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by

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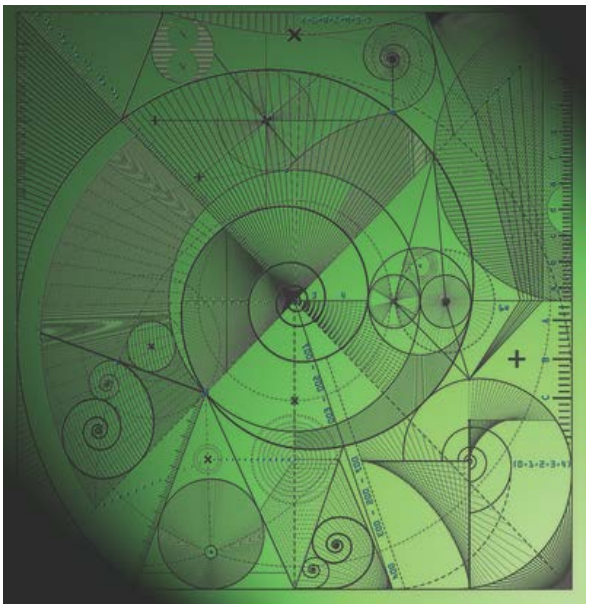

Review article: Advances in the study of boundary value problems for nonlinear integrable PDEs

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Abstract

In this review I summarise some of the most significant advances of the last decade in the analysis and solution of boundary value problems posed for integrable partial differential equations in two independent variables. These equations arise widely in mathematical physics, and in order to model realistic applications, it is essential to consider bounded domain and inhomogeneous boundary conditions.

I focus specifically on a general and widely applicable approach, usually referred to as the *Unified Transform* or *Fokas Transform*, that provides a substantial generalisation of the classical Inverse Scattering Transform. This approach preserves the conceptual efficiency and aesthetic appeal of the more classical transform approaches, but presents a distinctive and important difference. While the Inverse Scattering Transform follows the "separation of variables" philosophy, albeit in a nonlinear setting, the Unified Transform is a based on the idea of synthesis, rather than separation, of variables.

I will outline the main ideas in the case of linear evolution equations, and then illustrate their generalisation to certain nonlinear cases of particular significance.

1 Introduction

The Inverse Scattering Transform is one of the most celebrated advances in the study of nonlinear systems, pioneered at the end of the 1960's by Kruskal et al. [44] and consolidated throughout the 1970's by the work of many others [4, 5, 50, 57].
This transform is essentially a nonlinear version of the Fourier transform in one variable, and can be used to unravel the behaviour of many systems with the property that the nonlinearity is exactly balanced by other effects, such as dispersive effects. This implies that, in many important respects, the behaviour of the solutions of the system is highly regular. For example, when posed on an infinite spatial domain, these systems admit localised solutions (often referred to as solitons) that interact elastically - the interaction does not destroy the amplitude or speed of the solutions. More importantly still, localised initial conditions with sufficient energy will eventually evolve into a train of solitons, followed by a dispersive tail. These properties are remarkable for a nonlinear system, and were first described heuristically by Zabuski and Kruskal who observed this elastic interaction in numerical experiments modelling solutions of the Korteweg-deVries equation.

Systems with the particular properties described above are called *integrable*.

There are by now many review papers devoted to integrable systems, but these systems are not per-se the focus of this article. My aim is to discuss how far the *integrability structure* survives when the system is studied in a domain with boundaries, and in the presence of additional boundary conditions.

The Unified Transform of Fokas

The study of boundary value problems for integrable PDEs in the last fifteen years has motivated the development of a new powerful method in mathematical physics, usually referred to as *Unified Transform* or *Fokas Transform*. This approach combines the main insights of the Inverse Scattering Transform with elements of the theory of Riemann-Hilbert problems, and uses essentially complex analytic properties, for example to eliminate unknown boundary values from the solution representation. This method, proposed by Fokas and extensively developed by him, myself and others, has produced a substantial body of results and, unexpectedly, has led to a new perspective and results on the theory of boundary value problems also for linear PDEs in two independent variables. For an introduction to this theory and a summary of the main results, see [29].

There is no universally accepted definition of integrability for an infinite dimensional system such as the one described by a PDE. In this article, the property that will characterise an integrable PDE is the fact that such a PDE can be formulated as the compatibility condition of two linear ODEs, called a *Lax pair* associated with the given PDE. The Lax pair is a system involving an additional complex parameter, which I will refer to as the *spectral parameter*, and denote by λ, where $\lambda \in \mathbb{C}$. In the analogy with the Fourier transform, the spectral parameter plays the role of the independent variable in Fourier space.

The Unified Transform was developed starting from two fundamental observations. These observations, seemingly pertaining to separate fields, have in common the natural formulation in terms of a *Riemann-Hilbert problem*. By this here we mean the problem of reconstruction of a function of a complex variable, analytic everywhere off a given contour, from its known jump condition across the contour. A clear, non-technical review of Riemann-Hilbert problems in this context can be found in [47]. Such formulations are regarded as a universal, unifying element of a disparate variety of problems in mathematics and mathematical physics [21].
The two observations are the following:

1. An integrable PDE admits a Lax pair formulation. This formulation provides a highly nontrivial generalisation of the concept of separation of variables, valid for both linear and integrable nonlinear PDEs [40].

2. Integral transforms such as the Fourier transform can be derived via the spectral analysis of an ODE in the complex plane [31].

The main contribution of Fokas was to realise that the existence of a Lax pair makes it possible to solve both ODEs of this pair simultaneously. Just as the analysis of a single ODE yields a transform associated with one given independent variable, this simultaneous analysis yields a transform associated with two independent variables. Moreover, since this transform is associated with the particular Lax pair, it is also custom-made for the

given PDE. Combining this idea with the two observations above, it is possible to construct algorithmically a formal integral representation of the solution of a given boundary value problem for an integrable PDE - this construction is the basis of the Unified Transform approach. It is important to note that this representation generally involves contours in the complex plane; for the solution of the linear problem, it takes the form of an integral along such a complex contour.
The representation derived in this way has explicit dependence on the independent variables of the PDE, hence it is straightforward to show that it indeed represents a solution of the PDE. For evolution problems, it is a direct consequence of analyticity consideration that this representation also takes the given initial condition for $t = 0$. However, this representation involves *all* the boundary values of the solution. To obtain a fully determined solution of the boundary value problem, it is necessary to overcome the basic difficulty, inherent to all boundary value problems, that in general at least some of the boundary values cannot be prescribed independently, and are therefore unknown. Namely, the difficulty is in obtaining the effective elimination of the unknown boundary values from the representation, to obtain a solution of the PDE that satisfies the given boundary conditions.
However, already at this point, a significant advantage of the complex representation derived via a Riemann-Hilbert formulation is that, since it has an *explicit dependence* on the independent variables of the problem, asymptotic and qualitative information on the solution can be derived from it, even before the contribution of the unknown boundary values is eliminated.

