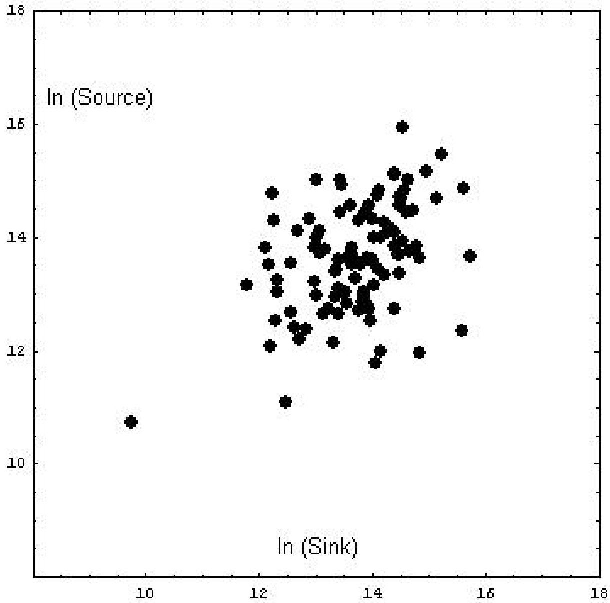


Figure 2. Source-communicability (rows sums of $B_K$) versus sink-communicability (column sums): $\alpha = 0.03$, natural log scale.

Immediately one can see that some vertices are good propagators (good message sources, reaching out to all other vertices) whilst others are good listeners (good message sinks, hearing everything from all other vertices). We can illustrate this using the data given in Figure 1: see Figure 2. A further question revolves around when the communicability is most sensitive to the $A_k$. Such critical moments may be useful in managing, or disturbing, the networks or the information being propagated through them.

It is not hard to see the applications in monitoring surveillance, and network management; but also in messaging propagation, marketing and persuasion. Viral or buzz marketing are the modern version of *word of mouth* concepts: these were perviously limited to Bass diffusion types of macroscopic models [20]. Now we have a whole new experimental and theoretical menagerie of evolving networks to characterize, callibrate and exploit.

To date social network data is available from a variety of sources (see [17, 18] for example, for mobile communication and social network data sets analyzed in [15]). Or CRAW-DAD, a Community Resource for Archiving Wireless Data At Dartmouth, a wireless network data resource for the research community [22]. These are useful practical resources (not least for publication purposes) but the really challenging data sets will be commercially held, and access usually requires some confidential agreement, or will be securely held by public bodies or regulators (and even secure access may require permissions).

## 3.2 Behaviour based inferencing for individuals

In some countries and economic regions there are leapfrogs in technology. In Africa there are presently two taking place. First the population has become connected by mobile telephony, there being no land line infrastructure, and the mass phone usage has skipped straight into the digital space. Second there is no rescalable conventional banking network: the masses simply cannot be traditionally banked. So people are becoming banked through their mobile phones. In Kenya after 2 years from launch M-Pesa has twice as many m-banking accounts as there are real bank accounts. In Africa it is estimated that there are 1B people who own mobile phones but no bank account.

Providing a simple bank account, enabling virtual transactions, is the relatively straightforward part: it is an operational challenge, not a conceptual or analytical one. What comes next will be more interesting. Banks, governments and financial NGOs all want to see more people financially included: this will in turn drive the economic development of the region. So they need to offer a range of financial services: overdrafts. personal loans, saving and investment schemes, insurances, business loans and so on. But there is no credit referencing available: conventional records and database assets (information such as the number and details of an applicant's loan accounts, details of slow payments, bankruptcies, the number of requests for new credit and so on) are largely nonexistent. So the challenge is to take what we have, that which cannot be manufactured or simply claimed by applicants. That is the digital data from their mobile accounts, internet accounts, or other digital footprints and use this as a surrogate for the owners' lifestyles. Then we must infer whatever it is that we need to know - in this case credit worthiness, reliability, and stability - from the digital profile.

Such challenges go far beyond behaviour based credit referencing. But this is starting to happen, and the emerging field is less than a decade old [21, 23]. Because people do not sit still (they get on with their lives) there needs to be a dynamical element in such classification and inferences. Static segmentations (lifestyle or life-stage, for example [30]) developed for conventional marketing over the past twenty years, and are reapplied infrequently, simply can not react fast enough to keep pace with the digital age. If we can use mobile telco databases (all transactions, calls, sms, data, incoming, outgoing, time of day, day of week, volatility, biases, tariff switching and so on) one can imagine a number of alternative analytical way to make inferences.

