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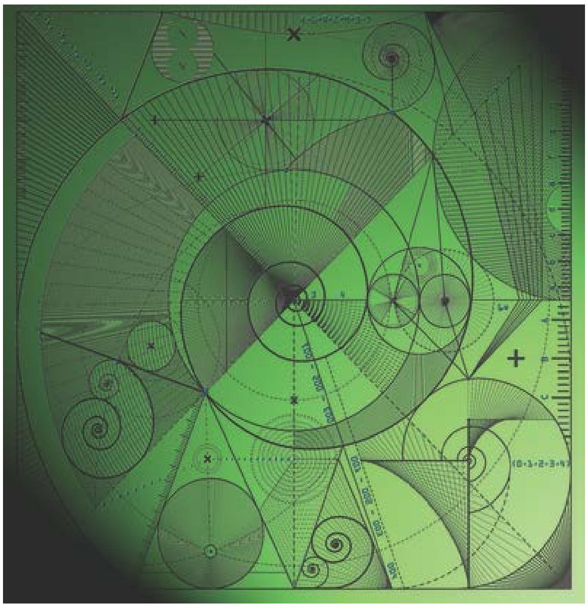

GENERALISED SOLUTIONS FOR FULLY NONLINEAR PDE SYSTEMS AND EXISTENCE-UNIQUENESS THEOREMS

NIKOS KATZOURAKIS

Abstract. We introduce a new theory of generalised solutions which applies to fully nonlinear PDE systems of any order and allows the interpretation of merely measurable maps as solutions without any further a priori regularity requirements. This approach bypasses the standard problems arising by the application of Distributions to PDEs and is not based on either duality or on integration by parts. Instead, our starting point builds on the probabilistic representation of limits of difference quotients via Young measures over compactifications of the spaces of derivatives. After developing some basic theory, as a first application we consider the Dirichlet problem and prove existence & uniqueness for fully nonlinear degenerate elliptic 2nd order systems, as well as existence for the ∞-Laplace system of vectorial Calculus of Varations in L^∞.

Contents

1. Introduction

It is well known that PDEs, either linear or nonlinear, in general do not possess classical solutions, in the sense that not all derivatives that appear in the equation may actually exist. The standard approach to this problem consists of looking for appropriately defined *generalised solutions* for which the hope is that at least

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existence can be proved given certain boundary conditions. Subsequent considerations typically include uniqueness, qualitative properties, regularity and numerics. This approach has been enormously successful but unfortunately only equations and systems with fairly special structure have been considered so far. A standing idea in this regard consists of using duality and integration-by-parts in order to interpret rigorously derivatives by “passing them to test functions”. This method which dates back to the 1930s ([S1, S2, So]) is basically restricted to divergence structure equations and systems. A more recent approach discovered in the 1980s is that of viscosity solutions ([CL]) which builds on the maximum principle as a device to “pass derivatives to test functions”. Although it applies mostly to single equations supporting the maximum principle, it has been hugely successful because it includes the fully nonlinear case.

In this paper we introduce a new theory of generalised solutions which applies to nonlinear PDE systems of any order. Our approach allows for *merely measurable* maps to be rigorously interpreted and studied as solutions of PDE systems, even fully nonlinear and with discontinuous coefficients. More precisely, let $p, n, N, M \in \mathbb{N}$, let also $\Omega \subseteq \mathbb{R}^n$ be an open set and

$$F \ : \ \Omega \times \Big(\mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s \times \cdots \times \mathbb{R}^{Nn^p}_s\Big) \longrightarrow \mathbb{R}^M \tag{1.1}$$

a Carathéodory map. Here $\mathbb{R}^{Nn}$ denotes the space of $N \times n$ matrices and $\mathbb{R}^{Nn^p}_s$ the space of symmetric tensors

$$\Big\{ \mathbf{X} \in \mathbb{R}^{Nn^p} \ \big| \ \mathbf{X}_{\alpha i_1 \dots i_p} = \mathbf{X}_{\alpha \sigma(i_1 \dots i_p)} \ , \ \alpha = 1, ..., N, \\ i_k = 1, ..., n, \ k = 1, ..., p, \ \sigma \text{ permutation } \Big\}$$

wherein the gradient matrix

$$Du(x) \ = \ \big(D_i u_\alpha(x)\big)^{\alpha=1,\dots,N}_{i=1,\dots,n}$$

and the pth order derivative

$$D^p u(x) \ = \ \Big(D^p_{i_1 \dots i_p} u_\alpha(x)\Big)^{\alpha=1,\dots,N}_{i_1,\dots,i_p \in \{1,\dots,n\}}$$

of (smooth) maps $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ are respectively valued. Obviously, $D_i \equiv \partial/\partial x_i$, $x = (x_1, ..., x_n)^\top$, $u = (u_1, ..., u_N)^\top$ and $\mathbb{R}^{Nn^1}_s = \mathbb{R}^{Nn}$. The present theory applies to measurable solutions of the system

$$F\big(\cdot, u, Du, ..., D^p u\big) \ = \ 0, \quad \text{on } \Omega, \tag{1.2}$$

without any further restrictions on F. Since we will not assume that the solutions are locally integrable on Ω, the derivatives Du, ..., $D^p u$ may not have any classical meaning, *not even in the sense of distributions.*

The starting point of our approach in not based either on duality or on the maximum principle. Instead, it builds on the probabilistic representation of the limits of difference quotients by using *Young* measures, also known as *parameterised* measures. The Young measures have been introduced in the 1930s in order to show existence of “relaxed” solutions to nonconvex variational problems for which the minimum may not be attained ([Y]) at a function. Today they are indispensable tools in Calculus of Variations and PDE theory ([E, M, P, FL]) and there is also a abstract topological theory for them ([CFV, FG, V]). The typical utility of Young measures to date has been to quantify the failure of strong convergence in approximating sequences due to the combination of phenomena of oscillations and/or concentrations ([DPM, KR]).

In the present framework, a version of Young measures is utilised in order to define generalised solutions of (1.2) by applying it to the *difference quotients* of the candidate solution. The exact definitions are thoroughly motivated later, but roughly the idea restricted to the first order case $p = 1$ of (1.1) is as follows: suppose that $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is a Lipschitz continuous strong a.e. solution of the PDE system

$$F(\cdot, u, Du) \,=\, 0, \quad \text{on } \Omega. \tag{1.3}$$

Then, u satisfies

$$F\Big(x, u(x), \lim_{h\to 0} D^{1,h}u(x)\Big) \,=\, 0, \qquad \text{a.e. } x \in \Omega,$$

where $D^{1,h}$ is the first difference quotients operator. Since F is continuous with respect to the gradient variable, this is equivalent to

$$\lim_{h\to 0} F\Big(x, u(x), D^{1,h}u(x)\Big) \,=\, 0, \qquad \text{a.e. } x \in \Omega.$$

The crucial observation is that the above statement makes sense if u is merely measurable. In order to represent this limit, we "embed" the difference quotients $D^{1,h}u$ into the space of Young measures over the Alexandroff compactification

$$\overline{\mathbb{R}}^{Nn} \,:=\, \mathbb{R}^{Nn} \cup \{\infty\}$$

(that is the set $\mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn}\big)$ of measurable probability-valued maps $\Omega \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^{Nn}\big)$, see Section 2 for the precise definitions) and consider instead the Dirac mass $\delta_{D^{1,h}u}$ at the difference quotients. By the weak* compactness of Young measures, there *always* exist probability-valued maps $\mathcal{D}u : \Omega \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^{Nn}\big)$ such that

$$\delta_{D^{1,h}u} \overset{*}{\rightharpoondown} \mathcal{D}u \quad \text{ in } \ \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn}\big), \qquad \text{as } h \to 0, \tag{1.4}$$

along subsequences (even if u is merely measurable). Then, by a convergence argument it follows that strong solutions of (1.3) satisfy

$$\int_{\overline{\mathbb{R}}^{Nn}} \Phi(X)\, F\big(x, u(x), X\big)\, d[\mathcal{D}u(x)](X) \,=\, 0, \qquad \text{a.e. } x \in \Omega, \tag{1.5}$$

for any compactly supported "test" function $\Phi \in C^0_c\big(\mathbb{R}^{Nn}\big)$ and any "diffuse derivative" $\mathcal{D}u$. We stress again that this last statement is *independent* of the regularity of u; the only extra piece of information the differentiability provides is that $\mathcal{D}u$ coincides a.e. on Ω with the Dirac mass δ_{Du} at the pointwise gradient Du. In the latter case, we recover strong solutions since we obtain

$$\int_{\overline{\mathbb{R}}^{Nn}} \Phi(X)\, F\big(x, u(x), X\big)\, d[\delta_{Du(x)}](X) \,=\, 0, \qquad \text{a.e. } x \in \Omega,$$

for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn}\big)$. Up to a minor technical adaptation of the concept (which is that we may need to consider special difference quotients with respect to appropriate frames depending on F) (1.4) *and* (1.5) *essentially constitute the definition of* ***$\mathcal{D}$-solutions*** [1] *in the special case of the* 1*st order system* (1.3) and will be the central notion of solution in this paper.

[1]We use the letter "$\mathcal{D}$-" as a shorthand of either of the modifiers "diffuse" or "dim" or "disintegration" because all of these terms are relatively descriptive of the notion. We leave it to the reader to decide for the interpretation of their preference.

Our motivation to introduce and study generalised solutions for nonlinear PDE systems primarily comes from the need to study the recently discovered ∞-Laplace system rigorously, which is the fundamental equation of Vectorial Calculus of Variations in the space L^∞. Calculus of Variations in L^∞ has a long history which started in the 1960s by Aronsson ([A1]-[A5]) who was the first to consider variational problems for supremal functionals of the form

$$E_\infty(u,\Omega) \,:=\, \big\| H(\cdot,u,Du)\big\|_{L^\infty(\Omega)}. \tag{1.6}$$

Aronsson introduced the appropriate notion of minimisers for such functionals and studied classical solutions of the respective equation which is the L^∞-analogue of the Euler-Lagrange equation. In the simplest case of $H(p)=|p|$ (the Euclidean norm on $\mathbb{R}^n$), the L^∞-equation is called the ∞-Laplacian and reads

$$\Delta_\infty u \,:=\, Du\otimes Du : D^2u \,=\, 0. \tag{1.7}$$

Since then, the field has undergone huge development due to both the intrinsic mathematical interest and the important for applications: minimisation of the maximum provides more *realistic* models when compared to the classical case of integral functionals where the average is minimised instead. A basic difficulty in the study of (1.6) is that (1.7) possesses singular solutions. Aronsson himself exhibited this in [A6, A7] and the field had to wait until the development of viscosity solutions for 2nd order equations in the early 1990s in order to study general solutions (see [C, BEJ, E, E2] and for a pedagogical introduction see [K8]).

Until recently, the study of supremal functionals was restricted exclusively to the scalar case of $N=1$ and to first order problems. The principal reason for this was the absence of an efficient theory of generalised solutions which would allow the rigorous study of non-divergence PDE systems or higher order equations, including those arising in L^∞. The foundations of the vector case of (1.6), including the discovery of the appropriate system version of (1.7), the correct vectorial minimality notion and the study of classical solutions have been laid in a series of recent papers of the author ([K1]-[K6]). In the simplest case of

$$E_\infty(u,\Omega) \,=\, \|Du\|_{L^\infty(\Omega)} \tag{1.8}$$

applied to Lipschitz maps $u:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^N$ (where the L^∞ norm is interpreted as the essential supremum of the Euclidean norm $|Du|$ on $\mathbb{R}^{Nn}$), the analogue of the Euler-Lagrange equation is the ∞-Laplace system:

$$\Delta_\infty u \,:=\, \Big(Du\otimes Du + |Du|^2[Du]^\perp\otimes I\Big) : D^2u \,=\, 0. \tag{1.9}$$

In the above, $[Du(x)]^\perp$ denotes the orthogonal projection on the orthogonal complement of the range of the $N\times n$ gradient matrix. In index form (1.9) reads

$$\sum_{\beta=1}^N\sum_{i,j=1}^n \Big(D_iu_\alpha\, D_ju_\beta + \,|Du|^2[Du]^\perp_{\alpha\beta}\,\delta_{ij}\Big)\, D^2_{ij}u_\beta \,=\, 0, \qquad \alpha=1,...,N,$$

$$[Du]^\perp := \,\mathrm{Proj}_{(R(Du))^\perp}.$$

An additional difficulty of (1.9) which is not present in the scalar case of (1.7) is that the nonlinear operator may have *discontinuous coefficients* even when applied to smooth maps because the new term involving $[Du(x)]^\perp$ depends on the dimension of the tangent space of $u(\Omega)$ at the point x ([K1, K6]). Almost simultaneously to [K1], Sheffield and Smart [SS] studied the relevant problem of vectorial Lipschitz extensions and derived a different singular version of "∞-Laplacian", which in the present setting amounts to changing in (1.8) from the Euclidean to the operator norm on $\mathbb{R}^{Nn}$.

A further motivation to introduce generalised solutions stems from the insufficiency of the current PDE approaches to handle even elliptic linear systems with rough coefficients. For example, if $\mathbf{A}$ is a continuous symmetric 4th order tensor on $\mathbb{R}^{Nn}$ satisfying the strict Legendre-Hadamard condition, then for the divergence form system

$$\sum_{\beta=1}^{N}\sum_{i,j=1}^{n} D_i\Big(\mathbf{A}_{\alpha i\beta j}(x)\, D_j u_\beta(x)\Big) \,=\, 0, \quad \alpha=1,...,N,$$

"everything" is known: existence-uniqueness of weak solutions, regularity, etc (see e.g. [GM]). On the other hand, for its non-divergence counterpart

$$\sum_{\beta=1}^{N}\sum_{i,j=1}^{n} \mathbf{A}_{\alpha i\beta j}(x)\, D^2_{ij} u_\beta(x) \,=\, 0, \quad \alpha=1,...,N, \tag{1.10}$$

"nothing" is known, not even what is a meaningful notion of generalised solution, unless $\mathbf{A}$ is $C^{0,\alpha}$ and strictly elliptic in which case a priori estimates guarantee that solutions of (1.10), if they exist, have to be smooth ([GM]). In particular, to the best of our knowledge there are *no general uniqueness theorems not even for strong solutions* of (1.10), unless $\mathbf{A}$ is monotone (i.e. diagonal: $\mathbf{A}_{\alpha i\beta j}=\delta_{\alpha\beta}A_{ij}$), in which case the system decouples to N independent equations.

In the present paper, after motivating, introducing and developing some basic theory of $\mathcal{D}$-solutions for general systems (Section 2), we apply it to two important problems. Accordingly, we first consider the Dirichlet problem for the ∞-Laplacian

$$\left\{ \begin{aligned} \Delta_\infty u \,&=\, 0, \quad \text{on } \Omega, \\ u \,&=\, g, \quad \text{on } \partial\Omega, \end{aligned} \right. \tag{1.11}$$

when $\Omega \subseteq \mathbb{R}^n$ is an open domain with finite measure, $n=N$ and $g \in W^{1,\infty}(\Omega,\mathbb{R}^n)$. In Section 3 we prove existence of $\mathcal{D}$-solutions $u \in W^{1,\infty}_g(\Omega,\mathbb{R}^n)$ to (1.11) with extra properties (Theorem 29, Corollary 32). *The question of uniqueness for* (1.11) *has already been answered negatively in* [K2] even when we restrict ourselves to the class of **smooth** solutions; in fact, (1.11) has infinitely-many C^∞ solutions even for $n=N=2$ on the unit disc and with g the identity (at least not without imposing extra constraints, see Theorem 30).

The idea of the proof has two main steps. We first apply the analytic counterpart of Gromov's Convex Integration in the form of the Dacorogna-Marcellini Baire Category method ([DM]) in order to prove existence of a $W^{1,\infty}$ solution to a 1st order differential inclusion associated to (1.11) (Subsection 3.1). Next, we characterise this map as a $\mathcal{D}$-solution to (1.9) by utilising the machinery of Section 2. In doing so we actually establish a general tool of independent interest which goes far beyond the ∞-Laplacian and provides a method to construct $\mathcal{D}$-solutions of "tangential equations" by solving differential inclusions (Theorem 33).

The second main question we consider in this paper concerns the existence and uniqueness of $\mathcal{D}$-solutions to the Dirichlet problem for the fully nonlinear system

$$\left\{ \begin{aligned} F(\cdot, D^2 u) \,&=\, f, \quad \text{on } \Omega, \\ u \,&=\, 0, \quad \text{on } \partial\Omega, \end{aligned} \right. \tag{1.12}$$

when $\Omega \Subset \mathbb{R}^n$ is a C^2 convex domain, $F : \Omega \times \mathbb{R}^{Nn^2}_s \longrightarrow \mathbb{R}^N$ is a Carathéodory map and $f \in L^2(\Omega, \mathbb{R}^N)$. The essential hypothesis guaranteeing well posedness is an appropriate *degenerate ellipticity* condition. Roughly, we require that F is "near" a degenerate linear system of the form (1.10) with $\mathbf{A}$ constant which satisfies the (weak) Legendre-Hadamard condition. The problem (1.12) has first been considered by Campanato [C1, C2, C3] under a strong uniform ellipticity assumption of Cordes type which roughly requires F to be "near" the Laplacian. Under this condition, (1.12) is well posed in $(H^2 \cap H^1_0)(\Omega, \mathbb{R}^N)$. Very recently, the author ([K9, K11] and also [K7]) has generalised the results of Campanato by proving well posedness under a new weaker ellipticity notion. The latter results for strong solutions of elliptic systems were stepping stones to the general approach we develop herein for $\mathcal{D}$-solutions of degenerate elliptic systems.

In Section 4 we prove existence of a unique $\mathcal{D}$-solution to (1.12) (Theorem 37). The proof is rather long and is based on the study of the Dirichlet problem for the linear system (1.10) *in the $\mathcal{D}$-sense* and on the hypothesis of degenerate ellipticity which acts as "perturbation device". The method for the linear problem involves approximation and a priori "degenerate elliptic" partial estimates (Theorem 40). The ellipticity hypothesis allows the passage to the nonlinear problem via a fixed point argument.

Well posedness of (1.12) is established in an appropriate functional "fibre space" tailored to the degenerate case ((4.4),(4.5)). The fibre space is an extension of the classical Sobolev space and consists of *partially* regular maps which possess weakly differentiable projections only along certain rank-one directions corresponding to the "directions of ellipticity" of F. Then we characterise the fixed point in the fibre space as the unique $\mathcal{D}$-solution of (1.12) which generally is not even $W^{1,1}_{\text{loc}}$. A particular difficulty is the satisfaction of the boundary condition under this low regularity since there is no standard trace operator. We also note that our ellipticity assumption on F is relatively strong (Definition 35), but even in the *scalar linear strictly elliptic* case, the Dirichlet problem for the single equation $\sum_{i,j=1}^n \mathbf{A}_{ij} D^2_{ij} u = f$ is not well posed (see e.g. [LU]).

We conclude this introduction by noting that the table of contents gives an idea of the organisation of the material in this paper, as well as where the reader may find further motivation of the main ideas and proofs. We hope that the systematic theory proposed herein will be the starting point for future developments. In particular, in the companion paper [K12] we consider the relevant problem of existence of $\mathcal{D}$-solutions to the Dirichlet problem of the vectorial equations of Calculus of Variations in L^∞ for (1.6) but $n = 1$. Therein we follow the "natural" approach of approximation by the Euler-Lagrange equations of the associated L^p functionals as $p \to \infty$. A central difficulty when following this route is that in the vector case *existence is a highly nontrivial matter* and a priori estimates are required because p-Harmonic limits are "good" solutions of (1.9) (see the remarks at end of Section 3 regarding a selection criterion of ∞-Harmonic maps). The analogue of [K12] for $n \geq 2$ will be considered in future work.

2. Theory of $\mathcal{D}$-solutions for fully nonlinear systems

2.1. **Preliminaries.** We begin with some introductory material needed for the rest of the paper which will be used throughout freely, perhaps without explicit reference to this subsection.

Basics. Let $n, N \in \mathbb{N}$ be fixed, which in this paper will always be the dimensions of domain and range respectively of our candidate solutions $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$. By Ω we will always mean an open subset of $\mathbb{R}^n$, even if it is not explicitly mentioned. Unless indicated otherwise, Greek indices $\alpha, \beta, \gamma, ...$ will run in $\{1, ..., N\}$ and latin indices $i, j, k, ...$ (perhaps indexed $i_1, i_2, ...$) will run in $\{1, ..., n\}$, even when the range is not given explicitly. The norms $|\cdot|$ appearing throughout will always be the Euclidean, while the Euclidean inner products will be denoted by either "$\cdot$" on $\mathbb{R}^n, \mathbb{R}^N$ or by ":" on tensor spaces, e.g. on $\mathbb{R}^{Nn}$ and $\mathbb{R}^{Nn^2}_s$ we have

$$|X|^2 \,=\, \sum_{\alpha,i} X_{\alpha i} X_{\alpha i} \,\equiv\, X : X, \qquad |\mathbf{X}|^2 \,=\, \sum_{\alpha,i,j} \mathbf{X}_{\alpha ij} \mathbf{X}_{\alpha ij} \,\equiv\, \mathbf{X} : \mathbf{X},$$

etc. The standard bases on $\mathbb{R}^n$, $\mathbb{R}^N$, $\mathbb{R}^{Nn}$ will be denoted by $\{e^i\}$, $\{e^\alpha\}$ and $\{e^\alpha \otimes e^i\}$. By introducing the *symmetrised tensor product*

$$a \vee b \,:=\, \frac{1}{2}\Big(a \otimes b \,+\, b \otimes a\Big), \quad a, b \in \mathbb{R}^n, \tag{2.1}$$

we will write $\big\{e^\alpha \otimes (e^{i_1} \vee ... \vee e^{i_p})\big\}$ for the standard basis of the $\mathbb{R}^{Nn^p}_s$. We will follow the convention of denoting vector subspaces of Euclidean spaces as well as the orthogonal projections on them by the same symbol. For example, if $\Sigma \subseteq \mathbb{R}^N$ is a subspace, we denote the projection map $\mathrm{Proj}_\Sigma : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ by just Σ and we have $\Sigma^2 = \Sigma^\top = \Sigma \in \mathbb{R}^{N^2}_s$. We will also systematically use the Alexandroff 1-point compactification of the space $\mathbb{R}^{Nn^p}_s$. Its metric will be the standard one which makes it homeomorphic to the sphere of the same dimension (via the stereographic projection which identifies $\{\infty\}$ with the north pole). We will denote it by

$$\overline{\mathbb{R}}^{Nn^p}_s \,:=\, \mathbb{R}^{Nn^p}_s \cup \{\infty\}.$$

We note that all balls and distances taken in $\mathbb{R}^{Nn^p}_s$ (which we will view as a metric vector space isometrically contained into $\overline{\mathbb{R}}^{Nn^p}_s$) will be the Euclidean.

Our measure theoretic and function space notation is either standard as e.g. in [E2, EG] or self-explanatory. For example, the modifier "measurable" will always mean "Lebesgue measurable", the Lebesgue measure on $\mathbb{R}^n$ will be denoted by $|\cdot|$, the s-Hausdorff measure by $\mathcal{H}^s$, the characteristic function of a set A by χ_A, the standard L^p spaces of maps $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \Sigma \subseteq \mathbb{R}^N$ by $L^p(\Omega, \Sigma)$ etc. Let us also record the following simple fact about measurable functions which is taken from [AM] and will be used later:

Lemma 1 (cf. [AM]). *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be a measurable mapping. Then, we have $f(\cdot + z) \longrightarrow f(\cdot + x)$ locally in measure as $z \to x \in \mathbb{R}^n$. Namely, for any $\varepsilon > 0$ and $E \subseteq \mathbb{R}^n$ with $|E| < \infty$, we have*

$$\lim_{z \to x} \left| \Big\{ y \in E \,:\, \big|f(y+z) - f(y+x)\big| > \varepsilon \Big\} \right| \,=\, 0.$$

The proof is an easy consequence of Luzin's theorem and of the fact that the translation of the characteristic function of a compact set is continuous in L^1.