To give a concrete example and illustrate the difficulty of dealing with unknown boundary values, consider the simplest third order *linear* PDE, namely

$$\partial_t q(x,t) + \partial_{xxx} q(x,t) = 0, \qquad x \in I, \quad 0 < t < T, \qquad q(x,0) = q_0(x) \tag{1.1}$$

where $q_0(x)$ is a given initial condition, assumed sufficiently smooth and decaying if I is infinite, I denotes a finite or infinite interval in $\mathbb{R}$, and T is a given positive constant.
When $I = \mathbb{R}$, hence the PDE is posed on the full line, assuming sufficiently rapid decay of all functions as $x \to \pm\infty$, one obtains a Cauchy initial value problem for (1.1). The standard approach for solving this involves taking the Fourier transform of this equation to obtain a first order ODE that can be solved explicitly:

$$q_t + \partial_x^n q = 0 \quad \textbf{PDE} \overset{\textbf{FT}}{\to} \textbf{ODE} \quad \frac{\partial \hat{q}(\lambda,t)}{\partial t} + (i\lambda)^n \hat{q}(\lambda,t) = 0, \qquad \left(\hat{q}(\lambda,t) = \int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda x} q(x,t) dx \right). \tag{1.2}$$

Solving the ODE and inverting, one finds

$$\hat{q}(\lambda,t) = \mathrm{e}^{-(i\lambda)^n t} \hat{q}(\lambda,0) \qquad \to \quad q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x + i\lambda^3 t} \hat{q}(\lambda,0) d\lambda.$$

Consider now the case that $I = [0,\infty)$, and that the boundary condition $q(0,t) = f_0(t)$ is prescribed, along with decay as $x \to \infty$. Following the same steps as for $I = \mathbb{R}$, one could take the Fourier transform of $q(x,t)$ on the half line, solve the resulting ODE (which now contains three boundary values at $x = 0$) and take the inverse transform of the result. This

procedure yields the following integral representation of the solution of (1.1) on the half-line:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x + i\lambda^3 t}\left[\int_0^{\infty} \mathrm{e}^{-i\lambda y} q_0(y)dy\right] d\lambda + \tag{1.3}$$

$$+ \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x + i\lambda^3 t}\left[\int_0^{t} \mathrm{e}^{-i\lambda^3 s}\left(q_{xx}(0,s) + i\lambda q_x(0,s) - \lambda^2 f_0(s)\right) ds\right] d\lambda.$$

The first term in this representation is the contribution of the initial condition. This would be the only term present when solving the Cauchy value problem for decaying data, with the integration in y extending over $\mathbb{R}$. In this term, the x and t dependence is explicit through the exponential term.

The second term involves the boundary values of the solution at $x = 0$, but two of the boundary values involved in the integrand are not directly available. This is the generic case: to guarantee that the boundary value problem admits a unique solution, only one condition can be prescribed at this boundary, and therefore in general two boundary values are unknown [34, 35].

This difficulty can be overcome, for PDE of second order in x, by using the sine or cosine transform. However, this is not possible for PDE involving third order derivatives. In addition, even using sine or cosine transform, the t-dependence of the resulting expression is not explicit, as t appears in the integration interval as well as in the exponential part of the integrand. Last but not least, the integral expressions derived by using sine or cosine transform are *not uniformly convergent when* $x \to 0$.

In contrast, the Unified Transform yields the following representation for the solution of the same problem:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x + i\lambda^3 t}\left[\int_0^{\infty} \mathrm{e}^{-i\lambda y} q_0(y)dy\right] d\lambda + \tag{1.4}$$

$$+ \quad \frac{1}{2\pi}\int_{\partial D^+} \mathrm{e}^{i\lambda x + i\lambda^3 t}\left[\int_0^{T} \mathrm{e}^{-i\lambda^3 s}\left(q_{xx}(0,s) + i\lambda q_x(0,s) - \lambda^2 f_0(s)\right) ds\right] d\lambda,$$

where ∂D^+ is the contour in $\mathbb{C}^+$ defined by $Re(i\lambda^3) = 0$, see Figure 1.

This formulation has fully explicit x and t dependence, and it is uniformly convergent at the boundary $x = 0$, a fact that plays an important role also in devising efficient numerical schemes [24, 25]. Moreover, by considering the transforms of the boundary values (with respect to t) as functions of the *complex* variable λ, it is possible to eliminate all unknown contributions - as will be detailed in Section 4.2. Performing this elimination effectively is the heart of the matter, and its resolution is most naturally understood once the representation derived is of the form (1.4). Indeed, this resolution cannot be obtained by confining the spectral parameter λ to be real, hence restricting attention to real transforms.

As an example, the solution of the particular boundary value problem for the PDE (1.1) obtained when $q(0,t) = f_0(t)$ is the prescribed boundary condition for $0 < t < T$, is given by

$$\begin{aligned} q(x,t) \quad = \quad & \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{i\lambda x + i\lambda^3 t}\hat{q}_0(\lambda)d\lambda \\ & + \frac{1}{2\pi}\int_{\partial D^+} \mathrm{e}^{i\lambda x + i\lambda^3 t}\left[(\omega - 1)\hat{q}_0(\omega^2\lambda) + (1-\omega^2)\hat{q}_0(\omega\lambda) + 3\lambda^2\tilde{f}_0(\lambda)\right] d\lambda, \end{aligned} \tag{1.5}$$