In the commercial space a data driven Markov modelling approach has been developed and piloted during 2009 in Tanzania [24], with the research completed within a technology start-up company. There the partitioning is via state definitions, defined over a customer-time period *feature space*. These may be optimized so as to fine-focus on (and hence anticipate) the major behavioral transitions that customers make in their individual behaviours. When behavioral state data is put together with existing loan data sets some of the defined states inevitably imply a very high rate of defaulting or incomplete repayments (four or five times the mean). Accomplishing such analysis in practice is a challenge as there are telecoms, information, and banking regulators to be satisfied within all territories.

Behaviour based inferencing requires some sort of hybrid approach, from unsupervised discrimination (say, finite mixture modelling [25] in a multidimensional space, using the EM algorithm [26] for example), to searching though competing models so as to optimize performance measures based on desired forecasting (using a some discrete search methodologies: genetic algorithms [27], for example). This procedure is currently experimental, and far from rigorous.

Away from m-banking and m-lending there are many other fields where dynamical behaviour based inferencing is important. Not least in monitoring populations for aberrant behaviour, and in direct one to one marketing. If subjects stray into certain behavioral states at some time then they can receive the attention, advice or messaging that is appropriate. Even online gaming requires real-time assessment (at online poker tables for example) that may indicate fraudulent behaviour. Bayesian approaches [28, 29], especially in multiple hypothesis testing, should be productive and provide actionable decision support.

Mathematical methods need to become more adaptive, so that models implicitly learn (the distinctive types of) behaviour and evolve with the subject/user base. Just like the networking problems, behavioral memory may become a big issue. So the state of the art moves from static segmentations, beloved of marketing companies, and customer relationship management groups (such at a Experion's Mosaic [30]), through to dynamical state transition models (now being trialled for m-banking by technology start ups [24]), through to intelligent, learning, evolving, individual memory dependent modelling (the next generation). Green shoots are appearing (see [31], for example).

Digital marketing, often called new media, requires (a) low cost access to subjects via pervasive telecoms and computing; and (b) mathematical sophistication to distill the subject base and target the right messages to the right people. The delivery of digital marketing like spam will become annoying and result in a loss of reputation and trust rather than the desired outcome. The mathematical modelling required should be refined enough to recognize new forms of behaviour emerging from within the crowd: not large by energy, but distinctive. Needles in the digital haystack: not obvious, in your face, tipping points. How will this happen?

Of course there is an obvious tension here between what is carried out with publicly funded research programmes and what is carried out commercially, in-confidence, building known-how, intellectual property and software implementations for corporations and investors. In the longer term much of these activities will drive research in our universities. It is clear that there needs to be a mixed economy of public and private research.

## 3.3 Simulation and agent based modelling

Agent based models (ABMs) provide a very natural simulation tool for complex systems - especially where the agent-to-agent interactions are dynamic. However their use should not be limited to phenomenological demonstrations (or existence and uniqueness proofs), where at best we see that simply interactions and stochastic dynamics at the microscale can result in sophisticated dynamics (phase changes, tipping points and so on) at a macro scale. Buchannan's book [4] provides a readable introduction and history.

The greatest challenge for ABMs lies in their deployment to represent business or social systems where there is a large amount of data available. It is the digital economy equivalent of data assimilation. We wish to match behaviour observed at the macros scale by (parts of) populations; but we must specify deterministic and stochastic processes, plus interactions , all at the micro (agent) scale. Only by meeting this challenge directly can we produce ABMs with sufficient sophistication and relevance that they can be deployed.

Of course having agents behave stochastically, irrationally, sub-optimally is essential in many applications. Hidden variables within ABMs representing sentiments, moods, modes of decision making, are a bonus. If calibrated we will gain the extra information (hidden class variables, simply not observable in the data) giving the reasons why or how proportions of the population took certain actions. Hence suitably calibrated ABMs yield insights and could be applied in new situations where the specifics of the agents' transaction or their options change, yet the hidden mode/sentiment variables remain. In that way one would hope to simulate and forecast the population response to new scenarios. Hence strategic scenario and planning methods may be the key application *if* the data assimilation problem and the hidden (mode) variables can be mastered.