General frames, derivative expansions, difference quotients. In what follows we will need to consider non-standard orthonormal frames of $\mathbb{R}^{Nn^p}_s$ and express derivatives $D^p u$ with respect to them. Let $\{E^1, ..., E^N\}$ be an orthonormal frame of $\mathbb{R}^N$ and suppose that for each $\alpha = 1, ..., N$ we have an orthonormal frame

$\{E^{(\alpha)1},...,E^{(\alpha)n}\}$ of $\mathbb{R}^n$. Given such bases, we will *always* equip the spaces $\mathbb{R}^{Nn}$ and $\mathbb{R}^{Nn^p}_s$ with the following induced orthonormal bases:

$$\begin{aligned} \mathbb{R}^{Nn} &= \mathrm{span}\big[\big\{E^{\alpha i}\big\}\big], & E^{\alpha i} &:= E^\alpha \otimes E^{(\alpha)i}, \\ \mathbb{R}^{Nn^p}_s &= \mathrm{span}\big[\big\{E^{\alpha i_1 \dots i_p}\big\}\big], & E^{\alpha i_1 \dots i_p} &:= E^\alpha \otimes \Big(E^{(\alpha)i_1} \vee ... \vee E^{(\alpha)i_p}\Big). \end{aligned} \tag{2.2}$$

Given such frames, let $D_{E^{(\alpha)i}}$ and $D^p_{E^{(\alpha)i_p}...E^{(\alpha)i_1}} = D_{E^{(\alpha)i_p}} \cdots D_{E^{(\alpha)i_1}}$ denote the usual directional derivatives of 1st and pth order along the respective directions. Then, the gradient Du of a map $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ can be written as

$$Du \,=\, \sum_{\alpha,i} \Big(E^{\alpha i} : Du\Big)\, E^{\alpha i} \,=\, \sum_{\alpha,i} \Big(D_{E^{(\alpha)i}}(E^\alpha \cdot u)\Big)\, E^{\alpha i} \tag{2.3}$$

and the pth order derivative $D^p u$ as

$$\begin{aligned} D^p u \,&=\, \sum_{\alpha,i_1,...,i_p} \Big(E^{\alpha i_1 \dots i_p} : D^p u\Big)\, E^{\alpha i_1 \dots i_p} \\ &=\, \sum_{\alpha,i_1,...,i_p} \Big(D^p_{E^{(\alpha)i_1}...E^{(\alpha)i_p}}(E^\alpha \cdot u)\Big)\, E^{\alpha i_1 \dots i_p}. \end{aligned} \tag{2.4}$$

We will also use the following notation for the *pth order Jet of u*:

$$D^{[p]}u \,:=\, \big(Du, D^2u, ..., D^p u\big).$$

Given $a \in \mathbb{R}^n$ with $|a| = 1$ and $h \in \mathbb{R} \setminus \{0\}$, when $x, x + ah \in \Omega$ the 1st difference quotient of u along the direction a at x will be denoted by

$$D^{1,h}_a u(x) \,:=\, \frac{u(x + ha) - u(x)}{h}. \tag{2.5}$$

By iteration, if $h_1, ..., h_p \neq 0$ the pth order difference quotient along $a_1, ..., a_p$ is

$$D^{p,h_p...h_1}_{a_p...a_1} u \,:=\, D^{1,h_p}_{a_p}\Big(\cdots \big(D^{1,h_1}_{a_1} u\big)\Big). \tag{2.6}$$

Young Measures. Let $E \subseteq \mathbb{R}^n$ be a measurable set and $\mathbb{K} \subseteq \mathbb{R}^d$ a compact subset of some Euclidean space, which we will later take to be $\overline{\mathbb{R}}^{Nn^p}_s$. Consider the L^1 space of strongly measurable maps valued in the (separable Banach) space $C^0(\mathbb{K})$ of real continuous functions over $\mathbb{K}$, in the standard Bochner sense:

$$L^1\big(E, C^0(\mathbb{K})\big).$$

For details about these spaces we refer e.g. to [FL, F, V] (and references therein). The elements of $L^1\big(E, C^0(\mathbb{K})\big)$ can be identified with the Carathéodory functions

$$\Phi \;:\; E \times \mathbb{K} \longrightarrow \mathbb{R}, \quad (x, X) \mapsto \Phi(x, X)$$

for which

$$\|\Phi\|_{L^1(E,C^0(\mathbb{K}))} \,:=\, \int_E \max_{X \in \mathbb{K}} \big|\Phi(x, X)\big|\, dx \,<\, \infty$$

and the identification is given by considering Φ as a map $E \ni x \mapsto \Phi(x, \cdot) \in C^0(\mathbb{K})$. The notion of Carathéodory functions is meant in the usual sense, that is for every $X \in \mathbb{K}$ the function $x \mapsto \Phi(x, X)$ is measurable and for a.e. $x \in E$ the function $X \mapsto \Phi(x, X)$ is continuous. The space $L^1\big(E, C^0(\mathbb{K})\big)$ is separable and the simple functions of this space (which are norm-dense) have the form

$$E \,\ni\, x \,\mapsto\, \sum_{i=1}^{q} \chi_{E_i}(x)\; \Phi_i \;\in\; C^0(\mathbb{K}),$$

where $E_1, ..., E_q$ are measurable disjoint subsets of E and $\Phi_i \in C^0(\mathbb{K})$. By using that the dual space of $C^0(\mathbb{K})$ is the space $\mathcal{M}(\mathbb{K})$ of real (signed) Radon measures on $\mathbb{K}$ endowed with the total variation norm, it can be shown (see e.g. [FL]) that

$$\big(L^1\big(E, C^0(\mathbb{K})\big)\big)^* \;=\; L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big).$$

The dual Banach space $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$ consists of measure-valued maps

$$E \;\ni\; x \longmapsto \vartheta(x) \;\in\; \mathcal{M}(\mathbb{K})$$

which are weakly* measurable, that is for any fixed Borel set $B \subseteq \mathbb{K}$, the function $E \ni x \mapsto [\vartheta(x)](B) \in \mathbb{R}$ is measurable. The norm of $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$ is

$$\|\vartheta\|_{L^\infty_{w^*}(E, \mathcal{M}(\mathbb{K}))} \;:=\; \underset{x \in E}{\text{ess sup}}\; \|\vartheta(x)\|\,(\mathbb{K})$$

where "$\|\cdot\|(\mathbb{K})$" denotes the total variation. The duality pairing

$$\langle \cdot, \cdot \rangle \;\;:\;\; L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big) \times L^1\big(E, C^0(\mathbb{K})\big) \longrightarrow \mathbb{R}$$

is given by

$$\langle \vartheta, \Phi \rangle \;:=\; \int_E \int_{\mathbb{K}} \Phi(x, X)\, d[\vartheta(x)](X)\, dx.$$

Since $L^1\big(E, C^0(\mathbb{K})\big)$ is separable, the unit ball of $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$ is sequentially weakly* compact. Hence, for any bounded sequence $(\vartheta^m)_1^\infty \subseteq L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$, there is a limit map ϑ and a subsequence of m's along which $\vartheta^m \overset{*}{\rightharpoonup} \vartheta$ as $m \to \infty$. Further, by the density of simple functions and linearity, for bounded sequences the weak* convergence $\vartheta^m \overset{*}{\rightharpoonup} \vartheta$ is equivalent for any fixed $\Phi \in C^0(\mathbb{K})$ to

$$\int_{\mathbb{K}} \Phi(X)\, d\big[\vartheta^m(\cdot) - \vartheta(\cdot)\big](X) \overset{*}{\rightharpoonup} 0, \qquad \text{in } L^\infty(E).$$

Definition 2 (Young Measures)**.** The space of Young (or Parameterised) Measures is the subset of the unit sphere of $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$ which consists of probability-valued weakly* measurable maps:

$$\mathscr{Y}(E, \mathbb{K}) \;:=\; \Big\{ \vartheta \;\in\; L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big) \;:\; \vartheta(x) \in \mathscr{P}(\mathbb{K}),\ \text{for a.e. } x \in E \Big\}.$$

Remark 3 (Properties of $\mathscr{Y}(E, \mathbb{K})$)**.** The following well known facts about Young measures will be extensively used hereafter (for proofs see e.g. [FG]):

i) [**weak* compactness**] The set of Young measures is convex and by the compactness of $\mathbb{K}$, it follows that it is sequentially weakly* compact in $L^\infty_{w^*}\big(E, \mathcal{M}(\mathbb{K})\big)$.

ii) [**weak* density**] The vector space of Lebesgue measurable mappings $v : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{K}$ can be embedded into $\mathscr{Y}(E, \mathbb{K})$ via the map $v \mapsto \delta_v$ which is given by $\delta_v(x) := \delta_{v(x)}$ and the embedding actually has weakly* dense image.

iii) [**weak* LSC**] We have the following one-sided characterisation of weak* convergence of Young measures:

$$\vartheta^m \overset{*}{\rightharpoonup} \vartheta \quad \text{as } m \to \infty, \ \ \text{in } \mathscr{Y}(E, \mathbb{K}) \quad \Longleftrightarrow \quad \langle \vartheta, \Psi \rangle \;\leq\; \liminf_{m \to \infty} \langle \vartheta^m, \Psi \rangle,$$

for any function

$$\Psi \;\;:\;\; E \times \mathbb{K} \;\longrightarrow\; (-\infty, +\infty]$$

which is bounded from below, measurable in $x \in E$ for all $X \in \mathbb{K}$ and LSC (lower semicontinuous) in $X \in \mathbb{K}$ for a.e. $x \in E$.

The next result is a minor variant of a classical result which we give together with its short proof because it plays a fundamental role in our setting.

Lemma 4. *Suppose $E \subseteq \mathbb{R}^n$ is measurable and $v^m, v^\infty : E \longrightarrow \mathbb{K}$ are measurable maps, $m \in \mathbb{N}$. Then, there exist subsequences $(m_k)_1^\infty$, $(m_l)_1^\infty$:*

$$\begin{aligned} &(1) \quad v^m \longrightarrow v^\infty \ \text{ a.e. on } E \quad \Longrightarrow \quad \delta_{v^{m_k}} \overset{*}{\rightharpoonup} \delta_{v^\infty} \text{ in } \mathscr{Y}(E,\mathbb{K}),\\ &(2) \quad \delta_{v^m} \overset{*}{\rightharpoonup} \delta_{v^\infty} \text{ in } \mathscr{Y}(E,\mathbb{K}) \quad \Longrightarrow \quad v^{m_l} \longrightarrow v^\infty \ \text{ a.e. on } E. \end{aligned}$$

Proof of Lemma 4. (1) If $v^m \longrightarrow v^\infty$ a.e. on E, by Remark 3 there is $(v^{m_k})_1^\infty$ such that $\delta_{v^{m_k}} \overset{*}{\rightharpoonup} \vartheta^\infty$ in $\mathscr{Y}(E,\mathbb{K})$. If $\Phi \in L^1\big(E, C^0(\mathbb{K})\big)$, we have

$$\int_E \Phi\big(x, v^{m_k}(x)\big)\, dx \longrightarrow \int_E \int_{\mathbb{K}} \Phi(x,X)\, d[\vartheta^\infty(x)](X)\, dx$$

and also, the L^1 bound $|\Phi(\cdot, v^{m_k})| \leq \max_{X \in \mathbb{K}} |\Phi(\cdot, X)|$ gives $\Phi(\cdot, v^{m_k}) \longrightarrow \Phi(\cdot, v^\infty)$ in $L^1(E)$. Hence, by uniqueness of limits $\vartheta^\infty = \delta_{v^\infty}$ a.e. on E.

(2) If $\delta_{v^m} \overset{*}{\rightharpoonup} \delta_{v^\infty}$ in $\mathscr{Y}(E,\mathbb{K})$, we choose $\Phi(x,X) := |X - v^\infty(x)|$ where $|\cdot|$ denotes the norm of $\mathbb{R}^d$ restricted to the compact set $\mathbb{K}$. Then, for any $\varepsilon > 0$

$$0 = \int_E \Phi(\cdot, v^\infty) = \lim_{m\to\infty} \int_E \Phi(\cdot, v^m) \geq \varepsilon \limsup_{m\to\infty} \Big| \big\{ |v^m - v^\infty| > \varepsilon \big\} \Big|.$$

Hence, $v^m \longrightarrow v^\infty$ in measure on E. $\square$

Lemma 4 shows that weak* convergence is actually relatively strong since, if the Young measures are given by functions then it is equivalent to a.e. convergence.

2.2. **Motivation of the notions.** We seek to find a meaningful notion of generalised solution for fully nonlinear PDE systems which relaxes the notion of strong solution and does not require any more a priori regularity for the solution apart from measurability. We derive the notion in the instructive case of 2nd order systems. Suppose F is as in (1.1) with $p = 2$ and suppose $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is a $W^{2,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$ strong a.e. solution of the system

$$F(\cdot, u, Du, D^2u) = 0, \quad \text{on } \Omega. \tag{2.7}$$

By the standard equivalence between weak and strong L^1 derivatives, the difference quotients converge along subsequence a.e. on Ω to the weak derivatives. Hence, we have

$$F\Big(\cdot, u, \lim_{m\to\infty} D^{1,h_m}u, \lim_{m',m''\to\infty} D^{2,h_{m'}h_{m''}}u\Big) = 0,$$

a.e. on Ω. Here $D^{1,h}$, $D^{2,kh}$ stand for the usual difference quotient operators whose components with respect to standard basis $D^{1,h}_{e^i}$, $D^{2,kh}_{e^ie^j}$ are given by (2.5), (2.6). Since F is a Carathéodory map, the limits commute with the nonlinearity:

$$\lim_{m,m',m''\to\infty} F\Big(\cdot, u, D^{1,h_m}u, D^{2,h_{m'}h_{m''}}u\Big) = 0, \tag{2.8}$$

a.e. on Ω. The crucial observation is that (2.8) *is independent of the weak differentiability of* u and makes sense if u is merely measurable. How can we represent these limits and turn them into a handy definition? Going back to (2.7), we observe that u is a strong solution of (2.7) if and only if it satisfies

$$\int_{\mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s} \Phi(X, \mathbf{X})\, F(\cdot, u, X, \mathbf{X})\, d\big[\delta_{(Du, D^2u)}\big](X, \mathbf{X}) = 0, \quad \text{a.e. on } \Omega,$$

for any compactly supported "test" function $\Phi \in C^0_c\big(\mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s\big)$. This gives the idea that we can embed the difference quotient maps

$$\big(D^{1,h_m}u, D^{2,h_{m'}h_{m''}}u\big) \ : \ \Omega \longrightarrow \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s$$

into the spaces of Young measures and consider instead

$$\delta_{D^{1,h_m}u} \ : \ \Omega \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^{Nn}\big), \qquad \delta_{D^{2,h_{m'}h_{m''}}u} \ : \ \Omega \longrightarrow \mathscr{P}\big(\overline{\mathbb{R}}^{Nn^2}_s\big)$$

over the Alexandroff compactifications. The reason we need to attach the point at ∞ and compactify the space is to get "tightness" and have weak* compactness. This compensates the possible loss of mass since the difference quotients of measurable maps may not converge in any classical sense. However, there do exist sequential weak* limits in the Young measures. It will be also more fruitful to take these limits *separately* (regardless of the order), because the resulting object will be a (fibre) product Young measure:

$$\delta_{\big(D^{1,h_m}u, D^{2,h_{m'}h_{m''}}u\big)} \overset{*}{\rightharpoonup} \mathcal{D}u \times \mathcal{D}^2u \quad \text{in } \mathscr{Y}\Big(\Omega, \overline{\mathbb{R}}^{Nn} \times \overline{\mathbb{R}}^{Nn^2}_s\Big), \tag{2.9}$$

subsequentially as $m, m', m'' \to \infty$. Then, for any Φ, (2.8) is equivalent to

$$\int_{\overline{\mathbb{R}}^{Nn} \times \overline{\mathbb{R}}^{Nn^2}_s} \Phi(X, \mathbf{X})\, F\big(\cdot, u, X, \mathbf{X}\big)\, d\Big[\delta_{\big(D^{1,h_m}u, D^{2,h_{m'}h_{m''}}u\big)}\Big](X, \mathbf{X}) \ \longrightarrow \ 0,$$

subsequentially as $m, m', m'' \to \infty$, a.e. on Ω. By using Lemma 18 that follows, we obtain

$$\int_{\overline{\mathbb{R}}^{Nn} \times \overline{\mathbb{R}}^{Nn^2}_s} \Phi(X, \mathbf{X})\, F\big(\cdot, u, X, \mathbf{X}\big)\, d\big[\mathcal{D}u \times \mathcal{D}^2u\big](X, \mathbf{X}) \ = \ 0, \quad \text{a.e. on } \Omega,$$

for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s\big)$. We note that this statement is *independent* of the regularity of the solution of (2.7). If u is weakly once differentiable on Ω, by using Lemma 4 we have $\mathcal{D}u = \delta_{Du}$ a.e. on Ω and the above simplifies to

$$\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(\mathbf{X})\, F\big(\cdot, u, Du, \mathbf{X}\big)\, d[\mathcal{D}^2u](\mathbf{X}) \ = \ 0, \quad \text{a.e. on } \Omega,$$

for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn^2}_s\big)$. In this case any "diffuse hessian" $\mathcal{D}^2u$ arises as

$$\delta_{D^{1,h}Du} \overset{*}{\rightharpoonup} \mathcal{D}^2u \quad \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big), \quad \text{as } h \to 0,$$

along subsequences. If further D^2u exists weakly on Ω, by applying Lemma 4 again we have $\mathcal{D}^2u = \delta_{D^2u}$ a.e. on Ω thus recovering strong solutions.

2.3. **Main definitions and analytic properties.** We begin by introducing difference quotients taken with respect to frames as in (2.2), (2.3), (2.4). The only difficulty is the complexity in the notation so for pedagogical reasons we give the 1st order case separately from the general pth order case.

Definition 5 (Difference quotients)**.** Suppose $\{E^1, ..., E^N\}$ is an orthonormal frame of $\mathbb{R}^N$ and for each $\alpha = 1, ..., N$ we have an orthonormal frame $\{E^{(\alpha)1}, ..., E^{(\alpha)n}\}$ of $\mathbb{R}^n$ while the spaces $\mathbb{R}^{Nn^p}_s$ are equipped with the frames of (2.2), $p \in \mathbb{N}$.

Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be any measurable map which we understand to be extended by zero on $\mathbb{R}^n \setminus \Omega$. Given any infinitesimal sequences

$$(h_m)_{m\in\mathbb{N}} \ \subseteq \mathbb{R} \setminus \{0\}, \qquad h_m \to 0 \text{ as } m \to \infty,$$
$$(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^p} \subseteq \big(\mathbb{R} \setminus \{0\}\big)^p, \quad h_{\underline{m}} = (h_{m^1}, ..., h_{m^p}), \quad h_{m^q} \to 0 \ \text{ as } m^q \to \infty,$$

we define the **1st and pth order difference quotients of u** (with respect to the fixed reference frames) arising from $(h_m)_{m\in\mathbb{N}}$ and $(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^p}$ as

$$\begin{aligned} D^{1,h_m}u \ &: \quad \Omega\subseteq\mathbb{R}^n \longrightarrow \mathbb{R}^{Nn}, \qquad m\in\mathbb{N},\\ D^{p,h_{\underline{m}}}u \ &: \quad \Omega\subseteq\mathbb{R}^n \longrightarrow \mathbb{R}^{Nn^p}_s, \qquad \underline{m}=(m^1,...,m^p)\in\mathbb{N}^p, \end{aligned}$$

given respectively by

$$\begin{aligned} D^{1,h_m}u \ &:= \sum_{\alpha,i}\Big[D^{1,h_m}_{E^{(\alpha)i}}(E^\alpha\cdot u)\Big]\ E^{\alpha i},\\ D^{p,h_{\underline{m}}}u \ &:= \sum_{\alpha,i_1,...,i_p}\Big[D^{p,h_{m^p}...h_{m^1}}_{E^{(\alpha)i_p}...E^{(\alpha)i_1}}(E^\alpha\cdot u)\Big]\ E^{\alpha i_1...i_p}. \end{aligned}$$

In the above, the notation in the brackets is as in (2.5), (2.6). Further, given an infinitesimal sequence with a trigonal matrix of indices

$$(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^{p^2}}\subseteq\big(\mathbb{R}\setminus\{0\}\big)^{p^2}, \quad \underline{m}=\left[\begin{array}{cccc} m^1_1 & 0 & 0 \ ... & 0\\ m^1_2 & m^2_2 & 0 \ ... & 0\\ \vdots & & \ddots & \vdots\\ m^1_p & m^2_p & ... & m^p_p \end{array}\right], \quad h_{m^q_p}\to 0 \text{ as } m^q_p\to\infty,$$

we will denote its nonzero row elements by

$$\underline{m}_q \ := \ (m^1_q,...,m^q_q) \ \in\mathbb{N}^q, \quad q \ = \ 1,...,p,$$

and we define the **pth order Jet $D^{[p],h_{\underline{m}}}u$ of difference quotients of u** (with respect to the fixed reference frames) arising from $(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^{p^2}}$ as

$$D^{[p],h_{\underline{m}}}u \ := \Big(D^{1,h_{\underline{m}_1}}u,\dots,D^{p,h_{\underline{m}_p}}u\Big) \quad : \quad \Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^{Nn}\times...\times\mathbb{R}^{Nn^p}_s.$$

Definition 6 (Multi-indexed convergence)**.** Let $\underline{m}$ be either a vector of indices in $\mathbb{N}^p$ or a lower trigonal matrix of indices in $\mathbb{N}^{p^2}$ as above. The expression

$$\underline{m}\longrightarrow\infty$$

is defined to mean successive convergence with respect to each index separately taken in the following obvious order:

$$\begin{aligned} \lim_{\underline{m}\to\infty} \ &:= \lim_{m^p\to\infty}\ ... \lim_{m^2\to\infty}\lim_{m^1\to\infty}, \qquad\qquad \underline{m}\in\mathbb{N}^p,\\ \lim_{\underline{m}\to\infty} \ &:= \lim_{m^p_p\to\infty}\ ... \lim_{m^2_2\to\infty}\lim_{m^1_2\to\infty}\lim_{m^1_1\to\infty}, \quad \underline{m}\in\mathbb{N}^{p^2}. \end{aligned}$$

Definition 7 (Diffuse derivatives and Jets)**.** Suppose we have fixed some reference frames as in Definition 5.

For any measurable map $u:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^N$, we define **diffuse gradients** $\mathcal{D}u$, **diffuse pth order derivatives** $\mathcal{D}^pu$ and **diffuse pth order Jets $\mathcal{D}^{[p]}u$ of u** as the subsequential limits of the difference quotients in the spaces of Young measures over the respective 1-point compactifications which arise along infinitesimal sequences:

$$\begin{aligned} &\delta_{D^{1,h_m}u}\overset{*}{\rightharpoonup}\mathcal{D}u, &&\text{in } \mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn}\big), &&\text{as } m\to\infty,\\ &\delta_{D^{p,h_{\underline{m}}}u}\overset{*}{\rightharpoonup}\mathcal{D}^pu, &&\text{in } \mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn^p}_s\big), &&\text{as } \underline{m}\to\infty,\ \underline{m}\in\mathbb{R}^p,\\ &\delta_{D^{[p],h_{\underline{m}}}u}\overset{*}{\rightharpoonup}\mathcal{D}^{[p]}u, &&\text{in } \mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn}\times...\times\overline{\mathbb{R}}^{Nn^p}_s\big), &&\text{as } \underline{m}\to\infty,\ \underline{m}\in\mathbb{R}^{p^2}. \end{aligned}$$

Remark 8. As a consequence of the separate convergence, the pth order Jet is always a (fibre) product Young measure:

$$\mathcal{D}^{[p]}u \,=\, \mathcal{D}u \times \cdots \times \mathcal{D}^p u.$$

The weak* compactness of the spaces of Young measures readily implies the existence of plenty of diffuse derivatives for measurable mappings.

Lemma 9 (Existence of diffuse derivatives)**.** *Every measurable mapping $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ possesses diffuse derivatives of all orders, actually at least one for every choice of infinitesimal sequence.*

Remark 10 (Nonexistence of distributional derivatives)**.** Since we do not require our maps to be in $L^1_{\text{loc}}(\Omega, \mathbb{R}^N)$, they may not possess distributional derivatives.

In general diffuse derivatives *may not be unique* for nonsmooth maps. However, they are compatible with weak derivatives and a fortiori with classical derivatives:

Lemma 11 (Compatibility of weak and diffuse derivatives)**.** *If $u \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$, then the diffuse gradient $\mathcal{D}u$ is unique and*

$$\delta_{Du} \,=\, \mathcal{D}u, \quad \textit{a.e. on } \Omega.$$

More generally, if $q \in \{1, ..., p-1\}$ and $u \in W^{q,1}_{loc}(\Omega, \mathbb{R}^N)$, then $\mathcal{D}^{[q]}u$ is unique and

$$\mathcal{D}^{[p]}u \,=\, \delta_{(Du,\dots,D^q u)} \times \mathcal{D}^{q+1} \times \ldots \times \mathcal{D}^p u, \quad \textit{a.e. on } \Omega.$$

Proof of Lemma 11. It suffice to establish only the 1st order case. For any fixed $e \in \mathbb{R}^n$ we have $D^{1,h}_e u \longrightarrow D_e u$ in $L^1_{\text{loc}}(\Omega, \mathbb{R}^N)$ as $h \to 0$. We choose $e := E^{(\alpha)i}$ and $h := h_m$ to get

$$D^{1,h_m}_{E^{(\alpha)i}}(E^\alpha \cdot u) \longrightarrow D_{E^{(\alpha)i}}(E^\alpha \cdot u), \quad \text{in } L^1_{\text{loc}}(\Omega) \text{ as } m \to \infty.$$

Thus, by (2.3), (2.4) and Definition 5 we have $D^{1,h_m}u \longrightarrow Du$ a.e. on Ω as $m \to \infty$ along a subsequence. Application of Lemma 4 completes the proof. $\square$

Next we show that the diffuse gradient $\mathcal{D}u$ is a Dirac mass if and only if the map u is "differentiable in measure", a notion introduced and studied by Ambrosio and Malý in [AM]:

Definition 12 (Differentiability in measure, cf. [AM])**.** Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be measurable. We say that u is differentiable in measure on Ω with derivative the measurable map $\mathcal{L}Du : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn}$ if for any $\varepsilon > 0$ and $E \subseteq \Omega$ with $|E| < \infty$,

$$\lim_{y \to 0} \left| \left\{ x \in E \,:\, \left| \frac{u(x+y) - u(x) - \mathcal{L}Du(x)\, y}{|y|} \right| > \varepsilon \right\} \right| \,=\, 0.$$

The differentiability in measure arose in the study of the regularity of the flow map of ODEs driven by Sobolev vector fields (Le Bris and Lions, [BL]). In [AM] this notion is compared to the classical notion of approximate differentiability ([EG]). It follows that "$W^{1,1}_{\text{loc}} \Rightarrow BV_{\text{loc}} \Rightarrow$ Approximately diff. $\Rightarrow$ Diff. in measure" with all reverse implications failing in general.

Lemma 13 (Gradient in measure vs diffuse gradient)**.** *Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be measurable and suppose we have fixed some reference frames as in Definition 5.*

(a) If u is differentiable is measure with derivative $\mathcal{L}Du$, then the diffuse gradient $\mathcal{D}u \in \mathscr{Y}(\Omega, \overline{\mathbb{R}}^{Nn})$ is unique and

$$\mathcal{D}u \,=\, \delta_{\mathcal{L}Du}, \quad \textit{a.e. on } \Omega.$$

(b) If there exists a measurable map $U : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn}$ such that for any diffuse gradient $\mathcal{D}u \in \mathscr{Y}(\Omega, \overline{\mathbb{R}}^{Nn})$ we have

$$\mathcal{D}u \,=\, \delta_U, \quad \text{a.e. on } \Omega,$$

then it follows that u is differentiable in measure and $U = \mathcal{L}Du$ a.e. on Ω.