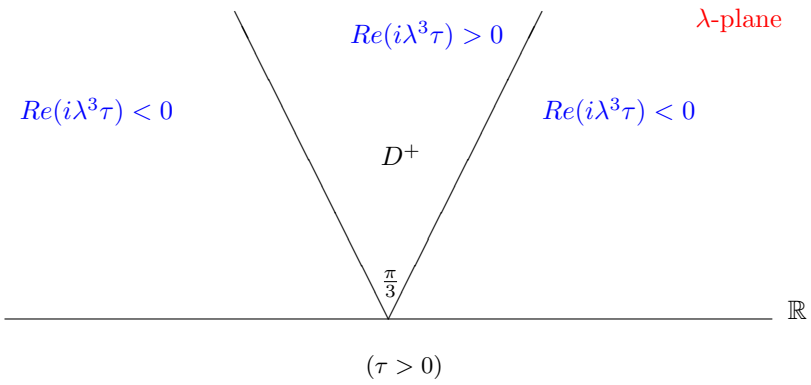

Figure 1: The domain D^+

with

$$\hat{q_0}(\lambda) = \int_0^\infty \mathrm{e}^{-i\lambda y} q(y,0)dy, \quad \lambda \in \mathbb{C}^-, \quad \tilde{f}_0(\lambda) = \int_0^T \mathrm{e}^{-i\lambda^3 s} f(s)ds, \qquad \omega = \mathrm{e}^{2\pi i/3}. \quad (1.6)$$

In this linear example, the expression (1.4) could be derived from (1.3) by simply considering analyticity properties with respect to the variable λ, extended from $\mathbb{R}$ to $\mathbb{C}$, and deforming the contour of integration. However, this deformation is not possible in the general nonlinear case. The general methodology to obtain the representation (1.4) is based instead on formulating and solving an associated Riemann-Hilbert problem, and this approach can indeed be extended to the case of nonlinear integrable evolution equations in one space variable, e.g. to the famous KdV or mKdV equation, posed on the domain $0 < x < \infty, \ \ 0 < t < T$:

$$(KdV) \qquad q_t + q_{xxx} + q_x + 6qq_x = 0, \quad (1.7)$$

$$(mKdV) \qquad q_t + q_{xxx} + \nu 6q^2 q_x = 0, \qquad \nu = \pm 1. \quad (1.8)$$

However, in this case, the elimination of the unknown boundary values is only as effective as in the linear case for special boundary conditions, called *linearisable* [26, 28]. For general boundary conditions, the characterisation can be obtained by a perturbation scheme effective to all orders [41, 42].

In this article, I present a summary of the main results obtained by this approach for boundary value problems, unifying the treatment of linear and integrable nonlinear PDEs in two independent variables. My aim is to review the main ideas and tools needed to carry out this programme successfully, and discuss its limitation, avoiding detailed proofs. For the details, the reader can refer to the papers cited throughout.
I will not include in my treatment of the nonlinear case any consideration of isolated singularities in the spectral variable λ, although soliton solutions arise precisely in connection with these singularities. This mechanism is well understood and it is not specific to the treatment of boundary value problems, which are my focus in this review [1, 3].

I will also leave out of my exposition the case of problems with periodicity in the variable x. The solution of this case, for integrable nonlinear evolution PDEs, was developed in the seventies, and it involves algebro-geometric techniques, through a formulation as a Riemann-Hilbert problem on a Riemann surface [13, 23, 48].

2 Integral transforms and Riemann-Hilbert problems

The Fourier transform provides the most effective way of solving initial value problems for linear evolution PDEs.
Consider for example the PDE $\partial_t q + \partial_x^n q = 0$. The solution "algorithm", assuming $q(x,0) = q_0(x) \in \mathcal{S}(\mathbb{R})$, is given by (1.2). Schematically, this algorithm is given by

$$q_0(x) \overset{\textbf{Direct map}}{\longrightarrow} \hat{q}_0(\lambda) \overset{\textbf{Inverse map}}{\longrightarrow} q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x - (i\lambda)^n t}\, \hat{q}_0(\lambda) d\lambda,$$

with $\hat{q}_0(\lambda)$ is the Fourier transform of $q_0(x)$.
This solution algorithm is generally applicable, it yields an exact solution representation and, on account of the explicit x and t dependence appearing in the solution representation, it contains qualitative information (in particular, asymptotic information for large t and x) even when $\hat{q}_0(\lambda)$ cannot be computed explicitly.

To generalise this approach, I begin by outlining the derivation of the Fourier transform on $\mathbb{R}$ within the more general framework of Riemann-Hilbert problems. Indeed, in the following two sections, I will give a derivation of the Fourier and Inverse Scattering transforms that relies on associating them with a suitable Riemann-Hilbert problem. In these sections, since the focus is on transforms of functions only in terms of the single variable x, any explicit mention of the additional time variable is dropped.

2.1 The Fourier inversion theorem via a Riemann-Hilbert problem

Let $q(x) \in \mathcal{S}(\mathbb{R})$ be a given, arbitrary function, and consider the following auxiliary ODE:

$$\mu_x - i\lambda\mu = q(x), \quad \mu = \mu(x,\lambda), \quad \lambda \in \mathbb{C}. \tag{2.1}$$

I interpret x as a parameter, and seek a solution $\mu(x,\lambda)$ bounded as $\lambda \to \infty$, $\forall \lambda \in \mathbb{C}$. It is easy to find two solutions bounded in $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively:

$$\mu_+(x,\lambda) = \int_{-\infty}^{x} \mathrm{e}^{i\lambda(x-y)} q(y) dy, \quad \text{bounded for } \mathrm{Im}(\lambda) \geq 0,$$

$$\mu_-(x,\lambda) = -\int_{x}^{\infty} \mathrm{e}^{i\lambda(x-y)} q(y) dy \quad \text{bounded for } \mathrm{Im}(\lambda) \leq 0.$$

In addition, the functions $\mu_\pm$ have the following properties:

- $\mu_\pm$ is analytic in $\mathbb{C}^\pm$
- $\mu_\pm = O\left(\frac{1}{\lambda}\right)$ as $\lambda \to \infty$ in $\mathbb{C}^\pm$,

- For $\lambda \in \mathbb{R}$, where both functions $\mu^{\pm}$ are defined, the difference is

$$(\mu_+ - \mu_-)(x,\lambda) = \mathrm{e}^{i\lambda x} \int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda y} q(y) dy := \mathrm{e}^{i\lambda x} \hat{q}(\lambda) \quad \lambda \in \mathbb{R} \tag{2.2}$$