## 3.4 Forecasting and social teleconnections

Can the chatter on social media provide the ability to track communities' interests in real time? Can one data set independent and, digitally at least, at a distance from another, be used to infer or predict the latter's behaviour (spending , transactions and action)? Could this inform public policy or commercial strategy?

In [32] the authors counted mentions of film titles on Twitter.com: using some 3 million tweets over 3 months. They found that the rate of tweeting about movies can accurately forecast the box office revenue of the film, but only after it is released. Film studios and others need this to gain an early estimate for the expected popularity of their products. The most accurate source to date is the Hollywood Stock Exchange, a market in which people can buy and sell virtual shares in actors, directors and individual movies. But now the rate of tweeting provides a much better forecast.

But does it work in reverse? Can one manage the demand for their film, product or service by directly influencing how people tweet about it?

Each week, millions search online for health information. There are flu-related searches during the flu season, allergy-related searches during the allergy season, and so on. Google Insights [33] now enables anybody to explore these trends. By counting the volume of flu related search queries made through Google, one may estimate how much flu is circulating in different countries and regions around the world [34]. There is similar recent work on the ability of Twitter on the swine flu outbreak in 2009, [35].

Knowledge about what, and how many, people are searching online; or of their behaviour within other social media; when fused with other data sets, representing possible consequences, yields a brand new opportunity. Indeed the new open data initiative (data.gov.uk) in the UK [36] may be a perfect, timely compliment to the (softer) digital social media behavioural data. Of particular interest should be the relationships between behaviour and consequences which are separated in time (by weeks and months) and across sectors (from education to economic growth, for example). Just like the idea of teleconnections within atmospheric science (the El Nino Southern Oscillation, for example) [37], these *social teleconnections* may provide actionable indicators, and suggestive of proto-models.

This activity could lead to a whole new branch of social modelling, with huge implications for government policy and commercial assessment. Below we show a simple example using data concerning the public interest in antisocial behaviour and ASBOs put together with the hard data on ASBOs actually issued. Prototype modelling ought to go far beyond analysis of the time series: we need the data to suggest concepts to be validated and models that produce testable hypotheses [38].


Figure 3. ASBOs - the public interest. Above: Google Insights measures searches from the UK [33]. Below: downloaded time line compared to "anti-social" searches and actual numbers of ASBO's issued [39]. Google's data is more recent, and higher resolution, than UK government stats, and it has annotations.

## 3.5 Customer relationship management and brand perception

In recent years many business have become more and more customer centric. For the retail and the fast moving consumer goods sectors (convenience stores, supermarkets and hypermarkets), within bricks and mortar stores and online, this paradigm change happened some years ago. Near to the end of the 1990's most retailers had settled on a category management approach which sought excellence and innovation in the way that product categories were sourced, ranged, marketed, priced and promoted: in short, how products appeared on the shelves. But this was not enough. In the customer centered approach, adopted in the early ti mid 2000s, the retailers switched to become focussed on how customers bought things, in combinations, so as to satiate their needs and desires. In short, how products were taken off the shelves.

The insights that enabled the switch were made available by mining and modelling the customers' disaggregated transaction data, especially longitudinal data such as loyalty card data or account data. Cross purchasing analysis at the basket level and the household level considering thousands of products simultaneously, Missions analysis (what different types of shopping were people doing in stores); the use of sub-stores or implants within stores (coffee shops, pharmacies, etc ), enhancements to the customer experience and participation, thematic initiatives such as well being, ethically sourced and premium quality offerings, and many other concepts were realized by the deployments of mathematical and statistical methods. Clustering vectors, the EM algorithm [26] for unsupervised discrimination, genetic algorithms for searching, supervised discrimination methods, Bayesian inference and real time updating for forecasting. Online product recommendation systems rely on the estimation of massive matrices of conditional purchasing probabilities. Such methods went far beyond the spreadsheet and the data was of a size that any analysis was often a challenge. Some of these topics are reviewed in [9] which gives a picture of where the customer centric retail has got to by mid 2005. Analysis and modelling are universally referred to as “analytics” in this field. Where possible the concepts and algorithms were picked up by retail software suppliers and analytics package providers: so there was no shortage to products and services offered into this space, and some convergence, with the larger software and service providers (such as Oracle, SAP, SAS, and so on) acquiring smaller boutique enterprises.