Proof of Lemma 13. (a) By choosing $y := hE^{(\alpha)i}$ in Definition 12 applied to the projection $E^\alpha \cdot u$ we get that $D^{1,h}_{E^{(\alpha)i}}(E^\alpha \cdot u) \longrightarrow E^{\alpha i} : (\mathcal{L}Du)$ as $h \to 0$ locally in measure on Ω. Thus, for any $h_m \to 0$, there is $h_{m_k} \to 0$ such that the convergence is a.e. on Ω, whence $\mathcal{D}u = \delta_{\mathcal{L}Du}$ by Lemma 4.

(b) We begin by observing a triviality: for any map $f : \mathbb{R}^n \to \mathbb{R}^N$ we have $f(y) \to l$ as $y \to 0$ if and only if for any $y_m \to 0$, there is $y_{m_k} \to 0$ such that $f(y_{m_k}) \to l$ as $k \to \infty$. We continue by noting that by Lemma 4 and our assumption we have that for any $h_m \to 0$ there is $h_{m_k} \to 0$ such that $D^{1,h_{m_k}}u \longrightarrow U$ a.e. on Ω, as $k \to \infty$. Hence, we obtain that $D^{1,h}u \longrightarrow U$ as $h \to 0$ (full limit), a.e. on Ω. Since a.e. convergence implies convergence locally in measure, we deduce that $U = \mathcal{L}Du$ a.e. on Ω, as desired. $\square$

The next notion of solution will be central in this work. For pedagogical reasons, we give it first for $W^{1,1}_{\text{loc}}$ solutions of 2nd order systems and then in the general case.

Definition 14 (Weakly differentiable $\mathcal{D}$-solutions of 2nd order systems)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open,

$$F \;:\; \Omega \times \Big(\mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s\Big) \longrightarrow \mathbb{R}^M$$

a Carathéodory map and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ a map in $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$. Suppose we have fixed some reference frames as in Definition 5 and consider the PDE system

$$F\big(\cdot, u, Du, D^2u\big) \,=\, 0, \quad \text{on } \Omega. \tag{2.10}$$

We say that u is a **$\mathcal{D}$-solution of** (2.10) when for any diffuse hessian of u arising from any infinitesimal sequence (Definition 7)

$$\delta_{D^{1,h_m}Du} \overset{*}{\rightharpoonup} \mathcal{D}^2u \quad \text{ in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big),$$

as $m \to \infty$, we have

$$\int_{\mathbb{R}^{Nn^2}_s} \Phi(\mathbf{X})\, F\big(\cdot, u, Du, \mathbf{X}\big)\, d[\mathcal{D}^2u](\mathbf{X}) \,=\, 0, \quad \text{a.e. on } \Omega,$$

for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn^2}_s\big)$.

Now we consider the general pth order case. For brevity, we will write

$$\underline{\mathbf{X}} \,\equiv\, (\mathbf{X}_1, ..., \mathbf{X}_p) \,\in\, \overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s.$$

Definition 15 ($\mathcal{D}$-solutions for pth order systems)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open,

$$F \;:\; \Omega \times \Big(\mathbb{R}^N \times \mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s\Big) \longrightarrow \mathbb{R}^M$$

a Carathéodory map and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ a measurable map. Suppose also we have fixed some reference frames as in Definition 5 and consider the PDE system

$$F\Big(x, u(x), D^{[p]}u(x)\Big) \,=\, 0, \quad x \in \Omega. \tag{2.11}$$

Then, we say that u is a **$\mathcal{D}$-solution of** (2.11) when for any diffuse pth order Jet of u arising from any infinitesimal sequence (Definition 7)

$$\delta_{D^{[p],h_{\underline{m}}}u} \overset{*}{\rightharpoondown} \mathcal{D}^{[p]}u \quad \text{in } \mathscr{Y}\Big(\Omega, \overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s\Big),$$

as $\underline{m} \to \infty$, we have

$$\int_{\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s} \Phi(\underline{\mathbf{X}})\, F\big(x, u(x), \underline{\mathbf{X}}\big)\, d\big[\mathcal{D}^{[p]}u(x)\big](\underline{\mathbf{X}}) \;=\; 0, \qquad \text{a.e. } x \in \Omega,$$

for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn} \times \cdots \times \mathbb{R}^{Nn^p}_s\big)$.

Note that Definition 14 can be deduced from Definition 15 by using Lemmas 11 and 4 and that the convergence is separate. These imply when $p = 2$ that $D^{2,h_{(m',m)}}u \longrightarrow D^{1,h_m}Du$ a.e. on Ω as $m' \to \infty$.

The following result asserts the fairly obvious fact that $\mathcal{D}$-solutions and strong solutions are compatible.

Proposition 16 (Compatibility of strong and $\mathcal{D}$-solutions)**.** *Let F a Carathéodory map as in (1.1) and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be a map in $W^{p,1}_{loc}(\Omega, \mathbb{R}^N)$ (or merely p-times differentiable in measure, Definition 12). Consider the PDE system*

$$F\Big(x, u(x), D^{[p]}u(x)\Big) \;=\; 0, \quad x \in \Omega.$$

Then, u is a $\mathcal{D}$-solution on Ω if and only if u is a strong a.e. solution on Ω.

Proof of Proposition 16. It is an immediate consequence of Lemma 11 (or Lemma 13) and the motivation of the notions (Subsection 2.2). ☐

Remark 17 (Absence of concentration measures)**.** The next estimate shows that "... $= 0$ a.e. on Ω" in Definition 15 is equivalent to "... $= 0$ in $L^\infty(\Omega)$". Namely, for any fixed Φ the left hand side is always a measurable function and no measures can arise mutually singular to the Lebesgue measure. Indeed, for a.e. $x \in \Omega$

$$\left| \int_{\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s} \Phi(\underline{\mathbf{X}})\, F\big(x, u(x), \underline{\mathbf{X}}\big)\, d\big[\mathcal{D}^{[p]}u(x)\big](\underline{\mathbf{X}}) \right| \le \Big(\sup_{\mathbb{R}^{Nn} \times \ldots \times \mathbb{R}^{Nn^p}_s} |\Phi| \Big) \max_{\mathrm{supp}(\Phi)} \big| F\big(x, u(x), \cdot\big) \big|.$$

Our next result is a simple yet powerful convergence result which we state and prove in the generality of Young measures. It will play an important role later in the construction of $\mathcal{D}$-solutions.

Lemma 18 (Convergence lemma)**.** *Suppose that $u^\infty, (u^\mu)_1^\infty$ are measurable maps $\Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ satisfying $u^\mu \longrightarrow u^\infty$ a.e. on Ω, as $\mu \to \infty$. Moreover, let $\mathbb{W}$ be a finite dimensional metric vector space, isometrically and densely contained into a compactification $\mathbb{K}$ of $\mathbb{W}$. Suppose also that we have Carathéodory maps*

$$F^\infty,\, F^\mu \;:\; \Omega \times \big(\mathbb{R}^N \times \mathbb{W}\big) \longrightarrow \mathbb{R}^M, \quad \mu \in \mathbb{N},$$

such that for a.e. $x \in \Omega$,

$$F^\mu(x, \cdot, \cdot) \longrightarrow F^\infty(x, \cdot, \cdot) \ \text{ in } C^0(\mathbb{R}^N \times \mathbb{W}), \quad \text{as } \mu \to \infty$$

and we also have Young measures $\vartheta^\infty, (\vartheta^\mu)_1^\infty \in \mathscr{Y}\big(\Omega, \mathbb{K}\big)$ such that

$$\vartheta^\mu \overset{*}{\rightharpoondown} \vartheta^\infty \ \text{ in } \ \mathscr{Y}\big(\Omega, \mathbb{K}\big), \quad \text{as } \mu \to \infty.$$

Then, if for a given $\Phi \in C^0_c(\mathbb{W})$ we have

$$\int_{\mathbb{K}} \Phi(\boldsymbol{X})\, F^\mu\big(x, u^\mu(x), \boldsymbol{X}\big)\, d[\vartheta^\mu(x)](\boldsymbol{X}) \,=\, 0, \quad a.e.\ x \in \Omega,$$

for all $\mu \in \mathbb{N}$, it follows that

$$\int_{\mathbb{K}} \Phi(\boldsymbol{X})\, F^\infty\big(x, u^\infty(x), \boldsymbol{X}\big)\, d[\vartheta^\infty(x)](\boldsymbol{X}) \,=\, 0, \quad a.e.\ x \in \Omega.$$

Proof of Lemma 18. It suffices to show that for any given fixed $\Phi \in C^0_c(\mathbb{W})$, we have that

$$\phi^\mu \,:=\, \sup_{\mathbf{X}\in\mathbb{W}} \Big|\Phi(\mathbf{X})\big[F^\mu(\cdot, u^\mu, \mathbf{X}) \,-\, F^\infty(\cdot, u^\infty, \mathbf{X})\big]\Big| \,\longrightarrow\, 0,$$

a.e. on Ω. Indeed, if this is the case, select as Φ the function of the assumption of the lemma and set $\Omega_R := \Omega \cap \mathbb{B}_R(0)$ for some fixed $R > 0$. Since $|\Omega_R| < \infty$, by Egoroff's theorem, we can find for each $j \in \mathbb{N}$ a measurable set $E_j \subseteq \Omega_R$ with $E_{j+1} \subseteq E_j$ and $|E_j| \leq 1/j$ such that

$$\|\phi^\mu\|_{L^\infty(\Omega_R \setminus E_j)} \longrightarrow 0, \quad \text{as } \mu \to \infty.$$

Then, by using the weak*-strong continuity of the pairing

$$L^\infty_{w^*}\Big(\Omega_R \setminus E_j, \mathcal{M}(\mathbb{K})\Big) \,\times\, L^1\Big(\Omega_R \setminus E_j, C^0(\mathbb{K})\Big) \longrightarrow \mathbb{R}$$

and that $L^\infty(\Omega_R \setminus E_j) \subseteq L^1(\Omega_R \setminus E_j)$, the convergence $\vartheta^\mu \overset{*}{\rightharpoonup} \vartheta^\infty$ in $\mathscr{Y}\big(\Omega_R \setminus E_j, \mathbb{K}\big)$ as $\mu \to \infty$ and our assumptions imply

$$\int_{\mathbb{K}} \Phi(\mathbf{X}) F^\infty\big(x, u^\infty(x), \mathbf{X}\big)\, d[\vartheta^\infty(x)](\mathbf{X}) \,=\, 0,$$

for a.e. $x \in \Omega_R \setminus E_j$. Then, we conclude by letting $j \to \infty$ and then taking $R \to \infty$. In order to establish that $\phi^\mu \to 0$ a.e. on Ω, we recall that $u^\mu \longrightarrow u^\infty$ a.e. on Ω and we fix an $x \in \Omega$ such that $u^\mu(x) \longrightarrow u^\infty(x)$. Then, we can find compact sets $C' \Subset \mathbb{R}^N$ and $C'' \Subset \mathbb{W}$ such that $u^\mu(x), u^\infty(x) \in C'$ and $\mathrm{supp}(\Phi) \subseteq C''$. By the convergence assumption on the maps F^μ, we have

$$\big\|F^\mu(x,\cdot) - F^\infty(x,\cdot)\big\|_{C^0(C'\times C'')} \longrightarrow 0, \quad \text{as } \mu \to \infty.$$

If $\omega^\infty_x \in C^0[0,\infty)$ denotes the modulus of continuity of $C' \ni \eta \mapsto F^\infty(x,\eta,\mathbf{X}) \in \mathbb{R}^M$ which can be chosen uniform with respect to $\mathbf{X} \in C''$, we have

$$\begin{aligned}
|\phi^\mu(x)| \,\leq\, & \sup_{\mathbf{X}\in C''} |\Phi| \left\{ \sup_{\mathbf{X}\in C''} \Big| F^\infty(x, u^\mu(x), \mathbf{X}) \,-\, F^\infty(x, u^\infty(x), \mathbf{X}) \Big| \right. \\
& \left. \qquad\qquad + \sup_{\mathbf{X}\in C''} \Big| F^\mu(x, u^\mu(x), \mathbf{X}) \,-\, F^\infty(x, u^\mu(x), \mathbf{X}) \Big| \right\} \\
\leq\, & \sup_{\mathbf{X}\in\mathbb{W}} |\Phi| \Big\{ \omega^\infty_x\big(|u^\mu(x) - u^\infty(x)|\big) \,+\, \big\|F^\mu(x,\cdot) - F^\infty(x,\cdot)\big\|_{C^0(C'\times C'')} \Big\} \\
=\, & o(1),
\end{aligned}$$

as $\mu \to \infty$, because $\omega^\infty_x(0^+) = 0$. Since this holds for a set of points $x \in \Omega$ of full measure, the conclusion follows and the lemma ensues. $\square$

The following result is a consequence of the convergence Lemma 18 and establishes that $\mathcal{D}$-solutions are well behaved under weak* convergence.

Proposition 19 (Convergence of $\mathcal{D}$-solutions). *Let $(u^\mu)_1^\infty$ be a sequence of maps where each $u^\mu : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is measurable and $u^\mu \longrightarrow u^\infty$ a.e. on Ω. Let also $(F^\mu)_1^\infty$ be a sequence of Carathéodory map (with the same dimensions as in (1.1)) and assume that each u^μ is a $\mathcal{D}$-solution of the system*

$$F^\mu\left(x, u^\mu(x), D^{[p]}u^\mu(x)\right) \,=\, 0, \quad x \in \Omega,$$

and that for a.e. $x \in \Omega$, $F^\mu(x,\cdot,\cdot) \longrightarrow F^\infty(x,\cdot,\cdot)$ uniformly on compact subsets as $\mu \to \infty$. If further every jet $\mathcal{D}^{[p]}u^\infty$ can be weakly approximated by a subsequence of the respective Jets $\mathcal{D}^{[p]}u^{\mu_\nu}$, then u^∞ is a $\mathcal{D}$-solution of*

$$F^\infty\left(x, u^\infty(x), D^{[p]}u^\infty(x)\right) \,=\, 0, \quad x \in \Omega.$$

Remark 20 (On stability). The reader should note that Proposition 19 is *not* a stability result, in the sense that we do not have compactness of diffuse jets as part of the conclusion. In fact, such a result is not possible without extra assumptions which would entail some sort of *a priori estimates*: for instance, consider the sequence $u^\mu(x) := \mu^{-1}\sin(\mu x)$, $x \in \mathbb{R}$. Then, $u^\mu \overset{*}{\longrightarrow} u^\infty$ in $W^{1,\infty}(\mathbb{R})$ where $u^\infty \equiv 0$. However, $\mathcal{D}u^\mu = \delta_{Du^\mu} \overset{*}{\longrightarrow} \vartheta$ in $\mathscr{Y}(\mathbb{R},\mathbb{R})$ as $\mu \to \infty$, where for a.e. $x \in \mathbb{R}$ $\operatorname{supp}(\vartheta(x)) = [-1,1]$ while $\vartheta(x) \neq \mathcal{D}u^\infty(x) = \delta_{\{0\}}$.

The next result gives equivalent formulations of the definition of $\mathcal{D}$-solutions. To this end we first need to introduce some further terminology.

Definition 21 (Reductions & cut offs). Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be a measurable map and F a Carathéodory map as in (1.1). Given $\vartheta \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn} \times ... \times \overline{\mathbb{R}}^{Nn^p}_s\big)$, we define the **reduced Young measure** ϑ_* as the next (fibre) restriction of ϑ:

$$\vartheta_*(x) \,:=\, \vartheta(x) \llcorner \left(\mathbb{R}^{Nn} \times ... \times \mathbb{R}^{Nn^p}_s\right), \qquad \text{a.e. } x \in \Omega.$$

Further, given a measurable map $U : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn} \times ... \times \mathbb{R}^{Nn^p}_s$ and $R > 0$, we define the **cut off of U associated to F** as:

$$[U]^R \,:=\, \left\{ \begin{array}{ll} U, & \text{on } \big\{|U| \leq R\big\}, \\ \mathbf{0}_F, & \text{on } \big\{|U| > R\big\}. \end{array} \right.$$

Here for each $R > 0$, $\mathbf{0}_F$ is a measurable selection of the set-valued mapping

$$\Omega \ni x \longmapsto \Big\{F\big(x, u(x), \cdot\big) = 0\Big\} \cap \mathbb{B}_R(0) \;\subseteq\; \left(\mathbb{R}^{Nn} \times ... \times \mathbb{R}^{Nn^p}_s\right) \setminus \{\emptyset\},$$

that is, for each $R > 0$, $\mathbf{0}_F$ is a measurable map $\Omega \longrightarrow \mathbb{R}^{Nn} \times ... \times \mathbb{R}^{Nn^p}_s$ satisfying

$$F\Big(x, u(x), \mathbf{0}_F(x)\Big) \,=\, 0, \quad \text{a.e. } x \in \Omega, \quad |\mathbf{0}_F(x)| \leq R.$$

The existence of measurable selections as above is a consequence of Aumann's theorem (for non-empty valued measurable maps as we have assumed above, see e.g. [FL]). If $F\big(x, u(x), \cdot\big)$ is linear, we may choose $\mathbf{0}_F \equiv 0$ with no R-dependence.

Proposition 22 (Equivalent definitions for $\mathcal{D}$-solutions). *Let F be a Carathéodory map as in (1.1) and $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ a measurable map. Then, the following are equivalent:*

(1) *The map u is a $\mathcal{D}$-solution of the PDE system*

$$F\left(x, u(x), D^{[p]}u(x)\right) = 0, \quad x \in \Omega.$$

(2) *All reduced pth order Jets of u satisfy the differential inclusion:*

$$\text{For a.e. } x \in \Omega, \quad \mathrm{supp}\Big(\mathcal{D}^{[p]}u_*(x)\Big) \subseteq \Big\{F\big(x,u(x),\cdot\big)=0\Big\}.$$

(3) *For any pth order Jet of u, we have*

$$\int_{\mathbb{R}^{Nn}\times\ldots\times\mathbb{R}^{Nn^p}_s} \big|F\big(x,u(x),\underline{\boldsymbol{X}}\big)\big|\, d\big[\mathcal{D}^{[p]}u(x)\big](\underline{\boldsymbol{X}}) \;=\; 0, \quad \text{a.e. } x\in\Omega.$$

(4) *For any pth order Jet of difference quotients of u and any $R>0$, we have*

$$F\left(x,u(x),\big[D^{[p],h_{\underline{m}}}u(x)\big]^R\right) \longrightarrow 0, \qquad \text{as } \underline{m}\to\infty,$$

for a.e. $x\in\Omega$ along subsequences.

(5) *For any pth order Jet of difference quotients of u and any $R>0$, we have*

$$\mathrm{dist}\left(\big[D^{[p],h_{\underline{m}}}u(x)\big]^R,\, \mathbb{B}_R(0)\cap\Big\{F\big(x,u(x),\cdot\big)=0\Big\}\right) \longrightarrow 0,$$

for a.e. $x\in\Omega$, as $\underline{m}\to\infty$ along subsequences.

If further F does not depend on $x, u(x)$, then (1)-(5) above are equivalent to:

(6) *For any pth order Jet of u, we have*

$$\int_{\mathbb{R}^{Nn}\times\ldots\times\mathbb{R}^{Nn^p}_s} \Psi(\underline{\boldsymbol{X}})\, F(\underline{\boldsymbol{X}})\, d\big[\mathcal{D}^{[p]}u\big](\underline{\boldsymbol{X}}) \;=\; 0, \quad \text{a.e. on } \Omega,$$

for any $\Psi\in\mathcal{A}$, where

$$\mathcal{A} := \left\{ \Psi\in C^0\Big(\mathbb{R}^{Nn}\times\ldots\times\mathbb{R}^{Nn^p}_s\Big) \;\middle|\; \limsup_{|\underline{\boldsymbol{X}}|\to\infty}\big|\Psi(\underline{\boldsymbol{X}})\big|\big(1+\big|F(\underline{\boldsymbol{X}})\big|\big) \;=\; 0 \right\}.$$

The presence of the reduced measures and of the truncations can be informally interpreted as follows: the mass which remains away from infinity (and does not escape) actually has to lie in the zero level set of the coefficients.

The proof of Proposition 22 does not rely on the particular structure of diffuse Jets and is an immediate consequence of the next general result.

Lemma 23. *All the equivalences of Proposition 22 remains true if more generally one replaces $D^{[p],h_{\underline{m}}}u$ by any measurable sequence*

$$U^m \;:\; \Omega\subseteq\mathbb{R}^n \longrightarrow \mathbb{R}^{Nn}\times\ldots\times\mathbb{R}^{Nn^p}_s, \quad m\in\mathbb{N},$$

and the respective Jet $\mathcal{D}^{[p]}u$ by any Young measure $\vartheta\in\mathscr{Y}\big(\Omega,\overline{\mathbb{R}}^{Nn}\times\cdots\times\overline{\mathbb{R}}^{Nn^p}_s\big)$ such that $\delta_{U^m}\overset{*}{\rightharpoonup}\vartheta$ as $m\to\infty$.

Proof of Lemma 23 & Proposition 22. We begin by showing (1)⇔(2), then we will establish that (5)⇒(4)⇒(3)⇒(2)⇒(5) and finally that (1)⇔(6).

(1)⇒(2): Suppose that $\delta_{U^m}\overset{*}{\rightharpoonup}\vartheta$ as $m\to\infty$ and we have

$$\int_{\overline{\mathbb{R}}^{Nn}\times\ldots\times\overline{\mathbb{R}}^{Nn^p}_s} \Phi(\underline{\mathbf{X}})\, F\big(x,u(x),\underline{\mathbf{X}}\big)\, d[\vartheta(x)](\underline{\mathbf{X}}) \;=\; 0, \quad \text{a.e. } x\in\Omega,$$

for any $\Phi\in C^0_c\big(\mathbb{R}^{Nn}\times\ldots\times\mathbb{R}^{Nn^p}_s\big)$, while the conclusion fails. To this end, we fix a point $x\in\Omega$ as above and suppose $\mathrm{supp}\big(\vartheta_*(x)\big)\not\subseteq\big\{F\big(x,u(x),\cdot\big)=0\big\}$. Then, there is some point

$$\underline{\mathbf{X}}_0 \;\in\; \Big(\mathbb{R}^{Nn}\times\ldots\times\mathbb{R}^{Nn^p}_s\Big)\setminus\Big\{F\big(x,u(x),\cdot\big)=0\Big\}$$

such that, for all $R>0$ we have $[\vartheta_*(x)]\big(\mathbb{B}_R(\underline{\mathbf{X}}_0)\big)>0$. Since $F\big(x,u(x),\cdot\big)$ is continuous and $F\big(x,u(x),\underline{\mathbf{X}}_0\big)\neq 0$, there exist $c_0,R_0>0$ and an index $\mu\in\{1,...,M\}$ such that

$$\big|F_\mu\big(x,u(x),\cdot\big)\big|\geq c_0>0, \quad \text{on } \ \mathbb{B}_{R_0}(\underline{\mathbf{X}}_0).$$

We now choose Φ such that

$$\chi_{\mathbb{B}_{R_0/2}(\underline{\mathbf{X}}_0)}\ \leq\ \Phi\ \leq\ \chi_{\mathbb{B}_{R_0}(\underline{\mathbf{X}}_0)}.$$

As a result, for this choice of Φ we have

$$\begin{aligned} 0\ &=\ \left|\int_{\mathbb{R}^{Nn}\times\ldots\times\mathbb{R}^{Nn^p}_s}\Phi(\underline{\mathbf{X}})\,F_\mu\big(x,u(x),\underline{\mathbf{X}}\big)\,d[\vartheta_*(x)](\underline{\mathbf{X}})\right| \\ &=\ \int_{\mathbb{B}_{R_0}(\underline{\mathbf{X}}_0)}\Phi(\underline{\mathbf{X}})\,\big|F_\mu\big(x,u(x),\underline{\mathbf{X}}\big)\big|\,d[\vartheta_*(x)](\underline{\mathbf{X}}) \\ &\geq\ c_0\,[\vartheta_*(x)]\big(\mathbb{B}_{R_0/2}(\underline{\mathbf{X}}_0)\big). \end{aligned}$$

The above contradiction establishes that the desired inclusion holds a.e. on Ω.