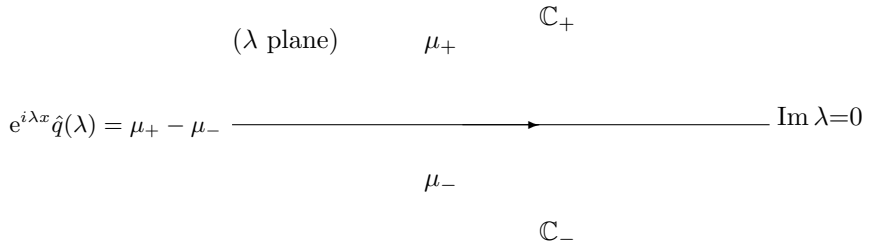

Figure 2: A Riemann-Hilbert problem on the real line

Hence, given $q(x)$, the transform $\hat{q}(\lambda)$ is obtained as the jump of the sectionally analytic solution μ such that $\mu = \mu^+$ for $\lambda \in \mathbb{C}^+$ and $\mu = \mu^-$ for $\lambda \in \mathbb{C}^-$.
Conversely, suppose $\hat{q}(\lambda)$ is given. Then the data above determine uniquely a scalar Riemann-Hilbert problem for μ, with jump on the real line given by (2.2). This Riemann-Hilbert problem can be solved explicitly by the so-called *Plemelj formula* [2]. This yields

$$\mu(x,\lambda) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{[\mu_+ - \mu_-](x,\zeta)}{\zeta - \lambda} d\zeta = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i\zeta x} \hat{q}(\zeta)}{\zeta - \lambda} d\zeta. \tag{2.3}$$

Hence one finds

$$q(x) = \mu_x - i\lambda\mu = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{i\zeta x} \hat{q}(\zeta) d\zeta, \ x \in \mathbb{R} \tag{2.4}$$

Note that the expression (2.4) for $q(x)$ in terms of $\hat{q}(\lambda)$ can also be derived starting from the asymptotic information

$$\mu^{\pm} \sim \frac{iq(x)}{\lambda}, \quad |\lambda| \to \infty \Longrightarrow q(x) = -i \lim_{\lambda \to \infty} \lambda \mu(x,\lambda).$$

The latter derivation generalises to the nonlinear case.

The moral is that *the first order ODE* $\mu_x(x,\lambda) - i\lambda\mu(x,\lambda) = q(x)$ *where* $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ *and* $q(x) \in \mathcal{S}(\mathbb{R})$ *is given, encodes the Fourier transform.*

The same idea can be used to give a derivation of other integral transforms, compute the associated inversion formula and prove rigorously its validity. This approach provides a new proof of the inversion formula for many classical transforms such as Mellin, Abel and Radon transforms, as well as the first proof of the inversion formula for the attenuated Radon transform, widely used in medical imaging [30, 52].

2.2 The Inverse Scattering Trasform: a nonlinear Fourier transform

The approach just described for formulating and inverting integral transforms has a nonlinear analogue. This nonlinear transform can be used to solve the initial value problem for integrable nonlinear evolution PDEs, in a way analogous to the use of Fourier transform in the linear case.
The starting point, rather than the scalar ODE (2.1), is a *matrix -valued* ODE. Namely, define the matrix Q in terms of the given arbitrary function $q(x) \in \mathcal{S}(\mathbb{R})$ (although much less regularity is required, see [6]) by

$$Q(x) = \begin{pmatrix} 0 & q(x) \\ \pm\overline{q(x)} & 0 \end{pmatrix}, \tag{2.5}$$

where $\bar{\cdot}$ denotes complex conjugation, and consider the ODE

$$M_x + i\lambda[\sigma_3, M] = QM \quad x \in \mathbb{R},\ \lambda \in \mathbb{C}; \qquad M(x,\lambda)\ a\ 2\times 2\ matrix, \tag{2.6}$$

where

$$[\sigma_3, M] = \sigma_3 M - M\sigma_3, \quad \sigma_3 = diag(1,-1). \tag{2.7}$$

One seeks a solution $M(x,\lambda)$ of this ODE well-defined for all $\lambda \in \mathbb{C}$. As for the linear case, one can define a solution $M^+(x,\lambda)$ well-defined and bounded for $\lambda \in \mathbb{C}^+$, indeed, such that $M^+ \to I$ as $x \to \infty$, and a solution $M^-(x,\lambda)$ well-defined and bounded for $\lambda \in \mathbb{C}^-$, with $M^- \to I$ as $x \to -\infty$. These matrix-valued solutions are not explicit, but characterised as the unique solution of a linear integral equation, namely

$$M^{\pm}(x,\lambda) = I - \int_x^{\pm\infty} \mathrm{e}^{-i\lambda(x-y)\widehat{\sigma_3}} Q(y) M^{\pm}(y,\lambda) dy, \qquad \lambda \in \mathbb{C}^{\pm}.$$

They satisfy a jump condition across $\mathbb{R}$ of the form

$$M^-(x,\lambda) = M^+(x,\lambda)\tilde{J}(x,\lambda), \qquad \lambda \in \mathbb{R}, \tag{2.8}$$

as well as asymptotic conditions $M^{\pm} = I + O\left(\frac{1}{\lambda}\right)$ as $|\lambda| \to \infty$. The jump $\tilde{J}(x,\lambda)$, defined for $\lambda \in \mathbb{R}$, is now matrix-valued. The entries of the 2×2 matrix $\tilde{J}(x,\lambda)$ are defined in terms of certain λ-transforms of the given function $q(x)$, called the *spectral data*, multiplied by explicit exponentials encoding the dependence on x:

$$\tilde{J}(x,\lambda) = \mathrm{e}^{i\lambda x\widehat{\sigma_3}} J(\lambda). \tag{2.9}$$

The notation expresses that the action of $\exp(x\widehat{\sigma_3})$ on a 2×2 matrix A is given by

$$\mathrm{e}^{x\widehat{\sigma_3}} A = \begin{pmatrix} a_{11} & \mathrm{e}^{2x} a_{12} \\ \mathrm{e}^{-2x} a_{21} & a_{22} \end{pmatrix}. \tag{2.10}$$