That sector is incredibly competitive. The quest for advantage between the rival suppliers (manufacturers), and the rival retailer chains themselves, is enduring and unending. The mathematical methods are also still moving on at a pace.

The lessons in Customer Relationship Management (CRM) learned by retailers have become relevant to a wide range of business for which longitudinal data is (or will be) available. Telecoms, banks, insurance companies have all seen their goods bought more competitively, and with purchasing decisions made more frequently. Their services behave more like fast moving, frequently bought, consumer goods than durables (how often did your father change his insurer thirty years ago?). They have embraced the quantitative methods of marketing devised for retail and consumer goods, sectors. Many seek behaviour based segmentations for their customers, such as the one depicted in Figure 3, below, based on their own specific indicators or services and product consumption by amount, category, frequency, volatility; and responses to inducements. But are these enough to get the most insight from the data?

Now the domestic energy sector is becoming digitized. All homes in the UK will shortly have smart meters providing 24/7 data on individual households usage for the first time; and we need to develop novel smart grid concepts, on the other side of the meters, to smooth out and manage both the demand and the household demand priorities. The analysis and management of usage will be the key to getting the most out of the existing network infrastructure, especially if demand increases radically to include the charging up of electric vehicles. Each sector that becomes customer centric on the back customer data has its own issues. Not least in the nature of the customer engagement and the market competition. Hence some combination of the mathematics deployed within retail will be relevant, while we expect that new concepts and methods will also come to the fore.

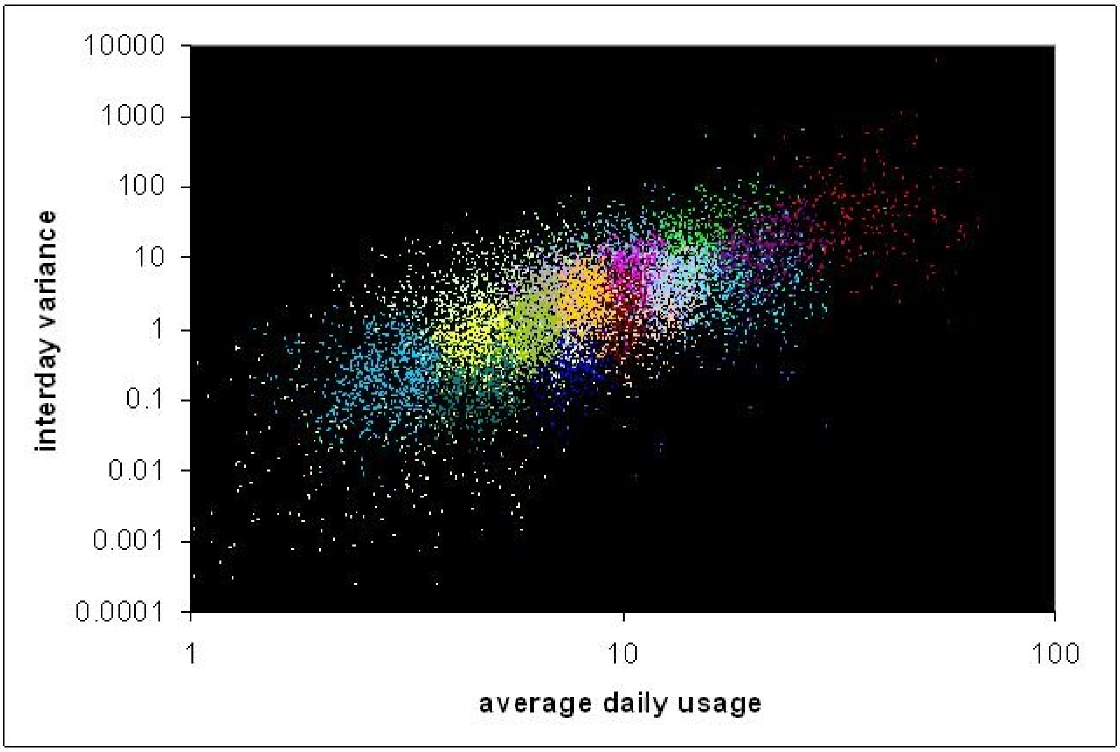


Figure 4. The unsupervised discrimination of 26 behavioral states based on 15 monthly customer key performance metrics, projected here into two dimensions.