(2)$\Rightarrow$(1): Suppose that $\mathrm{supp}\big(\vartheta_*(x)\big)\subseteq\big\{F\big(x,u(x),\cdot\big)=0\big\}$ for a.e. $x\in\Omega$. Then, for any $\Phi\in C^0_c\big(\mathbb{R}^{Nn}\times...\times\mathbb{R}^{Nn^p}_s\big)$ and any such x we have that $\Phi(\cdot)F\big(x,u(x),\cdot\big)$ vanishes $[\vartheta(x)]$-a.e. on $\overline{\mathbb{R}}^{Nn}\times...\times\overline{\mathbb{R}}^{Nn^p}_s$. Thus, for any such x we have

$$\int_{\overline{\mathbb{R}}^{Nn}\times\ldots\times\overline{\mathbb{R}}^{Nn^p}_s}\Phi(\underline{\mathbf{X}})\,F\big(x,u(x),\underline{\mathbf{X}}\big)\,d[\vartheta(x)](\underline{\mathbf{X}})\ =\ 0.$$

(5)$\Rightarrow$(4): Fix $R>0$. If suffices to show that for a.e. $x\in\Omega$, there is a strictly increasing modulus of continuity $\omega_{R,x}\in C^0[0,\infty)$ with $\omega_{R,x}(0)=0$ such that

$$\Big|F\big(x,u(x),\underline{\mathbf{X}}\big)\Big|\ \leq\ \omega_{R,x}\left(\mathrm{dist}\left(\underline{\mathbf{X}},\mathbb{B}_R(0)\cap\Big\{F\big(x,u(x),\cdot\big)=0\Big\}\right)\right),$$

when $\underline{\mathbf{X}}\in\overline{\mathbb{B}_R(0)}$. Indeed, in that case we conclude by choosing $\underline{\mathbf{X}}:=[U^m(x)]^R$. By continuity, indeed for a.e. $x\in\Omega$ there is a strictly increasing modulus of continuity $\omega_{R,x}$ such that

$$\Big|F\big(x,u(x),\underline{\mathbf{X}}\big)\ -\ F\big(x,u(x),\underline{\mathbf{Y}}\big)\Big|\ \leq\ \omega_{R,x}\big(|\underline{\mathbf{X}}-\underline{\mathbf{Y}}|\big),$$

when $\underline{\mathbf{X}},\underline{\mathbf{Y}}\in\overline{\mathbb{B}_R(0)}$. By choosing $\underline{\mathbf{Y}}$ such that $F\big(x,u(x),\underline{\mathbf{Y}}\big)=0$, we have

$$\begin{aligned}\Big|F\big(x,u(x),\underline{\mathbf{X}}\big)\Big|\ &\leq\ \inf_{F(x,u(x),\underline{\mathbf{Y}})=0,|\underline{\mathbf{Y}}|\leq R}\omega_{R,x}\big(|\underline{\mathbf{X}}-\underline{\mathbf{Y}}|\big)\\ &=\ \omega_{R,x}\left(\inf_{F(x,u(x),\underline{\mathbf{Y}})=0,|\underline{\mathbf{Y}}|\leq R}|\underline{\mathbf{X}}-\underline{\mathbf{Y}}|\right),\end{aligned}$$

as desired.

(4)$\Rightarrow$(3): We fix $R>0$ and any $\Phi\in C^0_c\big(\mathbb{R}^{Nn}\times...\times\mathbb{R}^{Nn^p}_s\big)$ such that

$$\chi_{\mathbb{B}_{R/2}(0)}\ \leq\ \Phi\ \leq\ \chi_{\mathbb{B}_R(0)}.$$

For any $k\in\mathbb{N}$, we set

$$\Omega_k\ :=\ \left\{x\in\Omega\cap\mathbb{B}_k(0)\ :\ \sup_{\mathbb{R}^{Nn}\times\ldots\times\mathbb{R}^{Nn^p}_s}\Phi(\cdot)\big|F\big(x,u(x),\cdot\big)\big|\leq k\right\}.$$

Then, $\Omega_k\subseteq\Omega_{k+1}$ and $|\Omega\setminus\Omega_k|\longrightarrow 0$ as $k\to\infty$. We also define

$$\Psi^k(x,\underline{\mathbf{X}})\ :=\ \Phi(\underline{\mathbf{X}})\big|F\big(x,u(x),\underline{\mathbf{X}}\big)\big|\chi_{\Omega_k}(x),\quad k\in\mathbb{N}.$$

Since $\delta_{U^m} \xrightharpoonup{*} \vartheta$ as $m \to \infty$ and Ψ^k is an admissible Carathéodory function, we have

$$\int_\Omega \Psi^k\big(x, U^m(x)\big)\, dx \ \longrightarrow\ \int_\Omega \int_{\overline{\mathbb{R}}^{Nn} \times \ldots \times \overline{\mathbb{R}}^{Nn^p}_s} \Psi^k\big(x, \underline{\mathbf{X}}\big) d[\vartheta(x)](\underline{\mathbf{X}})\, dx,$$

as $m \to \infty$. By assumption, we have that $F\big(\cdot, u, [U^m]^R\big) \longrightarrow 0$ a.e. on Ω as $m \to \infty$. By the properties of Φ and of the truncations, we have the identity

$$\Phi\left([U^m]^R\right) F\big(\cdot, u, [U^m]^R\big) \ =\ \Phi\left(U^m\right) F\big(\cdot, u, U^m\big)$$

valid a.e. on Ω. Together these last facts give that $\Psi^k(\cdot, U^m) \longrightarrow 0$ a.e. on Ω. Moreover, by using the bound $|\Phi^k| \leq k$ and that $|\Omega_k| < \infty$, the Dominated convergence theorem allows to infer that $\Psi^k(\cdot, U^m) \longrightarrow 0$ in $L^1(\Omega)$ as $m \to \infty$. Hence, by the above convergence and the definition of Φ, for a.e. $x \in \Omega_k$ we have that

$$\begin{aligned} 0 \ &=\ \int_{\overline{\mathbb{R}}^{Nn} \times \ldots \times \overline{\mathbb{R}}^{Nn^p}_s} \Psi^k\big(x, \underline{\mathbf{X}}\big)\, d[\vartheta(x)](\underline{\mathbf{X}}) \\ &=\ \int_{\overline{\mathbb{R}}^{Nn} \times \ldots \times \overline{\mathbb{R}}^{Nn^p}_s} \Phi(\underline{\mathbf{X}}) \big| F\big(x, u(x), \underline{\mathbf{X}}\big)\big|\, d[\vartheta(x)](\underline{\mathbf{X}}) \\ &\geq\ \int_{\mathbb{B}_{R/2}(0)} \big| F\big(x, u(x), \underline{\mathbf{X}}\big)\big|\, d[\vartheta(x)](\underline{\mathbf{X}}). \end{aligned}$$

The conclusion follows by letting $k \to \infty$ and then $R \to \infty$.

(3)$\Rightarrow$(2): We argue as in the case "(1)$\Rightarrow$(2)". Suppose that

$$\int_{\mathbb{R}^{Nn} \times \ldots \times \mathbb{R}^{Nn^p}_s} \big| F\big(x, u(x), \underline{\mathbf{X}}\big)\big|\, d[\vartheta(x)](\underline{\mathbf{X}}) \ =\ 0, \quad \text{a.e. } x \in \Omega,$$

while the conclusion fails. Fix $x \in \Omega$ for which the above holds and assume that $\mathrm{supp}\big(\vartheta_*(x)\big) \not\subseteq \big\{\big|F\big(x, u(x), \cdot\big)\big| = 0\big\}$. Then, there exists

$$\underline{\mathbf{X}}_0 \ \in\ \big(\mathbb{R}^{Nn} \times \ldots \times \mathbb{R}^{Nn^p}_s\big) \setminus \Big\{\big|F\big(x, u(x), \cdot\big)\big| = 0\Big\}$$

such that, for all $R > 0$ we have that $[\vartheta(x)]\big(\mathbb{B}_R(\underline{\mathbf{X}}_0)\big) > 0$. Since $\big|F\big(x, u(x), \cdot\big)\big|$ is continuous and $\big|F\big(x, u(x), \underline{\mathbf{X}}_0\big)\big| > 0$, there exist $c_0, R_0 > 0$ such that

$$\big|F\big(x, u(x), \cdot\big)\big| \geq c_0 > 0, \quad \text{on } \mathbb{B}_{R_0}(\underline{\mathbf{X}}_0).$$

Then, we have

$$0 \ =\ \int_{\mathbb{R}^{Nn} \times \ldots \times \mathbb{R}^{Nn^p}_s} \big| F\big(x, u(x), \underline{\mathbf{X}}\big)\big|\, d[\vartheta(x)](\underline{\mathbf{X}}) \ \geq\ c_0\, [\vartheta(x)]\big(\mathbb{B}_{R_0}(\underline{\mathbf{X}}_0)\big).$$

The above contradiction establishes the desired inclusion.

(2)$\Rightarrow$(5): We fix $R > 0$ and define the function

$$\Psi \ :\ \ \Omega \times \Big(\overline{\mathbb{R}}^{Nn} \times \cdots \times \overline{\mathbb{R}}^{Nn^p}_s\Big) \ \longrightarrow\ [0, \infty)$$

given by

$$\Psi(x, \underline{\mathbf{X}}) \ :=\ \chi_{\overline{\mathbb{B}_R(0)}}(\underline{\mathbf{X}})\, \mathrm{dist}\left(\underline{\mathbf{X}},\, \mathbb{B}_R(0) \cap \Big\{\big|F\big(x, u(x), \cdot\big)\big| = 0\Big\}\right).$$

Then, Ψ is measurable in x for all $\underline{\mathbf{X}}$ (this is a consequence of Aumann's theorem, see e.g. [FL]), upper semicontinuous in $\underline{\mathbf{X}}$ for a.e. x and also bounded. Hence, since

$\delta_{U^m} \overset{*}{\longrightarrow} \vartheta$ as $m \to \infty$, by Remark 3iii) and the definition of the reduced measure, we have that

$$\limsup_{m\to\infty} \int_\Omega \Psi\big(x, U^m(x)\big)\, dx \,\leq\, \int_\Omega \int_{\overline{\mathbb{R}}^{Nn}\times\cdots\times\overline{\mathbb{R}}_s^{Nn^p}} \Psi\big(x, \underline{\mathbf{X}}\big)\, d[\vartheta(x)](\underline{\mathbf{X}})\, dx$$

$$= \int_\Omega \int_{\overline{\mathbb{B}_R(0)}} \mathrm{dist}\left(\underline{\mathbf{X}},\, \mathbb{B}_R(0) \cap \Big\{\big|F\big(x, u(x), \cdot\big)\big| = 0\Big\}\right)\, d[\vartheta_*(x)](\underline{\mathbf{X}})\, dx.$$

By assumption we have that for a.e. $x \in \Omega$, the support of the measure $\vartheta_*(x) \llcorner \overline{\mathbb{B}_R(0)}$ is contained in the closed set

$$\overline{\mathbb{B}_R(0)} \cap \Big\{\big|F\big(x, u(x), \cdot\big)\big| = 0\Big\}$$

and the latter is a subset of the zero level set of the function $\Psi(x, \cdot)$. Hence, the last integral above vanishes and we obtain that $\Psi(\cdot, U^m) \longrightarrow 0$ in $L^1(\Omega)$ as $m \to \infty$. Further, in view of Definition 21, we have the identity

$$\Psi\big(x, U^m(x)\big) \,=\, \mathrm{dist}\left([U^m(x)]^R,\, \mathbb{B}_R(0) \cap \Big\{\big|F\big(x, u(x), \cdot\big)\big| = 0\Big\}\right),$$

which is valid for a.e. $x \in \Omega$ and by using it we obtain that

$$\int_\Omega \mathrm{dist}\left([U^m(x)]^R,\, \mathbb{B}_R(0) \cap \Big\{\big|F\big(x, u(x), \cdot\big)\big| = 0\Big\}\right)\, dx \,\longrightarrow\, 0,$$

as $m \to \infty$. The conclusion follows by passing to a subsequence.

(1)$\Leftrightarrow$(6): Obviously, (6) readily implies (1). Conversely, fix $\Psi \in \mathcal{A}$ and $\varepsilon > 0$. Then, for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn} \times \ldots \times \mathbb{R}^{Nn^p}_s\big)$, we have

$$\left|\int_{\mathbb{R}^{Nn}\times\ldots\times\mathbb{R}_s^{Nn^p}} \Psi(\underline{\mathbf{X}})\, F(\underline{\mathbf{X}})\, d[\vartheta(x)](\mathbf{X})\right| \leq \sup_{\mathbb{R}^{Nn}\times\ldots\times\mathbb{R}_s^{Nn^p}} \Big\{\big|\Psi - \Phi\big|\big(1 + |F|\big)\Big\}, \tag{2.12}$$

because $\vartheta(x)$ is a probability for a.e. $x \in \Omega$. By assumption we have that $\Psi(1 + |F|) \in C^0_0\big(\mathbb{R}^{Nn} \times \ldots \times \mathbb{R}^{Nn^p}_s\big)$ and hence we can find a compactly supported ϕ uniformly ε-close to $\Psi(1 + |F|)$. By choosing $\Phi := \frac{\phi}{1+|F|}$, the right hand side of (2.12) becomes less than ε. Hence, (6) ensues and so does the proposition. $\square$

Remark 24 (Nonlinear nature of diffuse derivatives)**.** In the context of classical PDE approaches (classical, strong, weak, distributional solutions), it is standard that the generalised derivative is a linear operation. However, without extra hypotheses this is generally **false for diffuse derivatives**. *Our approach is genuinely nonlinear and not a variant of classical developments.* As a consequence, we obtain that the sum of two $\mathcal{D}$-solutions to a certain linear equation is a $\mathcal{D}$-solution itself if at least one of the solutions is regular enough. Hence, the notions themselves are nonlinear even when we apply them to linear PDE.

In order to proceed further we need some notation.

Definition 25. Let $\mathbb{W}$ be a finite dimensional metric vector space isometrically and densely contained into a compactification $\mathbb{K}$ of $\mathbb{W}$. Let also $T_a : \mathbb{W} \to \mathbb{W}$ denote the translation operation given by $T_a b := b - a$. Given a probability $\vartheta \in \mathscr{P}(\mathbb{K})$, we define $\vartheta \circ T_a \in \mathscr{P}(\mathbb{K})$ by

$$\vartheta \circ T_a \,:=\, (\vartheta \circ T_a) \llcorner \mathbb{W} \,+\, \vartheta \llcorner (\mathbb{K} \setminus \mathbb{W}),$$

that is, for any Borel set $B \subseteq \mathbb{K}$, we set $\big(\vartheta \circ T_a\big)(B) = \vartheta\big((B \cap \mathbb{W}) - a\big) + \vartheta\big(B \setminus \mathbb{W}\big)$.

Definition 25 requires translation of the part contained in the vector space while the points "at infinity" are left intact. In the case of the 1-point compactification $\mathbb{K} = \mathbb{W} \cup \{\infty\}$, it says $\vartheta(B) = \vartheta\big((B \setminus \{\infty\}) - a\big) + \vartheta\big(\{\infty\}\big)$. Note that we may equivalently define $\vartheta \circ T_a \in \mathscr{P}(\mathbb{K})$ via duality:

$$\langle \vartheta \circ T_a, \Phi \rangle \, = \, \int_{\mathbb{W}} \Phi(a + X)\, d\vartheta(X) \, + \int_{\mathbb{K}\setminus\mathbb{W}} \Phi(X)\, d\vartheta(X), \qquad \Phi \in C^0(\mathbb{K}).$$

Proposition 26 (Diffuse derivatives & $\mathcal{D}$-solutions vs linearity)**.** *Let $u, v : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be measurable maps.*

a) If v is differentiable in measure on Ω with derivative $\mathcal{L}Dv$, (Def. 12), then

$$\mathcal{D}(u + v) \, = \, \mathcal{D}u \circ T_{\mathcal{L}Dv}, \quad \text{ a.e. on } \Omega,$$

where the diffuse Jets on both sides arise from the same infinitesimal sequence.

b) Consider the measurable maps

$$\boldsymbol{A}^q \, : \, \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn^q}_s \otimes \mathbb{R}^M, \qquad f, g \, : \, \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^M,$$

where $q = 1, ..., p$ and the linear pth order systems of PDE

$$\boldsymbol{A}(x) :: D^{[p]} u(x) \, = \, f(x), \qquad x \in \Omega$$
$$\boldsymbol{A}(x) :: D^{[p]} v(x) \, = \, g(x), \qquad x \in \Omega,$$

where $\boldsymbol{A} = (\boldsymbol{A}^1, ..., \boldsymbol{A}^p)$. If u, v are $\mathcal{D}$-solutions, then $u + v$ is a $\mathcal{D}$-solution of

$$\boldsymbol{A}(x) :: D^{[p]}(u + v)(x) \, = \, (f + g)(x) \qquad x \in \Omega,$$

when v is p-times differentiable in measure on Ω.

The notation "::" above is a convenient abbreviation of the multiple contraction

$$\sum_{\alpha_1, i_1} \mathbf{A}^1_{\mu;\alpha_1,i_1} D_{i_1} u_{\alpha_1} \, + \, ... \, + \sum_{\alpha_p, i^p_1 ... i^p_p} \mathbf{A}^p_{\mu;\alpha_p,i^1_1,...,i^p_p} D^p_{i^p_1 ... i^p_p} u_{\alpha_p}.$$

The proof is based on the next general lemma about Young measures.

Lemma 27. *Let $E \subseteq \mathbb{R}^n$ be a measurable set and $\mathbb{W}$ a finite dimensional metric vector space isometrically and densely contained into a compactification $\mathbb{K}$ of $\mathbb{W}$. If $U^m, V^m : E \subseteq \mathbb{R}^n \longrightarrow \mathbb{W}$ are sequences of measurable maps such that*

$$\delta_{U^m} \overset{*}{\rightharpoonup} \vartheta \ \text{ in } \mathscr{Y}(E, \mathbb{K}), \qquad V^m \longrightarrow V \ \text{ a.e. on } E,$$

as $m \to \infty$, then we have that $\delta_{U^m + V^m} \overset{*}{\rightharpoonup} \vartheta \circ T_V$ in $\mathscr{Y}(E, \mathbb{K})$, as $m \to \infty$.

Proof of Lemma 27. Fix $\phi \in L^1(E)$, $\Phi \in C^0(\mathbb{K})$ and $\varepsilon > 0$. Since Φ is uniformly continuous on the compact space $\mathbb{K}$, there is a bounded increasing modulus of continuity $\omega \in C^0[0, \infty)$ with $\omega(0) = 0$ such that $|\Phi(X) - \Phi(Y)| \leq \omega(|X - Y|)$ for all $X, Y \in \mathbb{K}$ and $\|\omega\|_{C^0(0,\infty)} < \infty$. Also, since $V^m \longrightarrow V$ a.e. on E, we have that $V^m \longrightarrow V$ μ-a.e. on E where μ is the finite measure $\mu(A) := \|\phi\|_{L^1(A \cap E)}$, $A \subseteq \mathbb{R}^n$. It follows that $V^m \longrightarrow V$ in μ-measure as well. Hence, we have

$$\begin{aligned} \left| \int_E \phi \Big[\Phi(U^m + V^m) - \Phi(U^m + V) \Big] \right| \, &\leq \, \int_E |\phi| \, \omega\big(|V^m - V|\big) \\ &\leq \, \|\omega\|_{C^0(0,\infty)} \, \mu\big(\{|V^m - V| > \varepsilon\}\big) \, + \, \omega(\varepsilon)\, \mu(E). \end{aligned}$$

By letting $m \to \infty$ and then $\varepsilon \to 0$, the density of the linear span of products of the form $\phi(x)\Phi(X)$ in $L^1\big(E, C^0(\mathbb{K})\big)$ and the definition of $\vartheta \circ T_V$ allow us to conclude. $\square$

Proof of Proposition 26. If suffices to establish b) and only for $p = 1$. By assumption, we have that $\mathbf{A}^1(x) : \mathcal{L}Dv(x) = g(x)$ and also that for any $\Phi \in C^0_c(\mathbb{R}^{Nn})$,

$$\int_{\overline{\mathbb{R}}^{Nn}} \Phi(X)\Big[\mathbf{A}^1(x) : X - f(x)\Big]\, d[\mathcal{D}u(x)](X) \,=\, 0,$$

both being valid for a.e. on $x \in \Omega$. Here $\mathcal{D}u$ is any diffuse gradient. We fix any point x as above and replace Φ by $\Phi\big(\,\cdot + \mathcal{L}Dv(x)\big)$. Then, we obtain

$$\int_{\overline{\mathbb{R}}^{Nn}} \Phi\big(X + \mathcal{L}Dv(x)\big)\Big[\mathbf{A}^1(x) : \big(X + \mathcal{L}Dv(x)\big) - f(x) - g(x)\Big]\, d[\mathcal{D}u(x)](X) \,=\, 0.$$

By the definition of $\mathcal{D}u \circ T_{\mathcal{L}Dv}$, we obtain

$$\int_{\overline{\mathbb{R}}^{Nn}} \Phi(Y)\Big[\mathbf{A}^1(x) : Y - (f+g)(x)\Big]\, d\big[\mathcal{D}u(x) \circ T_{\mathcal{L}Dv(x)}\big](Y) \,=\, 0.$$

By utilising part a), the conclusion ensues. $\square$

Example 28 (Nonlinearity of diffuse derivatives)**.** *Let $K \subseteq \mathbb{R}$ be a compact nowhere dense set of positive measure (e.g. $K = [0,1] \setminus (\cup_1^\infty (r_j - 3^{-j}, r_j + 3^{-j}))$ where $(r_j)_1^\infty$ is an enumeration of $\mathbb{Q} \cap [0,1]$). Then, for $u := \chi_K$ we have that $|D^{1,h}u(x)| \to \infty$ as $h \to 0$ for $x \in K$ and $u' = 0$ on $\mathbb{R} \setminus K$. Hence, by Lemma 4 along any $h_m \to 0$ we have $\mathcal{D}u(x) = \delta_{\{\infty\}}$ for a.e. $x \in K$. However, for $v := -u$, we have $\mathcal{D}(u+v) = \delta_{\{0\}}$ a.e. on $\mathbb{R}$, while $\mathcal{D}u = \mathcal{D}v = \delta_{\{\infty\}}$ a.e. on K.*

Comparison with distributional solutions. Let us conclude this section with an *informal* discussion of the relation between distributional and $\mathcal{D}$-solutions. Let us first compare distributional to diffuse derivatives. For any $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the distributional gradient Du can be weakly* approximated by difference quotients:

$$\langle \phi, Du \rangle \,=\, \lim_{m\to\infty} \int_{\mathbb{R}^n} \phi\, D^{1,h_m}u \,=\, \lim_{m\to\infty} \int_{\mathbb{R}^n} \phi \left(\int_{\mathbb{R}^n} X\, d\big[\delta_{D^{1,h_m}u}\big](X) \right),$$

$\phi \in C^\infty_c(\mathbb{R}^n)$. If "bar$_*$" denotes the barycentre of the restriction on $\mathbb{R}^n$, we have

$$\mathrm{bar}_*\left(\delta_{D^{1,h_m}u}\right) \overset{*}{\rightharpoonup} Du, \quad \text{ as } m \to \infty, \tag{2.13}$$

in the distributions $\mathscr{D}'\big(\mathbb{R}^n, \mathbb{R}^n\big)$. Along perhaps a further subsequence, we have

$$\delta_{D^{1,h_m}u} \overset{*}{\rightharpoonup} \mathcal{D}u, \;\text{ in } \mathscr{Y}\big(\mathbb{R}^n, \overline{\mathbb{R}}^n\big), \quad \text{ as } m \to \infty. \tag{2.14}$$

By juxtaposing (2.13) with (2.14), our interpretation is that *the barycentre of the (reduced) diffuse derivative is unique and equal to the distributional derivative*:

$$\mathrm{bar}_*(\mathcal{D}u) \,=\, Du.$$

Regarding the notions to solution, obviously $\mathcal{D}$-solutions are a more general theory than distributional solutions (and a fortiori than weak solutions) in the sense that they apply to more general PDEs and under weaker regularity requirements. However, the two theories are not immediately comparable on their common domain of L^1_{loc} solutions of linear systems with smooth coefficients. On the one hand, Proposition 26 and Example 28 point out a property which is not generally true for diffuse derivatives but is always true for distributional derivatives. On the other hand, there exist $\mathcal{D}$-solutions which are not distributional: for instance, $u = \chi_{(0,\infty)}$ is a $\mathcal{D}$-solution of $u' = 0$ on $\mathbb{R}$, while in the distributional sense it solves $u' = \delta_{\{0\}}$

(see also Remark 17). However, $\mathcal{D}$-solutions completely avoid the impossibility to multiply distributions. For example, if $\mathbf{A} \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$,

$$\mathbf{A} \cdot D^{1,h_m} u \;\begin{array}{l} \overset{*}{\not\rightharpoonup}\ \mathbf{A} \cdot Du, \quad \text{in } \mathscr{D}'\big(\mathbb{R}^n, \mathbb{R}^n\big), \quad \text{[not well defined!]} \\ \overset{*}{\rightharpoonup}\ \mathbf{A} \cdot \mathcal{D}u, \quad \text{in } \mathscr{Y}\big(\mathbb{R}^n, \overline{\mathbb{R}}^n\big), \quad \text{[well defined!]} \end{array}$$

Hence, for the equation $\mathbf{A} \cdot Du = 0$, solutions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ make perfect sense in our context by interpreting the equation as

$$\mathbf{A}(x) \cdot \int_{\overline{\mathbb{R}}^N} \Phi(X)\, X\, d[\mathcal{D}u(x)](X) \;=\; 0, \quad \text{a.e. } x \in \mathbb{R}^n,$$

for all $\Phi \in C^0_c(\mathbb{R}^n)$, while in the sense of distributions it is not well defined:

$$\text{“}\;\; \mathbf{A}(x) \cdot Du(x) \;=\; \mathbf{A} \cdot \mathrm{bar}\,(\mathcal{D}u_*(x)) \;=\; \mathbf{A}(x) \cdot \int_{\mathbb{R}^n} X\, d[\mathcal{D}u(x)](X) \;\;\text{”} \;=\; ?$$

We conclude this discussion by underlining the simplicity and handiness of our theory, as opposed to the more cumbersome algebraic theories of multiplication of distributions and the inconsistencies they present (e.g. [Co]).

3. $\mathcal{D}$-solutions of the ∞-Laplacian and tangent systems

In this section we establish our first main result concerning the existence of $\mathcal{D}$-solutions. We treat the Dirichlet problem for the ∞-Laplace system (1.9) which is the fundamental equation of vectorial Calculus of Variations in the space L^∞ and arises from the functional (1.8). A central ingredient in the proof of Theorem 29 below is a result of independent interest, Theorem 33 that follows, which provides a method of constructing nonsmooth $\mathcal{D}$-solutions to nonlinear systems by "differentiating an equation".