Hence given the matrix $Q(x)$, the jump condition (2.8) defines the transform $J(\lambda)$.
Conversely, given $J(\lambda)$, the jump and decay data above determine a Riemann-Hilbert problem for $M(x,\lambda)$ on the real line. Note that the jump matrix $J(\lambda)$ and the function $M(x,\lambda)$ are well-defined only modulo the possible existence of isolated singularities. However, the

role of such isolated singularities is well understood and we will ignore them for the purpose of this review, see also remark 5.2 below.
The difference with the linear case is that the jump condition in this case is multiplicative. The multiplicative, non-commutative structure of the Riemann-Hilbert problem is a manifestation of the nonlinearity of the equation, and it implies that the solution does not have an explicit expression analogous to the one in (2.3) given by the Plemelj formula. However, it is a classical result that the solution $M(x,\lambda)$ of the Riemann-Hilbert determined by the data above can be characterised as the solution of a linear singular integral equation, and its unique solvability can be rigorously proved, appealing to the symmetries forced on the system by the choice of the form of the matrix $Q(x)$ [10, 11, 12, 22].
From the expression for $M(x,\lambda)$ one must derive an expression for $q(x)$ and thus formulate the inverse transform. Recall that, in the linear case, $\mu \sim iq/\lambda$ as $\lambda \to \infty$. Similarly, from the expression for $M(x,\lambda)$, one can determine the arbitrary function q by computing the λ asymptotic of the $(1,2)$ element of the matrix $M(x,\lambda)$:

$$q(x) = 2i \lim_{|\lambda|\to\infty} (\lambda M_{12}(x,\lambda)).$$

The above procedure gives a nonlinear version of the Fourier transform, known as the *Inverse Scattering Transform (IST)*. The jump matrix $J(\lambda)$ is the analogue of the direct Fourier transform. Indeed, the direct and inverse transform diagram can be summarised as follows:
Direct transform - obtain $J(\lambda)$ from $q(x)$ via solving an ODE for $M(x,t,\lambda)$:

$$q \longrightarrow Q = \begin{pmatrix} 0 & q(x) \\ \pm\bar{q}(x) & 0 \end{pmatrix} \quad \longrightarrow \quad M:\ M_x + i\lambda[\sigma_3, M] = QM \longrightarrow \mathrm{e}^{i\lambda x\widehat{\sigma_3}} J = (M^+)^{-1} M^-.$$

Inverse transform - obtain $q(x)$ from $J(\lambda)$ via solving a Riemann-Hilbert problem for $M(x,t,\lambda)$:

$$J \longrightarrow M:\ M^+ = \mathrm{e}^{i\lambda x\widehat{\sigma_3}} J M^-,\ x \in \mathbb{R},\ M = I + O\left(\frac{1}{\lambda}\right) \longrightarrow \quad q = 2i \lim_{\lambda\to\infty} (\lambda M_{12}).$$

3 The structure of integrable PDEs: Lax pair formulation

The fundamental structural property that I take as characterising the integrability of PDEs is the fact that an integrable partial differential equation can be formulated as the compatibility condition of a pair of ordinary differential equations. This is not a rigorous definition [19], but it is certainly adequate for the purpose of this exposition. This pair is called the *Lax pair*, after the fundamental work of Lax in the early seventies [50, 57].
This pair is not unique, but for several evolution PDEs arising in mathematical physics, there is a specific choice involving ODEs that are of first order in both the independent variables x and t, and indeed usually the x part of the Lax pair is precisely the ODE associated with the Inverse Scattering Transform, namely the ODE given by equation (2.6) but including the time dependence on the right hand side, so that $Q = Q(x,t)$ now. (This first ODE in

the Lax pair may be altered by changes of variable to regularise the λ dependence, see for example the elliptic sine-Gordon case, equation (3.4) below).
A useful Lax pair for several important evolution PDEs takes the general form

$$\begin{cases} M_x + if_1(\lambda)[\sigma_3, M] = Q(x,t,\lambda)M, \\ M_t + if_2(\lambda)[\sigma_3, M] = \tilde{Q}(x,t,\lambda)M, \end{cases} \tag{3.1}$$

where σ_3 and the commutator $[\cdot,\cdot]$ are defined in (2.5), (2.7).
The particular form of the functions $f_i(\lambda)$, $Q(x,t,\lambda)$, $\tilde{Q}(x,t,\lambda)$ depends on the specific PDE. Throughout this review, I will refer to two of the most common and important integrable evolution PDEs, arising as models in mathematical physics, namely the nonlinear Schrödinger (NLS) and modified Korteweg-deVries (mKdV) equations. The case of this second- and third-order evolution equation illustrate the general approach for integrable evolution PDEs in one space variable. For these equations, the Lax pair is given by the following choices:

NLS

$$\begin{aligned} &\textit{equation}: \qquad iq_t + q_{xx} - 2\nu q_x|q|^2 = 0, \qquad \nu = \pm 1; \\ &\textit{Lax pair}: \qquad f_1(\lambda) = \lambda, \quad f_2(\lambda) = 2\lambda^2, \\ &\qquad\qquad Q = \begin{pmatrix} 0 & q(x) \\ \nu\bar{q}(x) & 0 \end{pmatrix}, \qquad \tilde{Q} = 2\lambda Q - iQ_x\sigma_3 \pm |q|^2\sigma_3; \end{aligned} \tag{3.2}$$

mKdV

$$\begin{aligned} &\textit{equation}: \qquad q_t + q_{xxx} - 6\nu q^2 q_x = 0, \qquad \nu = \pm 1; \\ &\textit{Lax pair}: \qquad f_1(\lambda) = \lambda, \quad f_2(\lambda) = 4\lambda^3, \\ &\qquad\qquad Q = \begin{pmatrix} 0 & q(x,t) \\ \nu q(x,t) & 0 \end{pmatrix}, \qquad \tilde{Q} = 2Q^3 - Q_{xx} - 2i\lambda[Q^2 + Q_x]\sigma_3 + 4\lambda^2 Q. \end{aligned} \tag{3.3}$$

There exist also integrable PDEs of elliptic type, with independent variables denoted x and y. The best-known such equation is the so-called elliptic sine-Gordon equation

$$\textit{equation}: \qquad q_{xx} + q_{yy} - \sin q(x,y) = 0. \tag{3.4}$$

For this PDE, a convenient Lax pair is given by [33, 37, 54]

$$\begin{aligned} \textit{Lax pair}: \qquad & M_x + \frac{1}{4i}\left(\lambda - \frac{1}{\lambda}\right)[\sigma_3, M] = Q(x,y,\lambda)M, \\ & M_y + \frac{1}{4}\left(\lambda + \frac{1}{\lambda}\right)[\sigma_3, M] = iQ(x,y,-\lambda)M, \end{aligned} \tag{3.5}$$

with

$$Q(x,y,\lambda) = \frac{i}{4}\begin{pmatrix} \frac{1}{\lambda}(1-\cos q) & q_x - iq_y + \frac{i\sin q}{\lambda}. \\ q_x - iq_y - \frac{i\sin q}{\lambda} & -\frac{1}{\lambda}(1-\cos q) \end{pmatrix}.$$

There is no general methodology for finding a Lax pair associated with a given nonlinear PDE, although the dressing method of Shabat and Zakharov can be used to derive an integrable PDE by starting from a Lax pair for a given linear PDE [29, 57].