Online purchasing has increased the competition - broadband penetration is on the rise, and online sales are increasing. Now you are only a few clicks away from your competition. Moreover the existence of market comparison aggregators has resulted in even fiercer pricing competition. For insurance, hotels, and gifting. But there is no reason for online sales to behave like sales in conventional shops all of the time: the next generation of analytics may need to reflect the two way interaction between buyers/bidders and sellers/suppliers (to support ebay,or taubau). The mathematical analysis of auctions of all flavours yet known, and yet to be discovered, needs to be more transparent and readily available. The convergence of purchasing, bidding, gaming and exploration is what will drive new forms of business to customers and peer to peer commerce. New business models will rely on multiplayer game and auctioning analysis.

The development of new products, new product formats and new services are designed to encourage loyalty and win share of total spend: but in practice a more aspiring, vociferous, and mobile community of consumers has the upper hand. For the goods suppliers who own the brands and manufacture consumer products this is a real issue. The power of the internet and the empowerment of users and consumers has resulted in an inversion. Who really defines the brands' positioning and their personalities? The consumers are increasingly to the fore. There is a whole industry now monitoring online what the users and consumers say or perceive about products and brands. They are in many cases, more innovative and more edgy than the brand managers and brand consultancies who previously held control. Your product is what ever people say it is.

## 3.6 Customer lifetime value

For many sectors and businesses customer churn is a key problem. When customers leave or defect to a competitor, what do they lose? They need to value present customers and see that the costs of acquisition may be dwarfed by the losses in total value from dissatisfied customers who trade down or who churn. So what comes next?

*Customer value* is a mature concept, but it is mostly conceptualized and talked about and is not often calculated within even most customer-centric businesses. Any modern business recognises that its key asset is the customer base, and its key activity is interacting with the customer. Businesses can evolve their product and service offerings (sometimes organically, sometimes radically), but the value of the business to its investors is based on how it might perform in the future, and where its future revenues will be earned. For many, their customer base is large and requires management and planning. Do they acquire their most valuable customers, or need to grow or develop them? Management science is alive to this issue and recent years have seen development of customer strategies. This includes the rise of CRM [40]: but awareness is not enough. Such businesses need to model, estimate, and manage thier customers' behaviour and responses to future initiatives and grow customer lifetime value (CLV). There is an excellent recent account of this topic, and potential solutions, given in [41]. This work also embeds the CLV modelling and calculation within a strategic and operational marketing framework.

Calculations of CLV, defined as the future sum of value or profits over the customer's future life, vary according to the sector a business is in [42]. They may employ some discounting of future revenues, or truncation at some suitable time horizon. Subscription services are particularly simple, whereas those situations where the time dependent customer value is highly behaviour driven and highly variable are much more difficult to model and manage.

Consider the following situation. Suppose a customer is represented by a probability density function $u(v,t)$ where $v$ denotes the net instantaneous value (rate of return) from the customer to the business (the rate of revenue or profit achievable) and $t$ is time. Then we might write

$$u_t(v,t) = -\lambda(v)u(v,t) + \int \phi(v,\hat{v})\lambda(\hat{v})u(\hat{v},t)\,d\hat{v}, \quad u(v,0) = \delta(v - v_0).$$

where $\lambda(v)$ denotes the rate at which a customer of value $v$ changes in value; the kernel $\phi(v,\hat{v})$ denotes conditional probability distribution that customers jumping from a rate $\hat{v}$ will convert to any new rate $v$. In state transition models with a finite number of states, this can reduce to a linear set of ODEs. If we also discretize time this reduces further to a discrete Markov model. In these simplifications one can estimate long term or lifetime value (with and without discounts for future revenues) using iterations of the transition matrix. Let us denote the solution of the above initial value problem by $u(t,v;v_0)$. Then the CLV for a customer starting at value $v_0$ today is simply

$$CLV(v_0) = \int_0^\infty \int v.u(t, v; v_0) dv\, dt.$$

Such models assume no long term memory, or customer age (or lethargy!). Within each state or at each value,$v$, all customer are mixed: people gradually trading up are mixed with those trading down. To avoid this it is useful to consider ways to define discrete states for which the state to state transitions are extremely sparse, and are dominated by a few key movements into and out of each state. The linearity of this model is also a drawback since there is no *following the crowd effect*, nor amplification of recent or popular transitions (copycat reinforcement). Finally the intended applications are not stationary: the products and whole offering will evolve, and there are seasons. So a simple transition matrix or kernel may *never* be valid.