Theorem 29 (Existence of ∞-Harmonic maps). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set with $|\Omega| < \infty$ and $n \geq 1$. Then, for any $g \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, the Dirichlet problem*

$$\left\{ \begin{aligned} \Big(Du \otimes Du + |Du|^2 [Du]^\perp \otimes I\Big) : D^2 u \;&=\; 0, \quad \text{on } \Omega, \\ u \;&=\; g, \quad \text{on } \partial\Omega, \end{aligned} \right. \tag{3.1}$$

has a $\mathcal{D}$-solution $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ in $W^{1,\infty}_g(\Omega, \mathbb{R}^n)$. In particular, u satisfies Definition 14 (with respect to the standard frames): for any diffuse hessian, we have

$$\int_{\overline{\mathbb{R}}^{nn^2}_s} \Phi(\boldsymbol{X}) \Big(Du \otimes Du + |Du|^2 [Du]^\perp \otimes I\Big) : \boldsymbol{X}\, d[\mathcal{D}^2 u](\boldsymbol{X}) \;=\; 0,$$

a.e. on Ω, for any $\Phi \in C^0_c\big(\mathbb{R}^{nn^2}_s\big)$, where

$$\delta_{D^{1,h_m} Du} \overset{*}{\rightharpoonup} \mathcal{D}^2 u \quad \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big), \quad \text{as } m \to \infty.$$

Unfortunately, as we proved in [K2], in general it is **impossible** to obtain uniqueness, **not even within the class of smooth solutions**.

Theorem 30 (Nonuniqueness of smooth ∞-Harmonic maps, [K2]). *Let $n = N \geq 2$ and consider the Dirichlet problem (3.1) where as Ω we take the punctured unit ball centred at the origin and as boundary condition we take $g(x) = x$. Then, (3.1) admits infinitely-many solutions in $C^\infty(\Omega, \mathbb{R}^n) \cap C^0(\overline{\Omega}, \mathbb{R}^n)$ which are diffeomorphisms from Ω to itself of the form $u(x) = e^{h(|x|^2)} x$ for certain $h \in C^\infty(0, \infty)$.*

Remark 31. Theorem 30 makes clear that uniqueness in the vectorial case is not an issue of defining a "proper" notion of generalised solution, since even classical solutions in general may not be unique. Instead, *extra conditions* need to be determined that will select a "good" solution. On the other hand, uniqueness is a well known property of the scalar problem (a celebrated theorem of Jensen, see e.g. [C] and also [K8]). Such phenomena are not exclusive to the ∞-Laplacian: for instance, the Dirichlet problem for the minimal surface system may have either non-existence or non-uniqueness in codimension greater than one (see [OL]), while for the minimal surface equation it is well posed.

In addition to Theorem 29, the next corollary will also be established in the course of its proof.

Corollary 32 (Multiplicity & geometric properties of $\mathcal{D}$-solutions)**.** *In the setting of Theorem 29, if $n \geq 2$ then (3.1) actually has an infinite set of solutions. Moreover, for any $M > \big\|(Dg^\top Dg)^{1/2}\big\|_{L^\infty(\Omega)}$ there is a $\mathcal{D}$-solution $u = u(M)$ satisfying*

$$|Du|^2 \,=\, nM^2, \quad \big|\det(Du)\big| \,=\, M^n, \quad \textit{a.e. on } \Omega. \tag{3.2}$$

Hence, the $\mathcal{D}$-solutions we construct have the geometric property of solving the vectorial Eikonal equation and having full rank.

3.1. **The idea of the proof.** Suppose that $u \in C^2(\Omega, \mathbb{R}^n)$ solves (1.9) and recall that $[Du]^\perp = \mathrm{Proj}_{(R(Du))^\perp}$. By contracting derivatives, we rewrite the system as

$$Du\, D\Big(\frac{1}{2}|Du|^2\Big) \,+\, |Du|^2[Du]^\perp \Delta u \,=\, 0. \tag{3.3}$$

By observing that the first summand of (3.3) is valued in the range $R(Du) \subseteq \mathbb{R}^N$ and the second summand is valued in $(R(Du))^\perp$, the ∞-Laplacian (3.3) decouples to the pair of independent systems

$$Du\, D\Big(\frac{1}{2}|Du|^2\Big) \,=\, 0, \quad |Du|^2[Du]^\perp \Delta u \,=\, 0. \tag{3.4}$$

Then, we obtain that *smooth* solutions of the 1st order differential inclusion

$$Du(x) \in \mathcal{K}_c, \quad \text{for } x \in \Omega,$$

where $c > 0$ is a parameter and

$$\mathcal{K}_c \,:=\, \Big\{X \in \mathbb{R}^{nn} \,:\, |X| \,=\, c,\, |\det(X)| \,>\, 0\Big\},$$

actually are ∞-Harmonic mappings: indeed, if $Du(\Omega) \subseteq \mathcal{K}_c$, then $|Du|^2 \equiv c^2$ and $\det(Du) \neq 0$ on Ω. Hence, in view of (3.4) we have that the 1st system is satisfied because $|Du|$ is constant (u is Eikonal) and the 2nd system is satisfied because u is a submersion (the codimension is zero), which forces $[Du]^\perp = 0$ on Ω. Hence, if we somehow could prove existence of a solution to the inclusion with the desired boundary data, it would yield a solution (3.1).

However, the preceding arguments make sense only for classical or strong solutions. The starting point of the proof of Theorem 29 is to use the Dacorogna-Marcellini Baire Category method [DM] in order to construct Lipschitz solutions of the inclusion with the given boundary data. Then, by using the machinery of $\mathcal{D}$-solutions we make the previous ideas rigorous for merely Lipschitz maps, which is the natural regularity class. Note also that our methodology is not variational and does not use the functional (1.8).

The next result is a tool which goes far beyond the scope of the ∞-Laplacian and allows to construct $\mathcal{D}$-solution of systems by solving differential inclusions.

Theorem 33 (Differentiating equations in the $\mathcal{D}$-sense)**.** *Let F be a Carathéodory map as in (1.1) which in addition is C^1 with respect to all its arguments and consider the p-th order system*

$$F\Big(x, u(x), D^{[p]}u(x)\Big) \,=\, 0, \quad x \in \Omega.$$

If $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is a strong a.e. solution to the system in $W^{p,\infty}_{loc}(\Omega, \mathbb{R}^N)$, then u is a $\mathcal{D}$-solution (Definition 15) to the "tangent" system

$$F_x\big(\cdot, u, D^{[p]}u\big) \,+\, F_\eta\big(\cdot, u, D^{[p]}u\big)\,Du \,+\, F_{\underline{\mathbf{X}}}\big(\cdot, u, D^{[p]}u\big) :: D^{[p+1]}u \,=\, 0,$$

on Ω (with respect to the usual frames).

For the notation "::" see Proposition 26. Theorem 33 is actually true for solutions which are merely $W^{p,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$ or just p-times differentiable in measure (Definition 12), but then we have to assume certain growth bounds on the derivatives of F.

We invite the reader to note the *simplicity* with which we pass to limits in the proof below within the framework of $\mathcal{D}$-solutions.

Proof of Theorem 33. It suffices to prove only the case of $p = 1$ and with no explicit u dependence, the general case following analogously. Hence we suppose that $u \in W^{1,\infty}_{\text{loc}}(\Omega, \mathbb{R}^N)$ solves

$$F\big(x, Du(x)\big) = 0, \quad \text{a.e. } x \in \Omega,$$

and we aim to show that

$$F_x\big(x, Du(x)\big) \,+\, F_X\big(x, Du(x)\big) : D^2u(x) \,=\, 0, \quad x \in \Omega,$$

in the $\mathcal{D}$-sense (Definition 14). For a.e. point $x \in \Omega$ such that $F\big(x, Du(x)\big) = 0$ and $h \neq 0$ small enough, Taylor's theorem implies for each i the identity

$$\begin{aligned}
&F_{x_i}\big(x, Du(x)\big) \,+\, F_X\big(x, Du(x)\big) : D^{1,h}_{e^i}Du(x) \\
&\quad = \,-\, D^{1,h}_{e^i}Du(x) : \int_0^1 \Big\{ F_X\Big(x + \lambda h e^i, Du(x) + \lambda\big[Du(x + he^i) - Du(x)\big]\Big) \\
&\qquad\qquad - F_X\big(x, Du(x)\big)\Big\}\, d\lambda \\
&\qquad - \int_0^1 \Big\{ F_{x_i}\Big(x + \lambda h e^i, Du(x) + \lambda\big[Du(x + he^i) - Du(x)\big]\Big) \\
&\qquad\qquad - F_{x_i}\big(x, Du(x)\big)\Big\}\, d\lambda.
\end{aligned} \tag{3.5}$$

We fix any infinitesimal sequence $(h_m)_{m=1}^{\infty} \subseteq \mathbb{R} \setminus \{0\}$ such that $h_m \to 0$ as $m \to \infty$ and observe that by the weak* compactness of Young measures, along perhaps a subsequence $h_{m_k} \to 0$ we have

$$\delta_{D^{1,h_{m_k}}Du} \overset{*}{\rightharpoonup} \mathcal{D}^2u \quad \text{in} \quad \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big), \quad \text{as } k \to \infty.$$

We now invoke Lemma 1 to infer that since $Du(\cdot + he^i) \longrightarrow Du$ locally in measure as $h \to 0$, there is a perhaps further subsequence denoted again by $(h_{m_k})_{k=1}^{\infty}$ such that for a.e. $x \in \Omega$ we have $Du(x + h_{m_k}e^i) \longrightarrow Du(x)$ as $k \to \infty$. Next, we set

$$G^{\infty}_i(x, \mathbf{X}) \,:=\, F_{x_i}\big(x, Du(x)\big) \,+\, \sum_{\beta,j} F_{X_{\beta j}}\big(x, Du(x)\big)\,\mathbf{X}_{\beta j i}$$

and for each $m \in \mathbb{N}$

$$
\begin{aligned}
G^m_i(x,\mathbf{X}) \,:=\, & F_{x_i}\big(x,Du(x)\big) \,+\, \sum_{\beta,j} F_{X_{\beta j}}\big(x,Du(x)\big)\,\mathbf{X}_{\beta j i} \\
& + \sum_{\beta,j}\mathbf{X}_{\beta j i}\int_0^1 \Big\{ F_{X_{\beta j}}\Big(x+\lambda h_m e^i, Du(x)+\lambda\big[Du(x+h_m e^i)-Du(x)\big]\Big) \\
& \qquad\qquad - F_{X_{\beta j}}\big(x,Du(x)\big)\Big\}\,d\lambda \\
& + \int_0^1 \Big\{ F_{x_i}\Big(x+\lambda h_m e^i, Du(x)+\lambda\big[Du(x+h_m e^i)-Du(x)\big]\Big) \\
& \qquad - F_{x_i}\big(x,Du(x)\big)\Big\}\,d\lambda.
\end{aligned}
$$

In view of C^1 regularity of F and that $Du(\cdot+h_{m_k}e^i) \longrightarrow Du$ a.e. on Ω as $k\to\infty$ (together with the Dominated convergence theorem and that $Du \in L^\infty_{\text{loc}}(\Omega,\mathbb{R}^{Nn})$), for a.e. $x\in\Omega$ we obtain

$$
G^{m_k}(x,\cdot) \longrightarrow G^\infty(x,\cdot) \quad \text{in } C^0\Big(\mathbb{R}^{Nn^2}_s,\mathbb{R}^M\Big), \quad \text{as } k\to\infty.
$$

Moreover, in view of the definition of G^m, the identity (3.5) gives

$$
G^m\Big(x, D^{1,h_m}Du(x)\Big) \,=\, 0 \quad \text{a.e. on } \Omega, \;\; m\in\mathbb{N}.
$$

Hence, for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn^2}_s\big)$ we have

$$
\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(\mathbf{X})\, G^{m_k}(x,\mathbf{X})\, d\big[\delta_{D^{1,h_{m_k}}Du(x)}\big](\mathbf{X}) \,=\, 0 \quad \text{a.e. } x\in\Omega,
$$

for $k\in\mathbb{N}$. The convergence Lemma 18 now implies

$$
\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(\mathbf{X})\, G^{\infty}(x,\mathbf{X})\, d\big[\mathcal{D}^2u(x)\big](\mathbf{X}) \,=\, 0, \quad \text{a.e. } x\in\Omega,
$$

for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn^2}_s\big)$ and any diffuse hessian $\mathcal{D}^2u$ arising from any infinitesimal sequence. Hence, u is a $\mathcal{D}$-solution of $G^\infty\big(x,D^2u(x)\big)=0$ on Ω and by the definition of G^∞ the proposition ensues. $\square$

3.2. **Proof of the main result.** Now we prove our first main existence result which is an easy consequence of Theorem 33 and of the existence results of [DM] for differential inclusions via the Baire Category method. The case $n=1$ is completely trivial (see [K1]), so we will henceforth assume $n\geq 2$.

Proof of Theorem 29 (and Corollary 32). Assume we are given $\Omega\subseteq\mathbb{R}^n$ with finite measure and $g\in W^{1,\infty}(\Omega,\mathbb{R}^n)$. We begin with the next:

Claim 34. *If $M > \big\|(Dg^\top Dg)^{1/2}\big\|_{L^\infty(\Omega)}$, there exists $u\in W^{1,\infty}_g(\Omega,\mathbb{R}^n)$ such that*

$$
\begin{aligned}
|Du|^2 \,&=\, nM^2, && \textit{a.e. on } \Omega,\\
|\det(Du)| \,&=\, M^n, && \textit{a.e. on } \Omega.
\end{aligned}
$$

Proof of Claim 34. Given a map $u:\Omega\subseteq\mathbb{R}^n\longrightarrow\mathbb{R}^n$ in $W^{1,\infty}_g(\Omega,\mathbb{R}^n)$, let $\lambda_i(Du)$ denote the ith singular value, that is the ith eigenvalue of $(Du^\top Du)^{1/2}$:

$$
\sigma\big((Du^\top Du)^{1/2}\big) \,=\, \big\{\lambda_1(Du),\dots,\lambda_n(Du)\big\}, \quad \lambda_i\leq\lambda_{i+1}.
$$

Fix an $M > 0$ as in statement and consider the Dirichlet problem:

$$\left\{ \begin{aligned} \lambda_i(Dv) \,&=\, 1, \qquad \text{a.e. in } \Omega, \quad i = 1, ..., n, \\ v \,&=\, g/M, \quad \text{on } \partial\Omega. \end{aligned} \right. \tag{3.6}$$

Then, we have the estimate

$$\begin{aligned} \big\|\lambda_n(Dg)\big\|_{L^\infty(\Omega)} \,&=\, \Big\| \max_{|e|=1} (Dg^\top Dg)^{1/2} : e \otimes e \Big\|_{L^\infty(\Omega)} \\ &\leq\, \big\|(Dg^\top Dg)^{1/2}\big\|_{L^\infty(\Omega)}. \end{aligned} \tag{3.7}$$

In view of the results of [DM], the estimate (3.7) implies that the required compatibility condition is satisfied in regard to the problem (3.6). Hence there is a strong solution v to (3.6) such that $v - (g/M) \in W^{1,\infty}_0(\Omega, \mathbb{R}^n)$ for the given M and the boundary data g. Finally, since $\lambda_i(Dv) = 1$ a.e. on Ω, by setting $u := Mv$ we have

$$|Du|^2 \,=\, M^2|Dv|^2 \,=\, M^2 \sum_i \lambda_i(Dv)^2 \,=\, nM^2, \quad \text{a.e. on } \Omega,$$

$$|\det(Du)| \,=\, M^n |\det(Dv)| \,=\, M^n \prod_i \lambda_i(Dv) \,=\, M^n, \quad \text{a.e. on } \Omega,$$

and in addition, $u - g \in W^{1,\infty}_0(\Omega, \mathbb{R}^n)$. The proof of the claim is complete. ☐

Now we may complete the proof the theorem. For the given boundary condition g, we fix an $M > 0$ as in the claim and consider one of its solutions $u \in W^{1,\infty}_g(\Omega, \mathbb{R}^N)$ which satisfies $|Du|^2 - nM^2 = 0$, a.e. on Ω. We set

$$F(X) \,:=\, |X|^2 - nM^2, \quad X \in \mathbb{R}^{Nn},$$

and apply Theorem 33 to infer that $u \in W^{1,\infty}_g(\Omega, \mathbb{R}^N)$ is a $\mathcal{D}$-solution to the tangent system $F_X(Du) : D^2u = 0$, that is for all i we have

$$\sum_{\beta,j} D_j u_\beta(x)\, D^2_{ij} u_\beta(x) \,=\, 0, \quad x \in \Omega, \text{ in the } \mathcal{D}\text{-sense}.$$

This means that for any diffuse hessian $\mathcal{D}^2u$ arising from any infinitesimal sequence

$$\delta_{D^{1,h_m}Du} \overset{*}{\longrightarrow} \mathcal{D}^2u \quad \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{nn^2}_s\big),$$

along any subsequence as $m \to \infty$, we have

$$\int_{\overline{\mathbb{R}}^{nn^2}_s} \sum_{\beta,j} D_j u_\beta(x)\, \Phi(\mathbf{X})\, \mathbf{X}_{\beta ij}\, d\big[\mathcal{D}^2u(x)\big](\mathbf{X}) \,=\, 0, \quad \text{a.e. } x \in \Omega,$$

for any $\Phi \in C^0_c\big(\mathbb{R}^{nn^2}_s\big)$. We multiply the above $D_i u_\alpha(x)$ and sum to obtain

$$\int_{\overline{\mathbb{R}}^{nn^2}_s} \sum_{\beta,j,i} \Phi(\mathbf{X})\, D_i u_\alpha(x)\, D_j u_\beta(x)\, \mathbf{X}_{\beta ij}\, d\big[\mathcal{D}^2u(x)\big](\mathbf{X}) \,=\, 0, \quad \text{a.e. } x \in \Omega.$$

Finally, by Claim 34 we have $\det(Du) \neq 0$ a.e. on Ω and as a result $Du(x)$ has rank equal to n in $\mathbb{R}^{nn}$, which implies that the orthogonal projection $[Du(x)]^\perp$ on the complement of the range of $Du(x)$ vanishes for a.e. $x \in \Omega$. Thus

$$\int_{\overline{\mathbb{R}}^{n \times n^2}_s} \sum_{\beta,i} \Phi(\mathbf{X})\, \big|Du(x)\big|^2 \big[Du(x)\big]^\perp_{\alpha\beta}\, \mathbf{X}_{\beta ii}\, d\big[\mathcal{D}^2u(x)\big](\mathbf{X}) \,=\, 0, \quad \text{a.e. } x \in \Omega,$$

for any $\Phi \in C^0_c\big(\mathbb{R}^{nn^2}_s\big)$ and any diffuse hessian $\mathcal{D}^2u$. The last two equalities imply that u is a $\mathcal{D}$-solution of the ∞-Laplacian and the theorem follows. ☐

We close this section we a discussion regarding the nonuniqueness problems related to the ∞-Laplace system.

A possible selection principle for Δ_∞. In view of Theorem 30 proved in [K2], among the many smooth solutions that (3.1) has, the boundary condition $g(x) = x$ is itself a solution. Moreover, it is the *only* solution which is a limit of p-Harmonic maps as $p \to \infty$: for each $p > 2$, the unique solution of the p-Laplacian $\Delta_p u = \mathrm{Div}\big(|Du|^{p-2}Du\big) = 0$ with data g on $\partial\Omega$ is g itself. On the other hand, in the scalar case *all* ∞-Harmonic functions arise as uniform limits of p-Harmonic functions (this is a consequence of Jensen's uniqueness theorem for the ∞-Laplacian and of the uniqueness for the p-Laplacian, see e.g. [C, K8] and references therein). Moreover, plenty of other examples seem to exhibit the same behaviour. Hence, we are led to the following conjecture regarding a selection ("entropy") principle of "good" solutions to the ∞-Laplace system:

Conjecture (Uniqueness for the Dirichlet problem for Δ_∞). *For any domain $\Omega \Subset \mathbb{R}^n$ with Lipschitz boundary and any $g \in W^{1,\infty}(\Omega, \mathbb{R}^N)$, the Dirichlet problem (3.1) has a unique $\mathcal{D}$-solution $u^\infty \in W^{1,\infty}_g(\Omega, \mathbb{R}^N)$ in the class of uniform subsequential limits of p-Harmonic mappings u^p as $p \to \infty$.*

Investigation of the validity of this conjecture is left for future work.

4. $\mathcal{D}$-solutions of fully nonlinear degenerate elliptic systems

Fix $n, N \geq 1$, let $\Omega \subseteq \mathbb{R}^n$ be an open set and

$$F \ : \ \Omega \times \mathbb{R}^{Nn^2}_s \longrightarrow \mathbb{R}^N$$

a Carathéodory map. In this section we establish our second main result, namely the existence of a unique $\mathcal{D}$-solution $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ to the Dirichlet problem

$$\left\{ \begin{array}{rl} F(\cdot, D^2 u) \, = \, f, & \text{in } \Omega, \\ u \, = \, 0, & \text{on } \partial\Omega, \end{array} \right. \tag{4.1}$$

when $f \in L^2(\Omega, \mathbb{R}^N)$ and F satisfies a degenerate ellipticity assumption which in general does not guarantee that solutions are even *once* weakly differentiable. This extends previous results of the author in the class of strong solution for (4.1) ([K9, K11]) under a stronger ellipticity notion than that we consider herein.

4.1. The idea of the proof.

The solvability of (4.1) in the class of $\mathcal{D}$-solutions is based on the study of the linearised system with constant coefficients

$$\left\{ \begin{array}{rl} \mathbf{A} : D^2 u \, = \, f, & \text{in } \Omega, \\ u \, = \, 0, & \text{on } \partial\Omega, \end{array} \right. \tag{4.2}$$

when $\mathbf{A}$ is a (perhaps degenerate) convex symmetric quadratic form and on a perturbation device provided by our ellipticity assumption for F. The latter allows to prove existence for (4.1) by proving existence for (4.2) and then using a fixed point argument in the guises of a classical theorem of Campanato ([C3]). In order to solve (4.2) in the $\mathcal{D}$-sense (and not just weakly) we impose a structural condition on $\mathbf{A}$ which allows to construct $\mathcal{D}$-solutions as maps having *twice weakly differentiable projections* along certain rank-one lines of $\mathbb{R}^{Nn}$. These are the "directions of strict ellipticity" of the system $\mathbf{A} : D^2 u = f$. We formalise this idea by introducing a "fibre" extension of the classical Sobolev spaces which consist of maps possessing only certain partial regularity along rank-one lines. Our fibre space counterparts which are adapted to the degenerate nature of the problem support feeble yet sufficient versions of weak compactness, trace operators and Poincaré inequalities for $\mathcal{D}$-solutions. The proof is completed by characterising the "fibre" object we have obtained via fixed point as the unique $\mathcal{D}$-solution of the Dirichlet problem (4.1) inside the fibre space.

4.2. **Fibre spaces, degenerate ellipticity and the main result.** Before stating our existence result we need some preparation. We will use the notation

$$\mathbf{A} \in \mathbb{R}^{Nn\times Nn}_s$$

to denote symmetric linear maps $\mathbf{A} : \mathbb{R}^{Nn} \longrightarrow \mathbb{R}^{Nn}$, i.e. 4th order tensors satisfying $\mathbf{A}_{\alpha i\beta j} = \mathbf{A}_{\beta j\alpha i}$ for all indices $\alpha, \beta = 1, ..., N$ and $i, j = 1, ..., n$. The notation

$$N\Big(\mathbf{A} : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}\Big), \quad N\Big(\mathbf{A} : \mathbb{R}^{Nn^2}_s \to \mathbb{R}^{N}\Big)$$

will be used to denote the nullspaces of $\mathbf{A}$ as linear map with domain and range those indicated in the brackets, i.e. when $\mathbf{A}$ acts respectively as

$$\mathbf{A}Q := \sum_{\alpha,\beta,i,j} \big(\mathbf{A}_{\alpha i\beta j} Q_{\beta j}\big)\, e^\alpha \otimes e^i, \qquad \mathbf{A} : \mathbf{X} := \sum_{\alpha,\beta,i,j} \big(\mathbf{A}_{\alpha i\beta j} \mathbf{X}_{\beta ij}\big)\, e^\alpha .$$

We will also use similar notation for the respective ranges with "R" instead of "N". If $\mathbf{A}$ is rank-one positive, i.e. if the respective quadratic form is rank-one convex

$$\mathbf{A} : \eta \otimes a \otimes \eta \otimes a \,=\, \sum_{\alpha,\beta,i,j} \mathbf{A}_{\alpha i\beta j} \eta_\alpha\, a_i\, \eta_\beta\, a_j \,\geq\, 0, \quad \eta \in \mathbb{R}^N,\ a \in \mathbb{R}^n,$$

we define

$$\begin{aligned}
\Pi \,&:=\, R\Big(\mathbf{A} : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}\Big) && \subseteq\, \mathbb{R}^{Nn},\\
\Sigma \,&:=\, \mathrm{span}\Big[\Big\{\, \eta \,\Big|\, \eta \otimes a \in \Pi \,\Big\}\Big] && \subseteq\, \mathbb{R}^{N},\\
\Xi \,&:=\, \mathrm{span}\Big[\Big\{\, \eta \otimes (a \vee b) \,\Big|\, \eta \otimes a,\ \eta \otimes b \in \Pi \,\Big\}\Big] && \subseteq\, \mathbb{R}^{Nn^2}_s,\\
\nu \,&:=\, \min_{|\eta|=|a|=1,\ \eta\otimes a \in \Pi} \Big\{ \mathbf{A} : \eta \otimes a \otimes \eta \otimes a \Big\} && >\, 0.
\end{aligned} \tag{4.3}$$

We will call ν the *ellipticity constant of $\mathbf{A}$*, bearing in mind that strictly speaking $\mathbf{A}$ may not be elliptic and the respective infimum over $\mathbb{R}^{Nn}$ may vanish. We also recall that we will use the same letters Π, Ξ, Σ to denote the subspaces as well as the orthogonal projections on them. Further, note that we may say "positive $\mathbf{A}$" meaning "non-negative $\mathbf{A}$", but "strictly positive" will always be used to clarify strictness.