3.1 The solution of the Cauchy problem for integrable PDEs in 2 independent variables using the Inverse Scattering Transform

I have introduced all the ingredients needed to extend the strategy for solving the Cauchy problem for linear evolution equations to nonlinear integrable evolution equations such as the NLS or mKdV equations. I focus on the particular example of the NLS equation (3.2) to discuss how this generalisation is obtained.
The Lax pair formulation of the NLS equation, given explicitly by

$$\begin{cases} M_x + i\lambda[\sigma_3, M] = QM \\ M_t + 2i\lambda^2[\sigma_3, M] = (2\lambda Q - iQ_x\sigma_3 - i\nu|q|^2\sigma_3)M \end{cases} \qquad \lambda \in \mathbb{C}, \tag{3.6}$$

where $M = M(x,t,\lambda)$, implies that $q(x,t)$ solves the PDE (3.2) if and only if for all $\lambda \in \mathbb{C}$ there exists an invertible matrix-valued function $M(x,t,\lambda)$ solving (3.6), In practice, the PDE is obtained by imposing the compatibility condition $M_{xt} = M_{tx}$.
The first ODE in this pair is precisely the ODE associated with the nonlinear Fourier transform. The second part of the Lax pair is used to determine the time evolution of $M(x,t,\lambda)$, so that it is possible to write down an implicit expression for the solution $M(x,t,\lambda)$ of the Lax pair, in terms of a given initial condition $q(x,0) = q_0(x) \in \mathcal{S}(\mathbb{R})$.
Conversely, given $J(\lambda)$ the function $q(x,t)$ can be represented in terms of the elements of the matrix $M(x,t,\lambda)$ as

$$q(x,t) = 2i \lim_{|\lambda|\to\infty} (\lambda M_{12}(x,t,\lambda)).$$

where the function $M(x,t,\lambda)$, sectionally analytic for $\lambda \in \mathbb{C}$, is the solution of the Riemann-Hilbert problem determined by

$$M^- = M^+ \mathrm{e}^{(i\lambda x - i\lambda^2 t)\widehat{\sigma_3}} J(\lambda),\ \lambda \in \mathbb{R}, \qquad M = I + O\left(\frac{1}{\lambda}\right)\ as\ |\lambda| \to \infty,$$

with the jump matrix $J(\lambda)$ given by

$$J(\lambda) = \begin{pmatrix} 1 & -\gamma(\lambda) \\ \pm\overline{\gamma(\lambda)} & 1 + |\gamma(\lambda)|^2 \end{pmatrix}, \quad \lambda \in \mathbb{R}; \qquad \gamma(\lambda) = \frac{b(\lambda)}{\overline{a(\lambda)}}, \tag{3.7}$$

and the spectral functions $a(\lambda)$, $b(\lambda)$ are defined by in terms of the given initial condition $q_0(x)$ as

$$(b(\lambda), a(\lambda)) = \lim_{x\to-\infty} (m(x,\lambda)_1, m(x,\lambda)_2), \qquad m(x,\lambda) = \begin{pmatrix} M_{21}(x,0,\lambda) \\ M_{22}(x,0,\lambda) \end{pmatrix}$$

(to simplify the exposition, here and in the sequel it is assumed that $a(\lambda)$ has no zeros).
The vector $m(x,\lambda)$, as the second column vector of the solution of the first ODE in the Lax pair at $t = 0$, satisfies

$$m_x(x,\lambda) + 2i\lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} m(x,\lambda) = \begin{pmatrix} 0 & q_0(x) \\ \pm\bar{q_0} & 0 \end{pmatrix} m(x,\lambda),$$

$$\lim_{x\to\infty} m(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The expressions above make it apparent that the spectral functions $a(\lambda)$, $b(\lambda)$ depend only on the initial condition $q_0(x)$, and that the time dependence only enters (explicitly) through the exponential term.

4 An integral transform for linear boundary value problems

My aim is to extend the approach of the Inverse Scattering Transform to boundary value problems. I have already mentioned the difficulty of solving a given boundary value problem using the Fourier transform, or the inverse scattering transform in the nonlinear case. This approach, that works well for the initial value problem when the time evolution is explicit and depends only on the initial condition, runs into the difficult problem of eliminating unknown boundary values. This problem is not purely nonlinear - it features also for linear PDEs. I will start therefore to consider how this difficulty can be resolved in the linear case.

4.1 Lax pairs for linear PDEs in two variables and the global relation

For second order PDEs, such as the heat or the linearised Schrödinger equation, the difficulty just described can be easily overcome by using sine or cosine transforms. For example, the unique solution of the Dirichlet problem for the linear Schrödinger equation

$$iq_t + q_{xx} = 0, \quad q(x,0) = q_0(x), \quad q(0,t) = f_0(t), \qquad x > 0, \;\; 0 < t < T,$$

is given in terms of $q_0(x)$ and $f(t)$ by

$$q(x,t) = \frac{2}{\pi} \int_0^\infty \mathrm{e}^{-i\lambda^2 t} \sin(\lambda x) \left[\int_0^\infty \sin(\lambda\xi) q_0(\xi) d\xi + i\lambda \int_0^t \mathrm{e}^{i\lambda^2 s} f_0(s) ds \right] d\lambda. \tag{4.1}$$

However this is specific to even-order PDEs and does not extend to odd-order ones. Indeed, the elimination of unknown boundary values for linear evolution PDE of odd order in the space variable x cannot be achieved merely by real linear combinations of sine/cosine transforms. For example, for the third order evolution PDE (1.1) a representation as effective as (4.1) is given by (1.5) which involves evaluation at points in the complex plane of the transform of the initial condition $q_0(x)$.

The given formula obviously satisfies $q(x,0) = q_0(x)$, but a subtle calculation is needed to show that $\lim_{x\to 0} q(x,t) = f_0(t)$. Indeed, since the expression above is not uniformly convergent at $x = 0$, one cannot simply take the limit under the integral sign. In addition, this lack of uniformity poses serious problems when designing a numerical scheme to approximate the boundary values.