Nevertheless the increase in the availability customer data (for both soft, behavioural, transactions and hard cash transactions) enabled by the digital society means that more companies and organizations will have the opportunity to be come customer centric and calculate some form of CLV. So we need better models that incorporate customers' soft history, their phase or age, the individual's (fairly recent) transaction history, some nonlinear peer to peer effects and external forcing, and nonstationarity. When calibrated such CLV models could forecast future income from the customer base (under various scenarios for financial planning), it might be used to drive marketing initiatives and justify investment in existing customers, it could be used to value the impacts of intended and unintended behavioral changes.

What makes one model superior to another? We not only need a more sophisticated class of models, we need frameworks for model optimization and model calibration based on the available historical data and the forecasting of the above uncertainties. Can such optimization be implicit in choice the state definitions/simplifications/resolution?

## 3.7 The role of delays in complex systems acting as information processors

The last topic is somewhat abstracted from its applications, but it is potentially of wide importance. Consider a complex systems of many units, each capable of passing some currency of *excitement*, or information, between each other. In the digital society the units are people and the currency may be ideas, perceptions, or insights; in the human brain the units might be close-by bundles of neurons where the currency is electrochemical stimuli. At each unit there will be some process which integrates over all possible incoming excitement, and which causes its own state to tend to change from excited to non excited or vice versa. We also assume that excitement cannot be transmitted instantaneously and that the units have well defined locations, so that the time taken for unit-to-unit excitation depends on the distance between them, resulting in delays.

Then we can ask how such systems might act as macroscopic information processing machines. When stimulated (by forcing terms) what kind of input-output responses do they exhibit? Here we argue that the existence of delays means that such systems can have a large number of discrete resonant modes and that they will exhibit effects analogous to mechanical resonance. Indeed these resonances are highly desirable since they will ensure that responses are quantized regardless of the distributed nature of the inputs.

The central point is that such systems exhibit this behaviour due to the interplay between the nature of the local and nonlocal coupling and the delay effects. Delays themselves are not enough as the coupling may be relatively localized and result in a dispersive (parabolic) type of systems. But when the interplay is suitable the resonances (controlled by the spectra of an associated operator) resemble those of a (damped) hyperbolic system, and the response surface is multimodal.

Here is a simple example. Consider a continuum problem with a homogeneous environment and translational symmetry. Let $u(x,t)$ denote the excitement of the unit at $x$, in one dimension, at time $t$. Then consider the Intergro Delay Differential Equation (IDDE)

$$u_t(x,t) = -u(x,t) + H\left(\int F(u(y,t-|x-y|/v)\phi(x-y)\,dy\right).$$

Here the kernel $\phi$ describes the local and nonlocal connections from all units at $y$ with the unit at $x$. $F$ describes the contribution passed from $y$ to $x$ as a function of the excitement at $y$, subject to a transmission speed $v > 0$, and thus time delay of $|x-y|/v$. $H$ represent the mapping if the total inputs received at $x$ into an increase in excitement. We have assumed a linear decay at $x$ in the absence of stimuli. $H$ and $F$ should be smotthe: certainly Lipschitz continuous.

Suppose $u = u_0$ is some uniform steady state solution. Then writing small perturbations in the form $u(x,t) - u_0 \sim e^{ik.x+\sigma t}$ we obtain the spectral equation for the local linearised system:

$$\sigma + 1 = constant.\mathcal{W}(k,\sigma)$$

where $\mathcal{W}(k,\sigma)$ is the Fourier transform of $e^{-|x|/v}\phi(x)$. In [43] such spectral equations are discussed with particular reference to applications of IDDEs. Such an equation for $\sigma(k)$, as $k$ varies along the real line, may have either a finite or infinite number of branches (corresponding to the generalised parabolic or hyperbolic sitruations respectively). The existence of delays ($v > 0$) is a necessary but not sufficient condition for an infinite number of branches. The behaviour of $\phi$ is important also. For an infinitity of branches we require the equation to be transcendental, and hence for $\mathcal{W}$ to have a factor in the form $e^{a\sigma}$, for some $a \neq 0$. It turns out this is possible for a wide variety of kernels. For example when $\phi(x) = \psi(x-a) + \psi(x+a)$, where $\psi$ is even and integrable.