The fibre Sobolev spaces. Given $\mathbf{A} \in \mathbb{R}^{Nn\times Nn}_s$ rank-one positive, let Σ, Π, Ξ be given by (4.3) and suppose that Π *is spanned by rank-one directions*. A sufficient condition regarding when this happens is when $\mathbf{A}$ is in a sense "decomposable", something we will require later in Definition 36 that follows. For simplicity, we treat only the L^2 2nd order case needed in this paper. Let us begin by identifying the space $W^{2,2}(\Omega, \mathbb{R}^N)$ with its isometric image $\tilde{W}^{2,2}(\Omega, \mathbb{R}^N)$ into a product of L^2 spaces:

$$\tilde{W}^{2,2}(\Omega, \mathbb{R}^N) \,\underset{\rightarrow}{\subsetneq}\, L^2\Big(\Omega, \mathbb{R}^N \times \mathbb{R}^{Nn} \times \mathbb{R}^{Nn^2}_s\Big)$$

via the map $u \mapsto (u, Du, D^2u)$. We define the **fibre Sobolev space** $\mathscr{W}^{2,2}(\Omega,\Sigma)$ as the Hilbert space

$$\mathscr{W}^{2,2}(\Omega,\Sigma) \,:=\, \overline{\mathrm{Proj}_{L^2\left(\Omega,\Sigma\times\Pi\times\Xi\right)}\, \tilde{W}^{2,2}(\Omega,\mathbb{R}^N)}^{\|\cdot\|_{L^2(\Omega)}} \tag{4.4}$$

with the natural induced norm (written for $W^{2,2}$ maps)

$$\|u\|_{\mathscr{W}^{2,2}(\Omega,\Sigma)} \,:=\, \big\|\Sigma\, u\big\|_{L^2(\Omega)} + \big\|\Pi\, Du\big\|_{L^2(\Omega)} + \big\|\Xi\, D^2u\big\|_{L^2(\Omega)}.$$

By utilising the Mazur theorem, $\mathscr{W}^{2,2}(\Omega,\Sigma)$ can be characterised in the following useful fashion

$$\mathscr{W}^{2,2}(\Omega,\Sigma) \,=\, \left\{ \begin{array}{l} \big(u, G(u), G^2(u)\big) \in\, L^2\big(\Omega, \Sigma\times\Pi\times\Xi\big) \;\Big|\; \exists\; (u^m)_1^\infty \subseteq \\ W^{2,2}(\Omega,\mathbb{R}^N): \quad \text{we have weakly in } L^2 \text{ as } m\to\infty \\ \text{that } \;\big(\Sigma\, u^m, \Pi\, Du^m, \Xi\, D^2u^m\big) \longrightarrow \big(u, G(u), G^2(u)\big) \end{array} \right\}.$$

We will call $G(u) \in L^2(\Omega,\Pi)$ the **fibre gradient** of u and $G^2(u) \in L^2(\Omega,\Xi)$ the **fibre hessian** of u.

It can be easily seen (by using integration by parts and that Σ,Π,Ξ are spanned by directions of the form η, $\eta\otimes a$ and $\eta\otimes(a\vee b)$ respectively) that the measurable maps $G(u), G^2(u)$ *depend only on* $u \in L^2(\Omega,\Sigma)$ *and not on the approximating sequence.*

Further, by using the standard properties of equivalence between strong and weak L^2 directional derivatives, we have that $G(u), G^2(u)$ can be characterised as "fibre" derivatives of u: for any directions $\eta\in\Sigma$, $\eta\otimes a\in\Pi$ and $\eta\otimes(a\vee b)\in\Xi$, we have

$$\begin{aligned} G(u) : (\eta\otimes a) \,&=\, D_a(\eta\cdot u),\\ G^2(u) : \big(\eta\otimes(a\vee b)\big) \,&=\, D^2_{ab}(\eta\cdot u) \,=\, D_b\big(G(u):(\eta\otimes a)\big), \end{aligned}$$

a.e. on Ω, where D_a, D^2_{ab} are the usual directional derivatives.

In general, the fibre spaces are strictly larger than their "non-degenerate" counterparts, since it is very easy to find singular examples which are not even $W^{1,1}_{\mathrm{loc}}$: take for instance $\mathbf{A} = \eta\otimes a\otimes\eta\otimes a$, $|a|=1$. Then, for any $f\in W^{2,2}(\mathbb{R})$, *any* $g\in C^0(\mathbb{R}^n)$ and $\zeta\in C^\infty_c(\mathbb{R}^n)$, the map

$$u(x) \,:=\, \zeta(x)\Big[f(a\cdot x) \,+\, g\big([I-a\otimes a]x\big)\Big]\eta$$

is an element of $\mathscr{W}^{2,2}(\Omega,\Sigma)$ arising from this $\mathbf{A}$, but $D_b(\eta\cdot u)$ may not exist in L^2 for any $b\perp a$.

Similarly to the 2nd order case, we may also define

$$\mathscr{W}^{1,2}_0(\Omega,\Sigma) \,:=\, \overline{\mathrm{Proj}_{L^2(\Omega,\Sigma\times\Pi)}\, \tilde{W}^{1,2}_0(\Omega,\mathbb{R}^N)}^{\|\cdot\|_{L^2(\Omega)}}, \tag{4.5}$$

equipped with the obvious respective norm $\|\cdot\|_{\mathscr{W}^{1,2}(\Omega)}$. Further functional properties of the fibre spaces (traces, Poincaré inequality) needed for the proof of the main result Theorem 37 will be discussed after its statement. The fibre space

$$\big(\mathscr{W}^{2,2}\cap\mathscr{W}^{1,2}_0\big)(\Omega,\Sigma)$$

is the appropriate setup within which we will obtain compactness and uniqueness of $\mathcal{D}$-solutions for the Dirichlet problems (4.1), (4.2), by utilising the necessary hypotheses introduced in the next paragraph.

Degenerate ellipticity and decomposability. Now we introduce our ellipticity hypothesis for (4.1) and a condition for tensors $\mathbf{A} \in \mathbb{R}_s^{Nn \times Nn}$ that will guarantee that their ranges Π are spanned by rank-one directions.

Definition 35 (Degenerate ellipticity)**.** We say that the Carathéodory map $F : \Omega \times \mathbb{R}_s^{Nn^2} \longrightarrow \mathbb{R}^N$ (or the system $F(\cdot, D^2u) = f$) is degenerate elliptic when there exists $\mathbf{A} \in \mathbb{R}_s^{Nn \times Nn}$ rank-one positive, constants $B, C \geq 0$ with $B + C < 1$ and a positive measurable function A satisfying $A, 1/A \in L^\infty(\Omega)$ such that

$$\Big|\mathbf{A} : \mathbf{Z} \,-\, A(x)\Big(F(x, \mathbf{X} + \mathbf{Z}) - F(x, \mathbf{X})\Big)\Big| \,\leq\, B\nu\,|\Xi\,\mathbf{Z}| \,+\, C\,|\mathbf{A} : \mathbf{Z}|,$$

for a.e. $x \in \Omega$ and all $\mathbf{X}, \mathbf{Z} \in \mathbb{R}_s^{Nn^2}$. We moreover require F to be valued in the subspace $\Sigma \subseteq \mathbb{R}^N$, i.e. $F(x, \mathbf{X}) \in \Sigma$, for a.e. $x \in \Omega$ and all $\mathbf{X} \in \mathbb{R}_s^{Nn^2}$.

Definition 35 is an extension to the degenerate elliptic realm of the strict ellipticity assumption introduced in [K11]. In the elliptic case we have $\Sigma = \mathbb{R}^N$, $\Pi = \mathbb{R}^{Nn}$ and $\Xi = \mathbb{R}_s^{Nn^2}$. We refer to [K9] for further material on the elliptic case. The special monotonic case of $\mathbf{A}_{\alpha i \beta j} = \delta_{\alpha\beta}\delta_{ij}$ and $A(x) = \text{const.}$ reduces to the classical notion introduced by Campanato ([C1, C2, C3]). It is easy to exhibit non-trivial examples of Carathéodory maps satisfying Definition 35, see Remark 38IV) that follows. It is quite restrictive, but even the *scalar linear strictly elliptic* case of (3.1) is not well posed (see e.g. [LU]) without extra assumptions.

Below is the structural hypothesis we will impose on tensor $\mathbf{A}$:

Definition 36 (Decomposability)**.** We will say that $\mathbf{A} \in \mathbb{R}_s^{Nn \times Nn}$ is *decomposable* when it can be written as

$$\mathbf{A}_{\alpha i \beta j} \,=\, B^1_{\alpha\beta} A^1_{ij} \,+\, \cdots \,+\, B^N_{\alpha\beta} A^N_{ij}$$

and:

i) The matrices $\{B^1, ..., B^N\} \subseteq \mathbb{R}_s^{N^2}$ are non-negative and their ranges $\Sigma^1, ..., \Sigma^N$ are mutually orthogonal in $\mathbb{R}^N$.

ii) The matrices $\{A^1, ..., A^N\} \subseteq \mathbb{R}_s^{n^2}$ are non-negative and if $\lambda^\gamma_{i_0}$ denotes the smallest positive eigenvalue of A^γ, the eigenspaces $N\big(A^\gamma - \lambda^\gamma_{i_0} I\big)$ have a non-trivial intersection in $\mathbb{R}^n$.

We will discuss certain implications of these hypotheses and some examples after the main result which we give right next.

$\mathcal{D}$-solutions for fully nonlinear degenerate elliptic systems. Now we state the principal result of this section followed by some relevant comments.

Theorem 37 (Existence and Uniqueness)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a strictly convex bounded domain with C^2 boundary and $F : \Omega \times \mathbb{R}_s^{Nn^2} \longrightarrow \mathbb{R}^N$ be a Carathéodory map which satisfies Definition 35 with respect to a decomposable tensor $\boldsymbol{A}$ (Definition 36). Let also the vector spaces Ξ, Π, Σ associated to $\boldsymbol{A}$ be as in* (4.3) *and assume $|F(\cdot, 0)| \in L^2(\Omega)$.*

Then, for any $f \in L^2(\Omega, \Sigma)$, the Dirichlet problem

$$\left\{ \begin{aligned} F(\cdot, D^2u) \,&=\, f, \quad \text{in } \Omega, \\ u \,&=\, 0, \quad \text{on } \partial\Omega, \end{aligned} \right.$$

has a unique $\mathcal{D}$-solution $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ (Definition 15) in the fibre space $(\mathscr{W}^{1,2}_0 \cap \mathscr{W}^{2,2})(\Omega, \Sigma)$ (given by (4.4), (4.5)*) with respect to certain orthonormal* *frames as in* (2.2) *depending only on* F. *In particular,* u *is well defined and vanishes* $\mathcal{H}^{n-1}$*-a.e. on* $\partial\Omega$ *and for any* $\Phi \in C^0_c\big(\mathbb{R}^{Nn^2}_s\big)$, *we have*

$$\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(\boldsymbol{X}) \Big(F(x,\boldsymbol{X}) - f(x)\Big)\, d[\mathcal{D}^2u(x)](\boldsymbol{X}) \,=\, 0, \quad a.e.\ x \in \Omega,$$

where $\mathcal{D}^2u$ *is any diffuse hessian of* u *arising from any infinitesimal subsequences:*

$$\delta_{D^{2,h_{\underline{m}}}u} \overset{*}{\longrightarrow} \mathcal{D}^2u \quad in\ \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big), \qquad as\ \underline{m} \to \infty.$$

Remark 38. I) [**Compatibility**] f has to be valued in the subspace Σ because this is a compatibility condition arising from the degeneracy of the problem. For example, the 2×2 system $\Delta u_1 = f_1$, $0 = f_2$ has no solution whatsoever in any weak sense unless $f_2 \equiv 0$.
II) [**Partial regularity**] The solution we obtain in Theorem 37 possess differentiable projections along certain rank-one lines, but in general this can not be improved further. For, choose any $f \in C^0(\overline{D})$ not weakly differentiable with respect to x_1 for any x_2 over the unit disc of $\mathbb{R}^2$. Then, the problem

$$D^2_{22}u \,=\, f \ \text{ on } D, \qquad u \,=\, 0 \ \text{ on } \partial D,$$

has the unique explicit $\mathcal{D}$-solution (which is not in $W^{1,1}_{\text{loc}}(\Omega)$)

$$u(x_1,x_2) \,=\, -v(x_1,x_2) \,+\, \int_{-\infty}^{x_2}\int_{-\infty}^{t_2} f(x_1,s_2)\, ds_2\, dt_2,$$

where for $(x_1,x_2) \in \overline{D}$,

$$\begin{aligned} v(x_1,x_2) \,&=\, \frac{x_2}{2\sqrt{1-x_1^2}}\left[w\Big(x_1,\sqrt{1-x_1^2}\Big) - w\Big(x_1,-\sqrt{1-x_1^2}\Big)\right] \\ &\qquad\qquad + \frac{1}{2}\left[w\Big(x_1,\sqrt{1-x_1^2}\Big) + w\Big(x_1,-\sqrt{1-x_1^2}\Big)\right], \\ w(x_1,x_2) \,&=\, \int_{-\infty}^{x_2}\int_{-\infty}^{t_2} f(x_1,s_2)\, ds_2\, dt_2. \end{aligned}$$

III) [**Decomposability**] Definition 36 trivialises when either $N=1$ or $n=1$ since any non-negative matrix $A \in \mathbb{R}^{n^2}_s$ or $B \in \mathbb{R}^{N^2}_s$ satisfies it. When $\max\{N,n\} \geq 2$, it is non-trivial, but in view of its constructive nature it is trivial to exhibit $\mathbf{A}$'s satisfying it. Also, any decomposable $\mathbf{A}$ must be non-negative: if $Q \in \mathbb{R}^{Nn}$,

$$\begin{aligned} \mathbf{A} : Q \otimes Q \,&=\, \sum_{\gamma,\alpha,\beta}\sum_{i,j} B^\gamma_{\alpha\beta}\, A^\gamma_{ij}\, Q_{\alpha i}\, Q_{\beta j} \\ &=\, \sum_{\gamma,\alpha,\beta,\kappa}\sum_{i,j,k} \Big((B^\gamma)^{1/2}_{\kappa\alpha}\, Q_{\alpha i}\, (A^\gamma)^{1/2}_{ik}\Big)\Big((B^\gamma)^{1/2}_{\kappa\beta}\, Q_{\beta j}\, (A^\gamma)^{1/2}_{jk}\Big) \,\geq\, 0. \end{aligned}$$

IV) [**Examples of nonlinearities**] Fix $\mathbf{A} \in \mathbb{R}^{Nn\times Nn}_s$ and an $f \in C^{0,1}\big(\mathbb{R}^{Nn^2}_s, \mathbb{R}^N\big)$ with constant $Lip(f)$. Then, for any positive A with $A, 1/A \in L^\infty(\Omega)$, the map

$$F(x,\mathbf{X}) \,:=\, \big(A(x)\big)^{-1}\Big[(1+\gamma)\mathbf{A} : \mathbf{X} \,+\, \Sigma f(\Xi\mathbf{X})\Big]$$

satisfies Definition 35 when $\nu|\gamma| + Lip(f) < \nu$. Linear examples satisfying Definition 35 are given by any $\mathbf{A} : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{Nn\times Nn}_s$ measurable such that

$$\big|\big(\mathbf{A} \,-\, A(x)\mathbf{A}(x)\big) : \mathbf{Z}\big| \,\leq\, B\nu\,|\Xi\,\mathbf{Z}|, \quad \mathbf{Z} \in \mathbb{R}^{Nn^2}_s,$$

for some $0 < B < 1$ and A positive such that $A, 1/A \in L^\infty(\Omega)$.
V) [**Partial monotonicity**] If F satisfies Definition 35 and Ξ (see (4.3)) satisfies

$$\Xi \supseteq N\Big(\mathbf{A} : \mathbb{R}^{Nn^2}_s \to \mathbb{R}^N\Big)^{\perp},$$

then the following "monotonicity" property holds true:

$$(4.6)\qquad \left\{ \begin{array}{l} \text{For a.e. } x \in \Omega,\ F(x,\cdot) \text{ is constant along the subspace } \Xi^{\perp}\text{:} \\ \qquad\qquad F(x,\mathbf{X}) \,=\, F(x, \Xi\,\mathbf{X}), \quad \mathbf{X} \in \mathbb{R}^{Nn^2}_s. \end{array} \right.$$

The above property of Ξ will turn out to be true when $\mathbf{A}$ satisfies Definition 36. To see (4.6), note that since $\Xi^{\perp} \subseteq N\big(\mathbf{A} : \mathbb{R}^{Nn^2}_s \to \mathbb{R}^N\big)$, for any $\mathbf{Z} \in \Xi^{\perp}$ we have $\mathbf{A} : \mathbf{Z} = 0$ and also $\Xi\,\mathbf{Z} = 0$. Hence, Definition 35 gives

$$\Big| - A(x)\Big(F(x, \mathbf{X} + \mathbf{Z}) - F(x, \mathbf{X})\Big)\Big| \,\leq\, 0, \qquad \mathbf{Z} \in \Xi^{\perp},\ \mathbf{X} \in \mathbb{R}^{Nn^2}_s.$$

Obviously, we also have $\mathbf{A} : \mathbf{X} = \mathbf{A} : (\Xi\,\mathbf{X})$. Observe that (4.6) is much weaker than the decoupling condition $F_\alpha(\mathbf{X}) = F_\alpha(\mathbf{X}_\alpha)$ required for vector-valued viscosity solutions.

Next we gather some properties of the fibre spaces essentially proved in [K10] but without the formalism of the fibre spaces.

Remark 39 (Basic properties of the fibre Sobolev space counterparts, cf. [K10])**.**
(I) [**Poincaré inequality**] For any $\Omega \Subset \mathbb{R}^n$, unit vectors a, η and $u \in W^{1,2}_0(\Omega, \mathbb{R}^N)$, we have

$$\|\eta \cdot u\|_{L^2(\Omega)} \,\leq\, \mathrm{diam}(\Omega)\,\big\|D_a(\eta \cdot u)\big\|_{L^2(\Omega)}.$$

(II) [**Norm equivalence**] The seminorm $\|G^2(\,\cdot\,)\|_{L^2(\Omega)}$ on the fibre space $(\mathscr{W}^{1,2}_0 \cap \mathscr{W}^{2,2})(\Omega, \Sigma)$ (see (4.4), (4.5)) is equivalent to its natural norm

$$\|\cdot\|_{\mathscr{W}^{2,2}(\Omega)} \,=\, \|\cdot\|_{L^2(\Omega)} + \|G(\,\cdot\,)\|_{L^2(\Omega)} + \|G^2(\,\cdot\,)\|_{L^2(\Omega)}.$$

(III) [**Trace operator**] If $\Omega \Subset \mathbb{R}^n$ is strictly convex and $a \in \mathbb{R}^n \setminus \{0\}$, then there is a closed set $E \subseteq \partial\Omega$ with $\mathcal{H}^{n-1}(E) = 0$ such that for any $\Gamma \Subset \partial\Omega \setminus E$, we have

$$\|v\|_{L^2(\Gamma, \mathcal{H}^{n-1})} \,\leq\, C\Big(\|v\|_{L^2(\Omega)} \,+\, \big\|D_a v\big\|_{L^2(\Omega)}\Big),$$

for some universal $C = C(\Gamma) > 0$ and all $v \in C^1(\overline{\Omega})$. Hence, there is a well-defined trace operator $T : \mathscr{W}^{1,2}(\Omega, \mathbb{R}^N) \to L^2_{\mathrm{loc}}(\partial\Omega \setminus E, \mathcal{H}^{n-1}; \mathbb{R}^N)$.

Before giving the proof of the main result, we need an important estimate. This is done in the next subsection.

4.3. **A priori degenerate hessian estimates.** Herein we establish an a priori estimate for strong solutions in $(W^{2,2} \cap W^{1,2}_0)(\Omega, \mathbb{R}^N)$ of a regularisation of

$$\mathbf{A} : D^2 u \,=\, f, \qquad \text{on } \Omega,$$

when $\mathbf{A}$ is decomposable. This is a generalisation of the elliptic estimate of [K11] (which extended the classical Miranda-Talenti identity) to the *degenerate* case.

Theorem 40 (Degenerate hessian estimate)**.** *Let $n, N \geq 1$ with $\Omega \subseteq \mathbb{R}^n$ a convex bounded C^2 domain. Suppose further that $\boldsymbol{A} \in \mathbb{R}^{Nn \times Nn}_s$ satisfies Definition 36. If Ξ, ν are as in (4.3), then for any $u \in (W^{2,2} \cap W^{1,2}_0)(\Omega, \mathbb{R}^N)$ and any $\varepsilon \geq 0$ we have the estimate*

$$\big\|\Xi\, D^2 u\big\|_{L^2(\Omega)} \,\leq\, \frac{1}{\nu}\,\big\|\boldsymbol{A}^{(\varepsilon)} : D^2 u\big\|_{L^2(\Omega)}$$

and also the property

$$\Xi \supseteq N\Big(\boldsymbol{A} : \mathbb{R}^{Nn^2}_s \to \mathbb{R}^N\Big)^{\perp}. \tag{4.7}$$

The tensor $\boldsymbol{A}^{(\varepsilon)}$ is the following (strictly) rank-one positive regularisation of $\boldsymbol{A}$:

$$\boldsymbol{A}^{(\varepsilon)}_{\alpha i \beta j} := \sum_{\gamma=0}^{N} B^{(\varepsilon)\gamma}_{\alpha\beta}\, A^{(\varepsilon)\gamma}_{ij},$$

$$B^{(\varepsilon)\gamma} := \begin{cases} B^{\gamma}, & \gamma = 1, ..., N, \\ \varepsilon I - \varepsilon\big(B^1 + \cdots + B^N\big), & \gamma = 0, \end{cases}$$

$$A^{(\varepsilon)\gamma} := \begin{cases} A^{\gamma} + \varepsilon I, & \gamma = 1, ..., N, \\ \varepsilon I, & \gamma = 0, \end{cases}$$

and B^{γ}, A^{γ} are the matrices appearing in Definition 36.

Note that in the vectorial case $N \geq 2$ of Theorem 40, the "correct" approximation in *not* the vanishing viscosity one, although it reduces to such when $N = 1$.

Proof of Theorem 40. The first step is to prove a weak version of the scalar case of the theorem.