I now consider the general case of linear, constant-coefficient evolution PDE.
Let the function $q(x,t)$ be a smooth function, decaying as $x \to \infty$, and satisfying

$$q_t + Sq = 0, \qquad 0 < x < \infty, \quad 0 < t < T, \tag{4.2}$$

where S is a linear, constant-coefficient differential operator of the form

$$S = w(-i\partial_x), \tag{4.3}$$

where

$$w(\lambda) = \alpha_0 + \alpha_1\lambda + ... + \alpha_n\lambda^n, \quad \alpha_k \in \mathbb{C}, \; k = 1, ...n, \tag{4.4}$$

with the condition that $Re\, w(\lambda) \geq 0$ for $\lambda \in \mathbb{R}$. This condition ensures that the pure initial value problem is well posed, excluding cases such as the "wrong" heat equation $q_t + q_{xx} = 0$, corresponding to $w(\lambda) = -\lambda^2$, whose one-parameter family of solutions $\exp(i\lambda x + \lambda^2 t)$, with $\lambda \in \mathbb{R}$, grow exponentially in time. Taking the limit as $\lambda \to \infty$, this condition can be recognised as a condition on the leading coefficient α_n. Namely, for n odd, it is enough to require the condition $\alpha_n = \pm i$, while for n even, the condition is that $Re\, \alpha_n \geq 0$.

To obtain a boundary value problem which is well posed in the sense of admitting a unique solution valid for all times, it is also necessary to prescribe an initial condition and an appropriate number of boundary conditions [60]. In this section, the boundary conditions I consider will be of the form

$$\beta_0 q(0,t) + \beta_1 \partial_x q(0,t) + .. + \beta_{n-1} \partial_x^{n-1} q(0,t) = f(t), \qquad \beta_j \in \mathbb{R}, \tag{4.5}$$

where $f(t)$ is an appropriate given function. It is also possible in general to consider more general (nonlinear or nonlocal) boundary conditions, [51].
In order to solve such a problem in the spirit of the Inverse Scattering Transform, I start from two observations.

(1) Linear PDEs in two independent variables admit a Lax pair formulation
This fact was first formalised in [31], and it is rather straightforward. To select a convenient Lax pair, one starts from the fact that the ODE $\mu_x - i\lambda\mu = q$ is associated with the Fourier transform. Hence one considers this ODE as one of the equations forming the Lax pair and chooses the PDE itself, i.e. $\mu_t + S\mu = 0$, as the second equation. This immediately guarantees compatibility, namely that $\mu_{xt} = \mu_{tx}$, if and only if the function $q(x,t)$ solves the PDE.
To arrive at a more convenient Lax pair, one eliminates the x derivatives in the second equation using the first one. It is easy to show that the quotient

$$P(\lambda, \zeta) = \frac{w(\lambda) - w(\zeta)}{\lambda - \zeta} \tag{4.6}$$

is a polynomial in both ζ and λ. Let $F(x,t,\lambda) = iP(\lambda, -i\partial_x)q(x,t)$, where $P(\lambda,\zeta)$ is given by (4.6). Then

$$F(x,t,\lambda) = \sum_{k=0}^{n-1} c_k(\lambda) \partial_x^k q(x,t), \tag{4.7}$$

where $c_k(\lambda)$ are (up to a factor i) the coefficients of the polynomial $P(\lambda, \cdot)$.
An explicit computation proves the following.

Proposition 4.1 *The PDE (4.2) is equivalent to the compatibility of the Lax pair*

$$\begin{cases} \mu_x - i\lambda\mu = q, \\ \mu_t + w(\lambda)\mu = F(x,t,\lambda) \end{cases} \tag{4.8}$$

where $F(x,t,\lambda)$ is given by (4.7). Equivalently, the PDE can be formulated in the divergence form

$$\left(\mathrm{e}^{-i\lambda x + w(\lambda)t} q(x,t)\right)_t - \left(\mathrm{e}^{-i\lambda x + w(\lambda)t} F(x,t,\lambda)\right)_x = 0. \tag{4.9}$$

Once the PDE is cast in this form, involving the auxiliary complex parameter λ, the stage is set for deriving an appropriate transform. As stated earlier, instead of using only one ODE, this unified transform is determined *by both ODEs in the Lax pair simultaneously.* This transform, valid only for the specific boundary value problem at hand, by construction involves both x and t as parameters.
In this way one derives the following formal representation for the solution of the PDE.

Proposition 4.2 (The formal solution representation) *Define*

$$\tilde{F}(\lambda,t)=\int_0^t \mathrm{e}^{w(\lambda)s}F(0,s,\lambda)ds=\sum_{k=0}^{n-1}c_k(\lambda)\tilde{f}_k(\lambda,t) \qquad \lambda\in\mathbb{C}. \tag{4.10}$$

with $F(x,t,\lambda)$ given by (4.7), and

$$\tilde{f}_k(\lambda,t)=\int_0^t \mathrm{e}^{w(\lambda)s}\partial_x^k q(0,s)ds. \tag{4.11}$$

A solution $q(x,t)$ of the PDE (4.2) is given by

$$q(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{i\lambda x-w(\lambda)t}\hat{q}_0(\lambda)d\lambda-\frac{1}{2\pi}\int_{\partial D^+}\mathrm{e}^{i\lambda x-w(\lambda)t}\tilde{F}(\lambda,T)d\lambda, \tag{4.12}$$

where $\hat{q}_0(\lambda)$ is given by (1.6), and

$$D^+=\{\lambda\in\mathbb{C}^+:\ Re\, w(\lambda)=0\}. \tag{4.13}$$

The expression (4.12) provides a solution of the PDE for almost any reasonable choice of the functions $\tilde{f}_k(\lambda,t)$. Moreover, this expression also satisfies the given initial condition as soon as these functions have the exponential dependence on λ given in (4.11). However this expression will only satisfy given boundary conditions if the set of (suitable transforms of) all boundary values is assumed a priori to satisfy a certain constraint, namely the global relation defined below. Such a set of boundary values is called *admissible*. Without this additional constraint, the expression (4.12) will not in general satisfy prescribed boundary conditions.
It is also a formal representation, as in order to compute all terms appearing in this representation, the initial condition $q(x,0)$ and all the *boundary values* of the solution $q(0,t)$,..., $\partial_x^{n-1}q(0,t)$ must be known. However, only a subset of these values can be prescribed. Hence this representation does not determine fully the solution of a given boundary value problem.

(2) The Lax pair formulation in the given domain implies the global relation
There is an additional, important deduction that can be made starting from the formulation (4.8) (or (4.9)). Namely, using Green's theorem in the domain $\Omega=\{(x,t)\in\mathbb{R}^2: 0<x<\infty,\ \ 0<t<T\}$, one finds that appropriate integral transforms of the solution and of its x derivatives along $\partial\Omega$ must satisfy an algebraic relation. This relation is generally known as the *global relation*, as it constitutes a constraint on the transforms of *all initial and boundary values of the solution.* This constraint must be satisfied for all values of $\lambda\in\mathbb{C}$ for which the transforms are well defined.