Now consider perturbing the above IDDE with a forcing term of the form $e^{i(\omega r+k.x)}$. The response surface will have maxima wherever the spectral branch gets locally close to the imaginary axis, at $i\omega$ say, where $\sigma(k)$'s real part has a local maximum, at the corresponding particular value for $k$. Below we show an example spectra for a linerised

IDDE (Figure 5), and a typical response surface (Figure 6) with the discrete resonant modes in the $(k, \omega)$ plane arising from various points on multiple spectral branches.

The useful thing about resonance is that it acts as a filter in real-time and produces a discrete set of possible outputs. This provides food for thought in applications. How do our brains process information? How does society respond to perturbations? How diverse and sophisticated can input and output responses be? The linear theory presented here suggests delays and nonlocal spatial coupling can be highly sophisticated, and perhaps provide tunable, or learning, response surfaces.

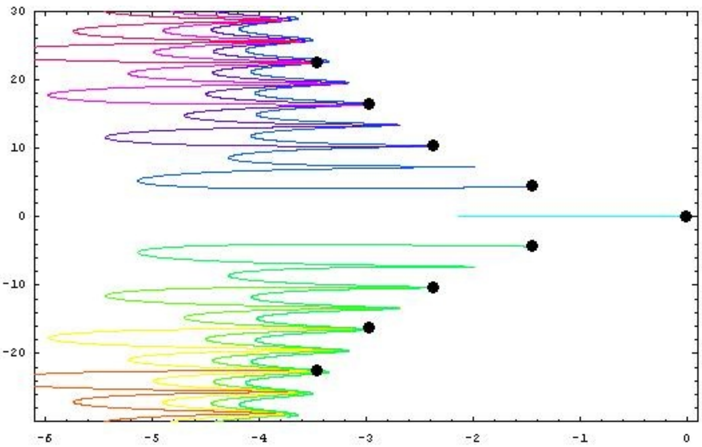


Figure 5. Infinite spectral branches, $\sigma(k)$, arising for an IDDE

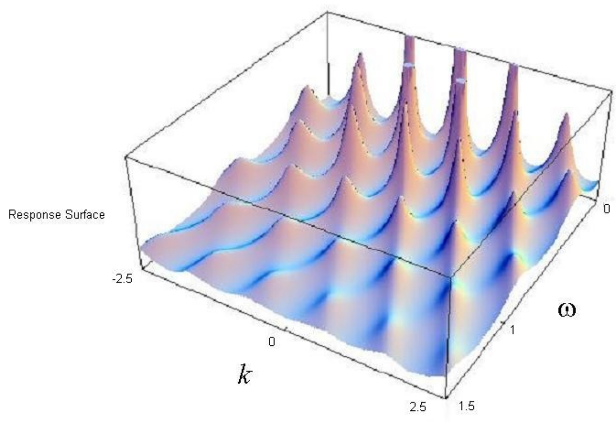


Figure 6. Amplitude of response, as surface over the $k - \omega$ plane.

# 4 Looking forwards

We have made the case for the Digital Society, and in particular for mathematical modelling to be an integrated part of this multidisciplinary effort. Future work within these fields will require leadership, scholarship and some courage for those willing to commit. Such mathematicians should rightly expect to face a critical examination from within the mathematics community: but their efforts will be welcomed (already and in the future) by the end users, and those students and researchers wishing to have careers within these digital sectors of the information economy.

We have also discussed how a mixed community of public and private research will be necessary and probably desirable. We have given a number of specific examples - not just of current applications of mathematics within the digital society, but of question posed by thinking about the digital society that require novel mathematical responses.

This work will be progressed over the next ten years and we should anticipate the *Mathematics for the Digital Society* will become a feature of many undergraduate courses, as well a cause of impact through collaborative work between academia and exploiters - the new businesses that will emerge, exploiting the future digital society and economy within a globalized, connected world.

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