Claim 41. *Let $\Omega \Subset \mathbb{R}^n$ be C^2 and convex and let also $A \geq 0$ in $\mathbb{R}^{n^2}_s$. Then, there exists a subspace $H \subseteq \mathbb{R}^{n^2}_s$ such that*

$$H \supseteq N\left(A : \mathbb{R}^{n^2}_s \to \mathbb{R}\right)^{\perp}$$

and for any $u \in (W^{2,2} \cap W^{1,2}_0)(\Omega)$ and any $\varepsilon \geq 0$ we have the estimate

$$\big\|HD^2 u\big\|_{L^2(\Omega)} \leq \frac{1}{\nu(A)} \big\|A : D^2 u + \varepsilon \Delta u\big\|_{L^2(\Omega)}$$

where

$$\nu(A) := \min_{|a|=1,\, a \in T} \{A : a \otimes a\}, \quad T := R\big(A : \mathbb{R}^n \to \mathbb{R}^n\big).$$

Proof of Claim 41. By the Spectral theorem, we can find a diagonal matrix Λ with entries $0 \leq \lambda_1 \leq ... \leq \lambda_n$ and $O \in O(n)$ such that $A = O\Lambda^{1/2}\,(O\Lambda^{1/2})^{\top}$ and

$$\Lambda = \left[\begin{array}{c|ccc} 0 & & 0 & \\ \hline & \lambda_{i_0} & & 0 \\ 0 & & \ddots & \\ & 0 & & \lambda_n \end{array}\right].$$

Evidently, $\big\{\lambda_1, ..., \lambda_n\big\} = \big\{0, ..., 0, \lambda_{i_0}, ..., \lambda_n\big\}$ are the eigenvalues of A and λ_{i_0} is the smallest positive eigenvalue. We also fix $\varepsilon \geq 0$ and set

$$\Theta := \big(\Lambda + \varepsilon I\big)^{1/2}, \quad \Gamma := O\Theta. \tag{4.8}$$

Then, since A equals $O\Lambda O^{\top}$ and Θ is symmetric, we have

$$A + \varepsilon I = O\Lambda O^{\top} + O(\varepsilon I)O^{\top} = O\Theta\,(O\Theta)^{\top} = \Gamma\Gamma^{\top} \tag{4.9}$$

and also

$$\nu(A) = \lambda_{i_0}$$

($\nu(A)$ is defined in the statement). We now define the subspaces of $\mathbb{R}^{n^2}_s$

$$
\begin{aligned}
H^0 &:= \left\{ X \in \mathbb{R}^{n^2}_s \ : \ X \ = \ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & (X_{ij})^{j=i_0,\dots,n}_{i=i_0,\dots,n} \end{array}\right] \right\}, \\
H &:= \left\{ X \in \mathbb{R}^{n^2}_s \ : \ O^\top X O \in H^0 \right\}.
\end{aligned} \tag{4.10}
$$

We begin by establishing the following algebraic inequality:

$$
\left|\Theta X \Theta\right| \ \geq \ \nu(A) \left|H^0 X\right|, \quad X \in \mathbb{R}^{n^2}_s. \tag{4.11}
$$

Indeed, since $\Theta_{ij} = 0$ when $i \neq j$ and $\Theta_{ii} = \sqrt{\lambda_i + \varepsilon}$, in view of (4.10) we have

$$
\begin{aligned}
\left|\Theta X \Theta\right|^2 &= \sum_{i,j,k,l,p,q=1}^{n} \big(\Theta_{ik} X_{kl} \Theta_{lj}\big) \big(\Theta_{ip} X_{pq} \Theta_{qj}\big) \ = \\
&= \sum_{i,j=1}^{n} \big(\Theta_{ii} X_{ij} \Theta_{jj}\big)^2 \ \geq \ \sum_{i,j=i_0}^{n} \big(\lambda_i + \varepsilon\big) \left(X_{ij}\right)^2 \big(\lambda_j + \varepsilon\big) \ \geq \\
&\geq \ (\lambda_{i_0})^2 \sum_{i,j=i_0}^{n} (X_{ij})^2 \ = \ \nu(A)^2 \left|H^0 X\right|^2.
\end{aligned}
$$

Hence, (4.11) has been established. In order to conclude, the goal is to reduce to the classical Miranda-Talenti inequality (see [M, T, K11]) which says that for $U \Subset \mathbb{R}^n$ convex C^2 domain and any $v \in (W^{2,2} \cap W^{1,2}_0)(U)$, we have

$$
\left\|D^2 v\right\|_{L^2(U)} \ \leq \ \left\|\Delta v\right\|_{L^2(U)}. \tag{4.12}
$$

It suffices to assume that $\varepsilon > 0$ since the case $\varepsilon = 0$ follows by letting $\varepsilon \to 0$. Given a fixed $u \in C^2(\overline{\Omega}) \cap C^1_0(\Omega)$, we set

$$
U \ := \ \Gamma^{-1}\Omega, \quad v(x) \ := \ u(\Gamma x), \quad x \in U.
$$

Then, we have $D^2_{ij} v(x) = \sum_{p,q=1}^{n} D^2_{pq} u(\Gamma x) \, \Gamma_{pi} \, \Gamma_{qj}$ and hence, by (4.8) and (4.9) we obtain

$$
\begin{aligned}
D^2 v(x) \ &= \ \Gamma^\top D^2 u(\Gamma x) \, \Gamma \ = \ \Theta\Big(O^\top D^2 u(\Gamma x) \, O\Big)\Theta, \\
\Delta v(x) \ &= \ D^2 u(\Gamma x) : \Gamma\Gamma^\top \ = \ D^2 u(\Gamma x) : (A \, + \, \varepsilon I).
\end{aligned} \tag{4.13}
$$

We now claim that since Ω is a C^2 bounded convex domain, U is a C^2 bounded convex domain as well. Indeed, by (4.8) we have $\Gamma^{-1} = \Theta^{-1} O^\top$ and since $O^\top$ is an isometry, it suffices to show that $\Theta^{-1} V$ is convex, where $V := O^\top \Omega$. To see this, note that we can find a convex $F \in C^2(\mathbb{R}^n)$ such that $\{F < 0\} = V$. We set

$$
G(x) \ := \ F(\Theta x), \quad G \in C^2(\mathbb{R}^n).
$$

Then, we have

$$
D^2_{ij} G(x) \ = \ \sum_{p,q=1}^{n} D^2_{pq} F(\Theta x) \, \Theta_{pi} \, \Theta_{qj}
$$

and hence the convexity of F implies $D^2 G(x) \geq 0$. It follows that the sublevel set $\{G < 0\}$ is convex and as a consequence U is convex too:

$$
U \ = \ \Theta^{-1} V \ = \ \left\{\Theta^{-1} x \in \mathbb{R}^n \ : \ F(x) < 0\right\} \ = \ \left\{y \in \mathbb{R}^n \ : \ G(y) < 0\right\}.
$$

We may now apply the estimate (4.12) to v over $U \subseteq \mathbb{R}^n$ and by (4.12), (4.13) to obtain

$$\int_U \big|D^2u(\Gamma x) : (A + \varepsilon I)\big|^2 dx \geq \int_U \Big|\Theta\Big(O^\top D^2u(\Gamma x)\, O\Big)\Theta\Big|^2 dx$$
$$\overset{(4.11)}{\geq} \nu(A)^2 \int_U \Big|H^0\Big(O^\top D^2u(\Gamma x)\, O\Big)\Big|^2 dx.$$

By the change of variables $y := \Gamma x$ and by using that O is orthogonal, we obtain

$$\big\|D^2u : (A + \varepsilon I)\big\|_{L^2(\Omega)} \geq \nu(A) \Big\|O\, \Big(H^0\, \big(O^\top D^2u\, O\big)\Big)\, O^\top\Big\|_{L^2(\Omega)}. \tag{4.14}$$

Now we claim that the orthogonal projection on the subspace $H \subseteq \mathbb{R}^{n^2}_s$ is given by

$$H\, X = O\, \Big(H^0\, \big(O^\top X\, O\big)\Big)\, O^\top. \tag{4.15}$$

Once (4.15) has been established, the desired estimate follows from (4.14), (4.15) and a standard density argument in the Sobolev norm. Indeed, if K denotes the linear operator defined by the right hand side of (4.15), for any $X \in \mathbb{R}^{n^2}_s$ we have

$$\begin{aligned} K\big(K\, X\big) &= O\, \Big(H^0\, \Big(O^\top O\, \Big(H^0\, \big(O^\top X\, O\big)\Big)\, O^\top\, O\Big)\Big)\, O^\top = \\ &= O\, \Big(H^0 H^0 \big(O^\top X\, O\big)\Big)\, O^\top = O\, \Big(H^0 \big(O^\top X\, O\big)\Big)\, O^\top = K\, X. \end{aligned}$$

Hence, $K^2 = K$. Moreover, K is symmetric as a map $\mathbb{R}^{n^2}_s \longrightarrow \mathbb{R}^{n^2}_s$: by using that H^0 is symmetric, we have

$$\begin{aligned} (K\, X) : Y &= \Big(O\, \Big(H^0\, \big(O^\top X\, O\big)\Big)\, O^\top\Big) : Y = H^0\, \big(O^\top X\, O\big) : \big(O^\top Y\, O\big) = \\ &= \big(O^\top X\, O\big) : H^0\, \big(O^\top Y\, O\big) = X : \Big(O\, \Big(H^0\, \big(O^\top Y\, O\big)\Big)\, O^\top\Big) = \\ &= X : (K\, Y), \end{aligned}$$

for any $X, Y \in \mathbb{R}^{n^2}_s$. Hence, (4.15) follows. It remains to exhibit the claimed property of H. To this end, fix $X \perp H$. Then, we have that the projection of X on H vanishes and as a result of (4.15) we obtain $H^0(O^\top X\, O) = 0$. By recalling that $A = O\, \Lambda\, O^\top$, we have $A : X = \Lambda : (O^\top X\, O)$ and since Λ belongs to H^0, we conclude that $A : X = 0$. Hence, we have just proved that $H^\perp \subseteq N(A : \mathbb{R}^{n^2}_s \to \mathbb{R})$, which is the desired property of the subspace H. The claim has been established. $\square$

The next step is to characterise the subspace $H \subseteq \mathbb{R}^{n^2}_s$ of Claim 41 in terms of the range of A.

Claim 42. *In the setting of Claim 41, we have the identity*

$$H = span\Big[\Big\{a \vee b \;\Big|\; a,\, b \in R\big(A : \mathbb{R}^n \to \mathbb{R}^n\big)\Big\}\Big] = T \vee T.$$

Proof of Claim 42. We begin by observing that in view of (4.10), we have $H = O\, H^0 O^\top$ where $O \in O(n)$. Since

$$H^0 = \mathrm{span}\Big[\Big\{e^i \vee e^j \,\Big|\, i, j = i_0, ..., n\Big\}\Big]$$

we obtain that H has a basis consisting of matrices of the form $Oe^i \vee Oe^j$, $i,j = i_0,...,n$. We recall now that $A = O\Lambda O^\top$ where Λ is a diagonal matrix with entries the eigenvalues $\{0,...,0,\lambda_{i_0},...,\lambda_n\}$ of A. We define the vectors

$$a^i := Oe^i = \big(O_{1i},...,O_{ni}\big)^\top, \quad i = 1,...,n.$$

Then, $\{a^1,...,a^n\}$ is an orthonormal frame of $\mathbb{R}^n$ corresponding to the columns of the matrix A and is a set of eigenvectors of A. Since $\{a^{i_0},...,a^n\}$ correspond to the nonzero eigenvalues $\{\lambda_{i_0},...,\lambda_n\}$, the nullspace $N\big(A:\mathbb{R}^n\to\mathbb{R}^n\big)$ is spanned by $\{a^1,...,a^{i_0-1}\}$ and hence

$$R\big(A:\mathbb{R}^n\to\mathbb{R}^n\big) = \mathrm{span}[\{a^{i_0},...,a^n\}].$$

Since H has a basis of the form $\{a^i \vee a^j : i,j = i_0,...,n\}$, the claim follows. $\square$

Now we begin working towards the vector case $N \geq 2$. Let us first verify that $\mathbf{A}^{(\varepsilon)}$ is strictly rank-one positive. Indeed, if $0<\varepsilon<1$, $\eta\in\mathbb{R}^N$, $a\in\mathbb{R}^n$, we have

$$\begin{aligned}
\mathbf{A}^{(\varepsilon)} : \eta\otimes a\otimes\eta\otimes a \,&=\, \sum_{\gamma=0}^{N}\Big(B^{(\varepsilon)\gamma} : \eta\otimes\eta\Big)\Big(A^{(\varepsilon)\gamma} : a\otimes a\Big)\\
&\geq \min_{\delta=0,\dots,N}\Big(A^{(\varepsilon)\delta} : a\otimes a\Big)\left[\sum_{\gamma=0}^{N} B^{(\varepsilon)\gamma} : \eta\otimes\eta\right]\\
&\geq \varepsilon\,|a|^2\left[\sum_{\gamma=1}^{N} B^\gamma \,+\, \varepsilon\left(I-\sum_{\delta=1}^{N}B^\delta\right)\right] : \eta\otimes\eta\\
&\geq \varepsilon^2\,|\eta|^2|a|^2,
\end{aligned}$$

as claimed. The next step is to characterise the range Π of decomposable tensors $\mathbf{A}\in\mathbb{R}^{Nn\times Nn}_s$ in terms of the matrices B^γ, A^γ composing $\mathbf{A}$.

Claim 43. *Let $\Pi\subseteq\mathbb{R}^{Nn}$ be the range of $\mathbf{A}:\mathbb{R}^{Nn}\longrightarrow\mathbb{R}^{Nn}$ (see (4.3)). Then,*

$$\Pi \,=\, \bigoplus_{\gamma=1}^{N}\big(\Sigma^\gamma\otimes T^\gamma\big),$$

$$\begin{aligned}
\Sigma^\gamma \,&=\, R\left(B^\gamma:\mathbb{R}^N\to\mathbb{R}^N\right) \,\subseteq\, \mathbb{R}^N,\\
T^\gamma \,&=\, R\left(A^\gamma:\mathbb{R}^n\to\mathbb{R}^n\right) \,\subseteq\, \mathbb{R}^n.
\end{aligned} \tag{4.16}$$

Proof of Claim 43. We first observe that by Definition 36, $\Sigma^\gamma\perp\Sigma^\delta$ if $\gamma\neq\delta$ and this implies that $\Sigma^\gamma\otimes T^\gamma\perp\Sigma^\delta\otimes T^\delta$ if $\gamma\neq\delta$. Let now $Q\in\mathbb{R}^{Nn}$. Then, $\mathbf{A}:Q$ is given in index form by

$$\sum_{\beta,j}\mathbf{A}_{\alpha i\beta j}\,Q_{\beta j} \,=\, \sum_{\gamma,\beta,j} B^\gamma_{\alpha\beta}\,Q_{\beta j}\,A^\gamma_{ji}$$

which by (4.16) shows that $\Pi\subseteq\oplus_\gamma\big(\Sigma^\gamma\otimes T^\gamma\big)$. Conversely, let $R\in\oplus_\gamma\big(\Sigma^\gamma\otimes T^\gamma\big)$. Then, R can be written as

$$R \,=\, \sum_{\gamma,\kappa}\left(B^\gamma\eta^{\kappa\gamma}\right)\otimes\left(A^\gamma a^{\kappa\gamma}\right)$$

for some $\eta^{\kappa\gamma}\in\Sigma^\gamma$, $a^{\kappa\gamma}\in T^\gamma$. We note that

$$\left(B^\delta\otimes A^\delta\right)\left(\sum_\kappa \eta^{\kappa\gamma}\otimes a^{\kappa\gamma}\right) \,=\, 0, \quad \text{ if } \gamma\neq\delta,$$

because $\eta^{\kappa\gamma} \perp \Sigma^\delta$ if $\gamma \neq \delta$. We now define $Q := \sum_{\gamma,\kappa} \eta^{\kappa\gamma} \otimes a^{\kappa\gamma}$ and we claim that $\mathbf{A} : Q = R$. Indeed, we have

$$\sum_{\beta,j} \mathbf{A}_{\alpha i\beta j}\, Q_{\beta j} = \sum_{\delta,\beta,j} \big(B^\delta_{\alpha\beta}\, A^\delta_{ij}\big)\Bigg(\sum_{\kappa,\gamma} \eta^{\kappa\gamma}_\beta\, a^{\kappa\gamma}_j\Bigg) = \sum_{\delta,\beta,j} \big(B^\delta_{\alpha\beta}\, A^\delta_{ij}\big)\Bigg(\sum_{\kappa} \eta^{\kappa\delta}_\beta\, a^{\kappa\delta}_j\Bigg) =$$
$$= \sum_{\kappa,\delta,\beta,j} \Big(B^\delta_{\alpha\beta}\, \eta^{\kappa\delta}_\beta\Big)\Big(A^\delta_{ij}\, a^{\kappa\delta}_j\Big) \;=\; R_{\alpha i}.$$

This establishes that $\Pi \supseteq \oplus_\gamma\big(\Sigma^\gamma \otimes T^\gamma\big)$, therefore completing the proof. ☐

The next step is to find an upper bound of the ellipticity constant ν of $\mathbf{A}$ in terms of the matrices B^γ, A^γ.

Claim 44. *Let ν be given (4.3) and Σ^γ, T^γ by (4.16). Then, we have the estimate*

$$\nu \;\leq\; \Bigg(\min_\gamma \min_{\eta\in\Sigma^\gamma,\,|\eta|=1} \big\{B^\gamma : \eta\otimes\eta\big\}\Bigg)\Bigg(\min_\delta \min_{a\in T^\delta,\,|a|=1} \big\{A^\delta : a\otimes a\big\}\Bigg).$$

Proof of Claim 44. We begin by noting that on top of the decomposability we may further assume that all the matrices A^γ have the same smallest positive eigenvalue $\lambda^\gamma_{i_0}$ equal to 1 for all $\gamma = 1,...,N$ which is realised at a common eigenvector $\bar{a} \in \mathbb{R}^n$. Indeed, existence of $\bar{a}$ follows from Definition 36 since the eigenspaces $N\big(A^\gamma - \lambda^\gamma_{i_0} I\big)$ intersect for all γ at least along a common line in $\mathbb{R}^n$. Further, by replacing $\{B^1,...,B^N\}$, $\{A^1,...,A^N\}$ by the rescaled families $\{\tilde{B}^1,...,\tilde{B}^N\}$, $\{\tilde{A}^1,...,\tilde{A}^N\}$ where $\tilde{B}^\gamma := \lambda^\gamma_{i_0} B^\gamma$, $\tilde{A}^\gamma := (1/\lambda^\gamma_{i_0})A^\gamma$, we have that the new families have the same properties as the original and in addition all the new A^γ matrices have the same minimum positive eigenvalue normalised to 1. Hence, we may assume that $\mathbf{A}$ is decomposable and moreover

$$(4.17)\quad \exists\ \bar{a} \,\in\, \partial\mathbb{B}^n_1 \bigcap_{\gamma=1}^{N} T^\gamma \quad : \qquad \lambda^\gamma_{i_0} = \min_{a\in T^\gamma,\,|a|=1} \big\{A^\gamma : a\otimes a\big\} = A^\gamma : \bar{a}\otimes\bar{a} = 1,$$

for all $\gamma = 1,...,N$. By using (4.17), Claim 43 and that $\cup_\gamma\big(\Sigma^\gamma\otimes T^\gamma\big) \subseteq \oplus_\gamma\big(\Sigma^\gamma\otimes T^\gamma\big)$, we calculate

$$\begin{aligned}
\nu \;&=\; \min_{|\eta|=|a|=1,\,\eta\otimes a\in\Pi} \sum_\delta \Big(B^\delta : \eta\otimes\eta\Big)\Big(A^\delta : a\otimes a\Big)\\
&\leq\; \min_{|\eta|=|a|=1,\,\eta\otimes a\in\cup_\gamma(\Sigma^\gamma\otimes T^\gamma)} \sum_\delta \Big(B^\delta : \eta\otimes\eta\Big)\Big(A^\delta : a\otimes a\Big)\\
&\leq\; \min_\gamma \Bigg(\min_{|\eta|=|a|=1,\,\eta\otimes a\in\Sigma^\gamma\otimes T^\gamma} \sum_\delta \Big(B^\delta : \eta\otimes\eta\Big)\Big(A^\delta : a\otimes a\Big)\Bigg)\\
&\leq\; \min_\gamma \Bigg(\min_{|\eta|=1,\,\eta\in\Sigma^\gamma} \sum_\delta \Big(B^\delta : \eta\otimes\eta\Big)\Big(A^\delta : \bar{a}\otimes\bar{a}\Big)\Bigg)\\
&=\; \min_\gamma \min_{|\eta|=1,\,\eta\in\Sigma^\gamma} \sum_\delta \Big(B^\delta : \eta\otimes\eta\Big).
\end{aligned}$$

Since $B^\delta : \eta\otimes\eta = 0$ if $\eta\in\Sigma^\gamma$ for $\gamma\neq\delta$, by using (4.17) again we conclude that

$$\begin{aligned}
\nu \;&\leq\; \min_\gamma \min_{\eta\in\Sigma^\gamma,\,|\eta|=1} \big\{B^\gamma : \eta\otimes\eta\big\}\\
&=\; \Bigg(\min_\gamma \min_{\eta\in\Sigma^\gamma,\,|\eta|=1} \big\{B^\gamma : \eta\otimes\eta\big\}\Bigg)\Bigg(\min_\delta \min_{a\in T^\delta,\,|a|=1} \big\{A^\delta : a\otimes a\big\}\Bigg),
\end{aligned}$$

as desired. $\square$

Now we complete the proof of the theorem by using the previous claims. We define

$$\Xi \,:=\, \bigoplus_{\gamma}\Big(\Sigma^\gamma \otimes T^\gamma \vee T^\gamma\Big) \quad \subseteq\, \mathbb{R}^{Nn^2}_s, \tag{4.18}$$

and for brevity we set

$$\Xi^\gamma \,:=\, T^\gamma \vee T^\gamma \,\subseteq\, \mathbb{R}^{n^2}_s,$$

where Σ^γ, T^γ are as in (4.16). Fix a map $u \in C^2(\overline{\Omega},\mathbb{R}^N)\cap C^1_0(\Omega,\mathbb{R}^N)$. Then, for any indices $\gamma,\alpha = 1,...,N$, by the Claims 41, 42 applied to the scalar function $(\Sigma^\gamma u)_\alpha \in C^2(\overline{\Omega})\cap C^1_0(\Omega)$, we have the estimate

$$\int_\Omega \Big|\Xi^\gamma D^2(\Sigma^\gamma u)_\alpha\Big|^2 \,\leq\, \int_\Omega \Big|A^{(\varepsilon)\gamma} : D^2(\Sigma^\gamma u)_\alpha\Big|^2,$$

where we have used that $A^{(\varepsilon)\gamma} = A^\gamma + \varepsilon I$ (by the definition of $\mathbf{A}^{(\varepsilon)}$) and we have employed the normalisation of (4.17) which forces $\lambda^\gamma_{i_0} = \nu(A^\gamma) = 1$. By summing in α,γ, the above estimate and (4.18) give

$$\int_\Omega \big|\Xi\, D^2u\big|^2 \,=\, \int_\Omega \sum_\gamma \Big|\Sigma^\gamma\otimes \Xi^\gamma : D^2u\Big|^2 \,\leq\, \int_\Omega \sum_\gamma \Big|\Sigma^\gamma\big(D^2u : A^{(\varepsilon)\gamma}\big)\Big|^2. \tag{4.19}$$

We also set

$$C^{(\varepsilon)\gamma} \,:=\, \Sigma^\gamma\big(D^2u : A^{(\varepsilon)\gamma}\big), \quad \gamma = 1,...,N.$$

Then, (4.19) says

$$\int_\Omega \big|\Xi\, D^2u\big|^2 \,\leq\, \int_\Omega \sum_{\gamma=1}^N \big|C^{(\varepsilon)\gamma}\big|^2. \tag{4.20}$$

By the definition of $\mathbf{A}^{(\varepsilon)}$, we have that $B^{(\varepsilon)\gamma} \perp B^{(\varepsilon)\delta}$ for $\gamma \neq \delta$ in $\{0,1,...,N\}$. By using this fact, we calculate

$$\begin{aligned}
\big|\mathbf{A}^{(\varepsilon)} : D^2u\big|^2 &= \left(\sum_{\gamma=0}^N B^{(\varepsilon)\gamma}\left(D^2u : A^{(\varepsilon)\gamma}\right)\right)\cdot\left(\sum_{\delta=0}^N B^{(\varepsilon)\delta}\left(D^2u : A^{(\varepsilon)\delta}\right)\right)\\
&= \sum_{\gamma=0}^N \Big(B^{(\varepsilon)\gamma}\left(D^2u : A^{(\varepsilon)\gamma}\right)\Big)\cdot\Big(B^{(\varepsilon)\gamma}\big(D^2u : A^{(\varepsilon)\gamma}\big)\Big)\\
&= \Big|B^{(\varepsilon)0}\left(D^2u : A^{(\varepsilon)0}\right)\Big|^2 + \sum_{\gamma=1}^N \Big|B^{(\varepsilon)\gamma}\left(D^2u : A^{(\varepsilon)\gamma}\right)\Big|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\big|\mathbf{A}^{(\varepsilon)} : D^2u\big|^2 &\geq \sum_{\gamma=1}^N \Big|B^\gamma\left(D^2u : A^{(\varepsilon)\gamma}\right)\Big|^2 \;=\; \sum_{\gamma=1}^N \big|B^\gamma\, C^{(\varepsilon)\gamma}\big|^2 \;=\\
&\geq \sum_{\gamma=1}^N \max_{|\eta|=1}\Big(B^\gamma : \big(C^{(\varepsilon)\gamma}\otimes\eta\big)\Big)^2 \geq\\
&\geq \sum_{\gamma=1}^N \Big(B^\gamma : \Big(\mathrm{sgn}(C^{(\varepsilon)\gamma})\otimes \mathrm{sgn}(C^{(\varepsilon)\gamma})\Big)\Big)^2\big|C^{(\varepsilon)\gamma}\big|^2.
\end{aligned} \tag{4.21}$$

Hence, (4.21) gives

$$\big|\mathbf{A}^{(\varepsilon)} : D^2 u\big|^2 \geq \left(\min_{\delta=1,\ldots,N}\Big\{B^\delta : \Big(\mathrm{sgn}\big(C^{(\varepsilon)\delta}\big) \otimes \mathrm{sgn}\big(C^{(\varepsilon)\delta}\big)\Big)\Big\}\right)^2 \sum_{\gamma=1}^{N}\big|C^{(\varepsilon)\gamma}\big|^2$$

and as a result we obtain

$$\big|\mathbf{A}^{(\varepsilon)} : D^2 u\big|^2 \geq \left(\min_{\delta=1,\ldots,N}\ \min_{|\eta|=1,\,\eta\in\Sigma^\delta}\big\{B^\delta : \eta\otimes\eta\big\}\right)^2 \sum_{\gamma=1}^{N}\big|C^{(\varepsilon)\gamma}\big|^2. \tag{4.22}$$

By using the Claim 44 (and also the normalisation condition (4.17)), (4.22) gives

$$\int_\Omega \big|\mathbf{A}^{(\varepsilon)} : D^2 u\big|^2 \geq \nu^2 \int_\Omega \sum_{\delta=1}^{N}\big|C^{(\varepsilon)\delta}\big|^2. \tag{4.23}$$

Hence, by (4.23) and (4.20) we obtain the desired estimate for smooth u, the general case following by a standard density argument in the Sobolev norm. We complete the proof by showing that the subspace $\Xi \subseteq \mathbb{R}^{Nn^2}_s$ satisfies (4.7). Indeed, let $\mathbf{X} \perp \Xi$. Then, by (4.18) we have that $\mathbf{X}$ is normal to $\Sigma^\gamma \otimes H^\gamma$ for any $\gamma = 1, ..., N$, where we have used the obvious notation $H^\gamma := T^\gamma \vee T^\gamma$. Hence the projection of $\mathbf{X}$ on $\Sigma^\gamma \otimes H^\gamma$ vanishes: $(\Sigma^\gamma \otimes H^\gamma)\mathbf{X} = 0$. By Claim 41 we have that $A^\gamma : X = A^\gamma : (H^\gamma X)$ for any $X \in \mathbb{R}^{n^2}$. Hence, we get that $B^\gamma(\mathbf{X} : A^\gamma) = 0$ for all $\gamma = 1, ..., N$ and by summing in γ we obtain $\mathbf{A} : \mathbf{X} = 0$. Thus, we have shown that $\Xi^\perp \subseteq N\big(\mathbf{A} : \mathbb{R}^{Nn^2}_s \to \mathbb{R}^N\big)$, as desired. The theorem has been established. $\square$

4.4. **Proof of the main result.** Now we may finally establish our second main result by utilising the a priori estimate of subsection 4.3.

Proof of Theorem 37. The fist step is to prove existence of a map in the fibre space $(\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ solving in a certain sense the linear problem.

Claim 45. *In the setting of Theorem 37 and under the same assumptions, for any $f \in L^2(\Omega, \Sigma)$, there exists a unique $u \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ such that*

$$\boldsymbol{A} : G^2(u) = f, \quad \text{a.e. on } \Omega,$$

where $G^2(u)$ is the fibre hessian of u.

Proof of Claim 45. The proof is based on the approximation by strictly elliptic systems and relies on the stable estimate of Theorem 40. Let $\mathbf{A}^{(\varepsilon)}$ be the approximation of $\mathbf{A}$ of Theorem 40 and consider for a fixed $f \in L^2(\Omega, \Sigma)$ the system

$$\mathbf{A}^{(\varepsilon)} : D^2 u^\varepsilon = f, \quad \text{a.e. on } \Omega.$$

By standard lower semicontinuity and regularity results (see e.g. [D, GM]), the problem has for any $\varepsilon > 0$ a unique strong a.e. solution $u^\varepsilon \in (W^{2,2} \cap W^{1,2}_0)(\Omega, \mathbb{R}^N)$. By Theorem 40 and Remark 39, we have the uniform estimate

$$\big\|\Sigma\, u^\varepsilon\big\|_{L^2(\Omega)} + \big\|\Pi\, Du^\varepsilon\big\|_{L^2(\Omega)} + \big\|\Xi\, D^2 u^\varepsilon\big\|_{L^2(\Omega)} \leq \frac{C}{\nu}\,\|f\|_{L^2(\Omega)}$$

for some universal $C > 0$. By the definition of $(\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ ((4.4),(4.5)), there exists u such that $\big(\Sigma\, u^\varepsilon, \Pi\, Du^\varepsilon, \Xi\, D^2 u^\varepsilon\big) \longrightarrow \big(u, G(u), G^2(u)\big)$, along a sequence $\varepsilon_k \to 0$ in L^2. Now we pass to the weak limit in the equations. By the form of the approximation $\mathbf{A}^{(\varepsilon)}$ and Definition 36, we have

$$\sum_{\gamma=1}^{N} B^{(\varepsilon)\gamma}\Big(D^2u^\varepsilon : A^{(\varepsilon)\gamma}\Big) \,=\, f \,-\, B^{(\varepsilon)0}\big(D^2u^\varepsilon : A^{(\varepsilon)0}\big),$$

a.e. on Ω. By using that $B^{(\varepsilon)\gamma} = B^\gamma$ for $\gamma = 1, ..., N$ and that $B^{(\varepsilon)0} \perp B^1 + \cdots + B^N$, we may project the system above on the range of $B^1 + \cdots + B^N$ which we denote by Σ. Then, since $\Sigma f = f$ and $A^{(\varepsilon)\gamma} = A^\gamma + \varepsilon I$, we obtain

$$\sum_{\gamma=1}^{N} B^\gamma\Big(\varepsilon\Delta u^\varepsilon \,+\, D^2u^\varepsilon : A^\gamma\Big) \,=\, f,$$

a.e. on Ω. Moreover, by (4.7) (and in view of Remark 38), we deduce

$$\mathbf{A} : \big(\Xi\, D^2 u^\varepsilon\big) \,-\, f \,=\, -\varepsilon \sum_{\gamma=1}^{N} B^\gamma \Delta(\Sigma\, u^\varepsilon),$$

a.e. on Ω. Then, for any $\phi \in C^\infty_c(\Omega, \mathbb{R}^N)$, integration by parts gives

$$\int_\Omega \Big(\mathbf{A} : \big(\Xi\, D^2 u^\varepsilon\big) \,-\, f\Big)\cdot \phi \,=\, -\varepsilon \int_\Omega \sum_{\gamma=1}^{N} B^\gamma(\Sigma u^\varepsilon)\cdot \Delta\phi.$$

By letting $\varepsilon_k \to 0$, we obtain $\mathbf{A} : G^2(u) = f$, a.e. on Ω. We finally show uniqueness. Let $v, w \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ be two solutions of the system. Then, there are sequences $(v^m)_1^\infty, (w^m)_1^\infty \subseteq (W^{2,2} \cap W^{1,2}_0)(\Omega, \mathbb{R}^N)$ such that $v^m - w^m \longrightarrow v - w$ with respect to $\|\cdot\|_{\mathscr{W}^{2,2}(\Omega)}$ as $m \to \infty$. By assumption we have $\mathbf{A} : G^2(v - w) = 0$ a.e. on Ω, and hence

$$\mathbf{A} : D^2(v^m - w^m) \,=:\, f^m, \quad \text{a.e. on } \Omega,$$

and $f^m \to 0$ in $L^2(\Omega, \mathbb{R}^N)$ as $m \to \infty$. Hence, by Theorem 40 and Remark 39, we have

$$\|f^m\|_{L^2(\Omega)} \geq \nu\big\|\Xi : D^2(v^m - w^m)\big\|_{L^2(\Omega)} \geq C\big\|\Sigma(v^m - w^m)\big\|_{L^2(\Omega)}$$

and by letting $m \to \infty$ we see that $v \equiv w$, hence uniqueness ensues. □

An essential ingredient in order to pass from the linear to the non-linear problem is the next result of Campanato taken from [C3] (see also [K7]) which we recall for the convenience of the reader.

Lemma 46 (Campanato's bijectivity of near operators)**.** *Let $\mathfrak{X} \neq \emptyset$ be a set and $(X, \|\cdot\|)$ a Banach space. Let also $\mathscr{F}, \mathscr{A} : \mathfrak{X} \longrightarrow X$ be two mappings and suppose there is a $K \in (0, 1)$ such that*

$$\Big\|\mathscr{F}(u) - \mathscr{F}(v) \,-\, \big(\mathscr{A}(u) - \mathscr{A}(v)\big)\Big\| \,\leq\, K\,\big\|\mathscr{A}(u) - \mathscr{A}(v)\big\|$$

for all $u, v \in \mathfrak{X}$. Then, if $\mathscr{A}$ is bijective, $\mathscr{F}$ is bijective as well.

Now we employ Lemma 46 in order to show existence of a map in the fibre space $(\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ solving in a certain sense the nonlinear problem.

Claim 47. *In the setting of Theorem 37 and under the same assumptions, for any $f \in L^2(\Omega, \Sigma)$ there exists a unique $u \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ such that*

$$F\big(\cdot, G^2(u)\big) \,=\, f, \quad \text{a.e. on } \Omega,$$

where $G^2(u)$ is the fibre hessian of u.

Proof of Claim 47. For any fixed $u \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$, we have that $\mathbf{A} : G^2(u)$ is in $L^2(\Omega, \Sigma)$ because $G^2(u) \in L^2(\Omega, \Xi)$ and also $\mathbf{A} : \mathbf{X}$ lies is in $\Sigma \subseteq \mathbb{R}^N$ for any $\mathbf{X} \in \Xi \subseteq \mathbb{R}^{Nn^2}_s$. Moreover, by Definition 35 we have

$$\big|F\big(\cdot, G^2(u)\big)\big| \,\leq\, \left(\frac{(C+1)|\mathbf{A}| \,+\, B\,\nu}{\text{ess inf}_{x\in\Omega}[A(x)]}\right) |G^2(u)| \,+\, \big|F\big(\cdot, 0\big)\big|,$$

a.e. on Ω. Hence, $F\big(\cdot, G^2(u)\big)$ is in $L^2(\Omega, \Sigma)$ as well. The previous considerations imply that the maps

$$\begin{aligned}
\mathscr{A} \ &: \ (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma) \longrightarrow L^2(\Omega, \Sigma), \quad \mathscr{A}(u) \,:=\, \mathbf{A} : G^2(u),\\
\mathscr{F} \ &: \ (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma) \longrightarrow L^2(\Omega, \Sigma), \quad \mathscr{F}(u) \,:=\, F\big(\cdot, G^2(u)\big),
\end{aligned}$$

are well defined. By Claim 45, $\mathscr{A}$ is bijective. We complete the claim by showing that $\mathscr{F}$ is near $\mathscr{A}$ in the sense of Lemma 46 and then the bijectivity of $\mathscr{F}$ will conclude the proof. For any $u, v \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$, by Definition 35 and Theorem 40 we have

$$\begin{aligned}
&\Big\| A(\cdot) \Big(F\big(\cdot, G^2(u)\big) - F\big(\cdot, G^2(v)\big)\Big) \,-\, \mathbf{A} : \big(G^2(u) - G^2(v)\big) \Big\|_{L^2(\Omega)}\\
&\quad\leq\, B\,\nu \left\|G^2(u) - H(v)\right\|_{L^2(\Omega)} \,+\, C \left\|\mathbf{A} : \big(G^2(u) - G^2(v)\big)\right\|_{L^2(\Omega)}\\
&\quad\leq\, (B + C) \left\|\mathbf{A} : \big(G^2(u) - G^2(v)\big)\right\|_{L^2(\Omega)}.
\end{aligned}$$

Hence, $\hat{\mathscr{F}}(u) := A(\cdot)\, F\big(\cdot, G^2(u)\big)$ is bijective and since $A, 1/A \in L^\infty(\Omega)$, the same is true for $\mathscr{F}$. The claim ensues. $\square$

The next claim completes the proof of Theorem 37.

Claim 48. *In the setting of Claim 47 and under the same assumptions, there exists an orthonormal frame $\{E^1, ..., E^N\} \subseteq \mathbb{R}^N$ and for each $\alpha = 1, ..., N$ there is an orthonormal frame $\{E^{(\alpha)1}, ..., E^{(\alpha)n}\} \subseteq \mathbb{R}^n$ (both depending only on F) such that, the map $u \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ corresponding to $f \in L^2(\Omega, \Sigma)$ is the unique $\mathcal{D}$-solution of the system*

$$F\big(x, D^2u(x)\big) \,=\, f(x), \quad x \in \Omega,$$

in the fibre space $(\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$.

Remark 49 (Functional representation of diffuse hessians)**.** In a certain sense, Claim 48 says that because of our (strong) assumption on F, all the diffuse hessians of the $\mathcal{D}$-solution u when restricted on the subspace of non-degeneracies have a certain "functional" representation *inside the coefficients*, given by $G^2(u)$. That is, by decomposing $\mathbb{R}^{Nn^2}_s = \Xi \oplus \Xi^\perp$, the restriction of any $\mathcal{D}^2u \in \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big)$ on Ξ is given by the fibre hessian:

$$\mathcal{D}^2u(x) \llcorner \Xi \,=\, \delta_{G^2u(x)}, \qquad \text{a.e. } x \in \Omega.$$

Although such a simple representation might not possible in general (compare e.g. with Theorems 29, 33), it is expected that weaker versions of such results should be true (see also Proposition 13).

Proof of Claim 48. Step 1 (The frames). By (4.3) and (4.16) we have that there is an orthonormal frame $\{E^\alpha|\alpha\}$ of $\mathbb{R}^N$ and for each α there is a frame $\{E^{(\alpha)i}|i\}$ of $\mathbb{R}^n$ such that each of the mutually orthogonal subspaces $\Sigma^\gamma \subseteq \mathbb{R}^N$ is spanned by a subset of vectors E^α and for the same index γ, T^γ is spanned by $\{E^{(\alpha)i_0}, ..., E^{(\alpha)n}\}$ which is a set of eigenvectors of A^γ. By (4.3) and (4.18) there are also induced orthonormal frames of $\mathbb{R}^{Nn}$ and $\mathbb{R}^{Nn^2}_s$ consisting of matrices as in (2.2). These frames are such that a subset of the $E^{\alpha ij}$'s spans the subspace $\Xi \subseteq \mathbb{R}^{Nn^2}_s$ and the rest are orthogonal to Ξ.

Step 2 (Sufficiency). Let now $u \in (\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ be the map of Claim 47 which satisfies $F(\cdot, G^2(u)) = f$ a.e. on Ω. Let also us fix any infinitesimal sequence $(h_{\underline{m}})_{\underline{m}\in\mathbb{N}^2}$ with respect to the frames of Step 1 (see Definition 5) and let $\mathcal{D}^2 u$ be any diffuse hessian of u arising from this sequence

$$\delta_{D^{2,h_{\underline{m}}}u} \overset{*}{\rightharpoonup} \mathcal{D}^2 u \quad \text{in } \mathscr{Y}\big(\Omega, \overline{\mathbb{R}}^{Nn^2}_s\big), \qquad \text{as } \underline{m} \to \infty$$

perhaps along subsequences. By the characterisation of the fibre hessian $G^2(u) \in L^2(\Omega, \Xi)$ in terms of directional derivatives of projections (Subsection 4.2), we have

$$G^2(u) \, = \sum_{\alpha,i,j \,:\, E^{\alpha ij}\in\Xi} \Big(G^2(u) : E^{\alpha ij}\Big) E^{\alpha ij}, \qquad \text{a.e. on } \Omega, \tag{4.24}$$

because the projection of $G^2(u)$ along $E^{\alpha ij}$ is non-zero only for those $E^{\alpha ij}$ spanning Ξ. Since F is a Carathéodory map and $F\big(x, G^2(u)(x)\big) = f(x)$ for a.e. $x \in \Omega$, by (4.24) and in view of (2.4) we get

$$F\left(x, \sum_{\alpha,i,j \,:\, E^{\alpha ij}\in\Xi} \Big[D^{2,h_{m_1^2} h_{m_2^2}}_{E^{(\alpha)i}E^{(\alpha)j}}\big(E^\alpha\cdot u\big)\Big](x) E^{\alpha ij}\right) \longrightarrow f(x),$$

for a.e. $x \in \Omega$ as $\underline{m} \to \infty$. By Remark 38V), the above is equivalent to

$$F\Big(x, D^{2,h_{\underline{m}}}u(x)\Big) \, = \, F\left(x, \sum_{\alpha,i,j} \Big[D^{2,h_{m_1^2} h_{m_2^2}}_{E^{(\alpha)i}E^{(\alpha)j}}\big(E^\alpha\cdot u\big)\Big](x) E^{\alpha ij}\right) \longrightarrow f(x),$$

for a.e. $x \in \Omega$, as $\underline{m} \to \infty$. We set

$$f^{\underline{m}}(x) \, := \, F\Big(x, D^{2,h_{\underline{m}}}u(x)\Big) \, - \, f(x)$$

and note that we have $f^{\underline{m}} \longrightarrow 0$, a.e. on Ω as $\underline{m} \to \infty$. By the above, for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn^2}_s\big)$ we have

$$\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(\mathbf{X}) \Big[F(x,\mathbf{X}) - \big(f(x) + f^{\underline{m}}(x)\big)\Big] \, d\big[\delta_{D^{2,s_{\underline{m}}}u(x)}\big](\mathbf{X}) \, = \, 0, \qquad \text{a.e. } x \in \Omega.$$

Since $f^{\underline{m}} \to 0$ a.e. on Ω as $\underline{m} \to \infty$, we apply the Convergence Lemma 18 to obtain

$$\int_{\overline{\mathbb{R}}^{Nn^2}_s} \Phi(\mathbf{X}) \big[F(x,\mathbf{X}) - f(x)\big] \, d\big[\mathcal{D}^2 u(x)\big](\mathbf{X}) \, = \, 0, \qquad \text{a.e. } x \in \Omega,$$

for any $\Phi \in C^0_c\big(\mathbb{R}^{Nn^2}_s\big)$. Hence, the map u of Claim 47 is a $\mathcal{D}$-solution of (4.1).

Step 3 (Necessity). We now finish the proof by showing that any $\mathcal{D}$-solution w of (4.1) with respect to the frames of Step 1 which lies in the fibre space $(\mathscr{W}^{2,2} \cap \mathscr{W}^{1,2}_0)(\Omega, \Sigma)$ actually coincides with the map u of Claim 47. By Theorem 22, we have that the $\mathcal{D}$-solution w can be characterised by the property that for any $R>0$, the cut off associated to F (see Definition 21) satisfies

$$F\Big(x,\big[D^{2,h_{\underline{m}}}w(x)\big]^{R}\Big)\longrightarrow f(x),\qquad \text{a.e. } x\in\Omega,$$

as $\underline{m}\to\infty$. By using Remark 38V), we have for any $R>0$ that

$$F\Big(x,\big[\Xi\, D^{2,h_{\underline{m}}}w(x)\big]^{R}\Big)\longrightarrow f(x),\qquad \text{a.e. } x\in\Omega,$$

as $\underline{m}\to\infty$. Since w is in $(\mathscr{W}^{2,2}\cap\mathscr{W}^{1,2}_0)(\Omega,\Sigma)$, by using the properties of the fibre space we get that $\Xi\, D^{2,h_{\underline{m}}}w\longrightarrow G^2(w)$ in L^2 and hence a.e. on Ω along perhaps further subsequences. By passing to the limit as $\underline{m}\to\infty$ and then as $R\to\infty$, we obtain that $F(\cdot,G^2(w))=f$, a.e. on Ω. Hence, $w\equiv u$ and the claim ensues. $\Box$

By recalling Remark 39 regarding the boundary trace values of maps in the fibre space, we conclude that the proof of Theorem 37 is now complete. $\Box$

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References

AF. R. Adams, J. Fournier, *Sobolev spaces*, Academic Press, 2nd edition, 2003.

AM. L. Ambrosio, J. Malý, *Very weak notions of differentiability*, Proceedings of the Royal Society of Edinburgh A 137 (2007), 447 - 455.

A1. G. Aronsson, *Minimization problems for the functional $sup_x F(x, f(x), f'(x))$*, Arkiv für Mat. 6 (1965), 33 - 53.

A2. G. Aronsson, *Minimization problems for the functional $sup_x F(x, f(x), f'(x))$ II*, Arkiv für Mat. 6 (1966), 409 - 431.

A3. G. Aronsson, *Extension of functions satisfying Lipschitz conditions*, Arkiv für Mat. 6 (1967), 551 - 561.

A4. G. Aronsson, *On the partial differential equation $u_x^2u_{xx}+2u_xu_yu_{xy}+u_y^2u_{yy}=0$*, Arkiv für Mat. 7 (1968), 395 - 425.

A5. G. Aronsson, *Minimization problems for the functional $sup_x F(x, f(x), f'(x))$ III*, Arkiv für Mat. (1969), 509 - 512.

A6. G. Aronsson, *On Certain Singular Solutions of the Partial Differential Equation $u_x^2u_{xx}+2u_xu_yu_{xy}+u_y^2u_{yy}=0$*, Manuscripta Math. 47 (1984), no 1-3, 133 - 151.

A7. G. Aronsson, *Construction of Singular Solutions to the p-Harmonic Equation and its Limit Equation for $p=\infty$*, Manuscripta Math. 56 (1986), 135 - 158.

AC. J.P. Aubin, A. Cellina, *Differential Inclusions*, Springer-Verlag, 1984.

BEJ. E. N. Barron, L. C. Evans, R. Jensen, *The Infinity Laplacian, Aronsson's Equation and their Generalizations*, Transactions of the AMS, Vol. 360, Nr 1, Jan 2008.

BL. C. Le Bris, P. L. Lions, *Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications*, Ann. di Mat. Pura ed Appl., 183 (2004) 97 - 130.

CC. L.A. Caffarelli, M.G. Crandall, *Distance Functions and Almost Global Solutions of Eikonal Equations*, Communications in PDE 03, 35, 391-414 (2010).

C1. S. Campanato, *A history of Cordes Condition for second order elliptic operators*, in "Boundary value problems for partial differential equations and applications", J.L. Lions et al (editors), Paris: Masson. Res. Notes Appl. Math 29, (1983), 319 - 325.

C2. S. Campanato, *A Cordes type condition for nonlinear non variational systems*, Rendiconti Accad. Naz. delle Scienze detta dei XL, vol. 198 (1989).

C3. S. Campanato, *On the condition of nearness between operators*, Analli di Mat. Pura Appl. IV, Vol. CLXVII (1994), 243 - 256.

CFV. C. Castaing, P. R. de Fitte, M. Valadier, *Young Measures on Topological spaces with Applications in Control Theory and Probability Theory*, Mathematics and Its Applications, Kluwer Academic Publishers, 2004.

Co. J.F. Colombeau, *New Generalized Functions and Multiplication of distributions*, North Holland, 1983.

C. M. G. Crandall, *A visit with the ∞-Laplacian*, in *Calculus of Variations and Non-Linear Partial Differential Equations*, Springer Lecture notes in Mathematics 1927, CIME, Cetraro Italy 2005.

CL. M. G. Crandall, P.L. Lions, *viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. 277, 1, 142 (1983).

D. B. Dacorogna, *Direct Methods in the Calculus of Variations*, 2nd Edition, Volume 78, Applied Mathematical Sciences, Springer, 2008.

DM. B. Dacorogna, P. Marcellini, *Implicit Partial Differential Equations*, Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, 1999.

DPM. R.J. DiPerna, A.J. Majda, *Oscillations and concentrations in weak solutions of the incompressible fluid equations*, Commun. Math. Phys. 108, 667 - 689 (1987).

E. L.C. Evans, *Weak convergence methods for nonlinear partial differential equations*, Regional conference series in mathematics 74, AMS, 1990.

E2. L.C. Evans, *Partial Differential Equations*, AMS, Graduate Studies in Mathematics Vol. 19, 1998.

EG. L.C. Evans, R. Gariepy, *Measure theory and fine properties of functions*, Studies in advanced mathematics, CRC press, 1992.

FG. L.C. Florescu, C. Godet-Thobie, *Young measures and compactness in metric spaces*, De Gruyter, 2012.

F. G.B. Folland, *Real Analysis: Modern techniques and their applications*, 2nd edition, Wiley-Interscience, 1999.

FL. I. Fonseca, G. Leoni, *Modern methods in the Calculus of Variations: L^p spaces*, Springer Monographs in Mathematics, 2007.

GM. M. Giaquinta, L. Martinazzi, *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*, Publications of the Scuola Normale Superiore 11, Springer, 2012.

K. N. Katzourakis, *A Hölder Continuous Nowhere Improvable Function with Derivative Singular Distribution*, SEMA Journal 59 (2012), 37 - 51.

K1. N. Katzourakis, *L^∞-Variational Problems for Maps and the Aronsson PDE system*, J. Differential Equations, Volume 253, Issue 7 (2012), 2123 - 2139.

K2. N. Katzourakis, *Nonuniqueness in Vector-valued Calculus of Variations in L^∞ and some Linear Elliptic Systems*, Communications on Pure and Applied Analysis, Vol. 14, 1, 313 - 327 (2015).

K3. N. Katzourakis, *Optimal ∞-Quasiconformal Immersions*, ESAIM Control, Opt. and Calc. Var., to appear (2015) DOI: http://dx.doi.org/10.1051/cocv/2014038.

K4. N. Katzourakis, *On the Structure of ∞-Harmonic Maps*, Communications in PDE, Volume 39, Issue 11 (2014), 2091 - 2124.

K5. N. Katzourakis, *∞-Minimal Submanifolds*, Proceedings of the Amer. Math. Soc., 142 (2014) 2797-2811.

K6. N. Katzourakis, *Explicit 2D ∞-Harmonic Maps whose Interfaces have Junctions and Corners*, Comptes Rendus Acad. Sci. Paris, Ser.I, 351 (2013) 677 - 680.

K7. N. Katzourakis, *Existence and Uniqueness of Global Solutions to Fully Nonlinear First Order Elliptic Systems*, Nonlinear Analysis 115, 50 - 61 (2015).

K8. N. Katzourakis, *An Introduction to viscosity Solutions for Fully Nonlinear PDE with Applications to Calculus of Variations in L^∞*, Springer Briefs in Mathematics, 2015, DOI 10.1007/978-3-319-12829-0.

K9. N. Katzourakis, *On the Dirichlet Problem for Fully Nonlinear Elliptic hessian Systems*, Ann. Sc. Norm. Sup. Pisa, in press.

K10. N. Katzourakis, *On Linear Degenerate Elliptic PDE Systems with Constant Coefficients*, Adv. in Calculus of Variations, DOI: 10.1515/acv-2015-0004, published online June 2015.

K11. N. Katzourakis, *Existence and uniqueness of global strong solutions to fully nonlinear second order elliptic systems*, ArXiv preprint, http://arxiv.org/pdf/1408.5423.pdf.

K12. N. Katzourakis, *Existence of Generalised Solutions to the Equations of Vectorial Calculus of Variations in L^∞*, ArXiv preprint, http://arxiv.org/pdf/1502.01179.pdf.

KR. J. Kristensen, F. Rindler, *Characterization of generalized gradient Young measures generated by sequences in $W^{1,1}$ and BV*, Arch. Rational Mech. Anal. 197, 539 - 598 (2010) and *erratum* Arch. Rational Mech. Anal. 203, 693 - 700 (2012).

LU. O. Ladyzhenskaya, N. Uraltseva, *Linear and quasilinear elliptic equations*, Academic press (Mathematics in Science & Engineering) Vol. 46, 1968.

M. S. Müller, *Variational models for microstructure and phase transitions*, Lecture Notes in Mathematics 1783, Springer, 85-210, 1999.

OL. H. B. Lawson Jr., R. Osserman, *Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system*, Acta Mathematica, Volume 139, 1, 1-17 (1977).

P. P. Pedregal, *Parametrized Measures and Variational Principles*, Birkhäuser, 1997.

SS. S. Sheffield, C.K. Smart, *Vector Valued Optimal Lipschitz Extensions*, Comm. Pure Appl. Math., Vol. 65, Issue 1, January 2012, 128 - 154.

S1. L. Schwartz, *Théorie des distributions 1 - 2*, Hermann, 1951.

S2. L. Schwartz, *Sur l'impossibilité de la multiplications des distributions*, C.R.Acad. Sci. Paris 239, 847848 (1954).

So. S.L. Sobolev, *Méthode nouvelle á rásoudre le probléme de Cauchy pour les équations linéaires hyperboliques normales*, Mat. Sbornik 1, 3972 (1936).

T. L. Tartar, *An introduction to Sobolev and Interpolation spaces*, Lecture Notes of the Unione Matematica Italiana 3, Springer, 2007.

V. M. Valadier, *Young measures*, in "Methods of nonconvex analysis", Lecture Notes in Mathematics 1446, 152-188 (1990).

Y. L.C. Young, *Generalized curves and the existence of an attained absolute minimum in the calculus of variations*, Comptes Rendus de la Societe des Sciences et des Lettres de Varsovie, Classe III 30, 212 - 234 (1937).

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