Proposition 4.3 (The global relation) *Consider the PDE (4.2), and let $\tilde{F}(\lambda, t)$ be given by (4.10).*
Consider also the Fourier transform in x of the solution $q(x,t)$ at time t:

$$\hat{q}(\lambda, t) = \int_0^\infty \mathrm{e}^{-i\lambda y} q(y,t) dy, \qquad 0 \le t \le T, \quad \lambda \in \mathbb{C}^-. \tag{4.14}$$

Then for every $t \in (0,T)$, the functions $\hat{q}_0(\lambda)$ given by (1.6), $\tilde{F}(\lambda,t)$ and $\hat{q}(\lambda,t)$ satisfy the global relation

$$\tilde{F}(\lambda, t) = \hat{q}_0(\lambda) - \mathrm{e}^{w(\lambda)t} \hat{q}(\lambda, t), \quad \lambda \in \mathbb{C}^-. \tag{4.15}$$

The restriction $\lambda \in \mathbb{C}^-$ in (4.15) is needed for the integral defining the terms $\hat{q}_0(\lambda)$ and $\hat{q}(\lambda,t)$ to be well defined. In general, the functions involved in the global relation may have isolated singularities in λ, that can arise from specific boundary conditions or for problems posed on bounded intervals, and whose residues play an important role in the explicit characterisation of the solution. I will touch on this point later, for the case of problems posed on bounded intervals. See [51] for other examples.

The global relation is a necessary condition that the boundary and initial values must satisfy. It has been shown to be also a sufficient condition, in the sense that if a set of smooth (and decaying in x) initial and boundary values satisfies it, then the function given by (4.12) satisfies the PDE (4.2) and takes the given values at $x = 0$ and $t = 0$, see [7, 8]. In particular, and in contrast with the representation given as an integral along $\mathbb{R}$, the integral (4.12) is uniformly convergent at the boundary $x = 0$.

Remark 4.1 The global relation, in the linear context, is simply a different way of interpreting the equation obtained by taking the Fourier Transform of the PDE (in the evolution case, with respect to the space variable x). Indeed, instead of viewing (4.15) as an equation for the transform $\hat{q}(t,\lambda)$, I ignore this unknown term and view the equation as a relation between all initial and boundary values.
This shift in perspective is crucial, and is justified a posteriori by the fact that the terms involving the (unknown) solution $q(x,t)$ turn out to be analytic inside the contour of integration in the final representation and hence do not contribute to it. This is the key property that can be exploited by allowing λ to be complex, and can be proved to be valid in general, and not just for this particular PDE.

Remark 4.2 Using the usual Fourier transform, it is easy to derive a formal integral representation similar to (4.12) but with the contour ∂D^+ replaced by the real axis $\mathbb{R}$.
However, it is important to note that
(1) the contour ∂D^+ arises as the jump contour of the Riemann-Hilbert problem associated with the Lax pair formulation of the PDE;
(2) if the integration is performed along $\mathbb{R}$, *it is not possible to use the global relation to eliminate the unknown boundary values. This is the crucial advantage of moving the analysis away from $\mathbb{R}$ and into the λ complex plane.*

Similar results holds for linear elliptic PDEs, see [29].

4.2 The determination of the spectral data - generalised Dirichlet to Neumann map

To formulate and solve a boundary value problem for a PDE of the form (4.2), it is necessary to

(a) determine how many boundary conditions should be prescribed at the boundary in order to guarantee the existence of a unique solution of the problem;

(b) derive a solution representation that involves only the prescribed boundary conditions, and not all boundary values as in (4.12).

The answer to the question posed in (a) yields the number of boundary values that must be determined as part of the effective solution of the boundary value problem.
The determination of these unknown boundary values yields the answer to (b). I refer to the expression for the unknown boundary values in terms of the prescribed data as the *generalised Dirichlet to Neumann map.*

Since the Lax pair yields naturally a formal representation of the solution as an integral in the complex λ plane, one considers the global relation as an equation for $\lambda \in \mathbb{C}$. This is the crucial point for the proof of the main results below. I discuss the case of evolution problems posed on a half-line and on a finite interval.

Boundary value problems posed on a half line

The starting point of the Unified Transform approach to answering the two questions (a), (b) above is the global relation. The general result summarised below can be found in [29].

Proposition 4.4 *Let*

$$N = \begin{cases} n/2 & n \text{ even} \\ (n-1)/2 & n \text{ odd}, \ \alpha_n = i \\ (n+1)/2 & n \text{ odd}, \ \alpha_n = -i \end{cases} \tag{4.16}$$

A boundary value problem for the PDE (4.2)

$$q_t + Sq = 0, \quad x > 0, \ 0 < t < T, \qquad q(x,0) = q_0(x) \in \mathcal{S}(\mathbb{R}^+),$$

where S is defined by (4.3), admits a unique solution if and only if N independent boundary conditions of the form (4.5) are prescribed at $x = 0$.

For such a boundary value problem, the unique smooth solution, decaying as $x \to \infty$, can always be expressed by an integral in the complex plane, with integrand depending *explicitly* on the prescribed boundary conditions. Moreover, the contribution of the terms in the global relation that depend on $q(x,T)$ can always be discarded using analyticity arguments.
To avoid technicalities, I give the general statement in the case that the N prescribed conditions are the first N boundary functions $q(0,t)$, ..., $\partial_x^{N-1} q(0,t)$.

Theorem 4.1 *Consider the PDE*

$$q_t + Sq = 0, \quad x > 0, 0 < t < T; \qquad q(x,0) = q_0(x), \ x > 0;$$

where S is given by (4.3), $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$ is a given function, and there are N prescribed boundary conditions $\partial_x^j q(x,t)|_{x=0} = f_j(t)$, $j = 0, ..., N-1$, where N is given by (4.16). Consider the global relation (4.15), and write it in the form

$$\sum_{k=N}^{n-1} c_k(\lambda)\tilde{f}_k(\lambda,t) = \hat{q}_0(\lambda) - K(\lambda,t) - \mathrm{e}^{w(\lambda)t}\hat{q}(\lambda,t), \quad \lambda \in \mathbb{C}^+, \quad 0 < t \leq T, \tag{4.17}$$

where the functions $\tilde{f}_k$ are given by (4.11) and $K(\lambda,t)$ is the **known** *function*

$$K(\lambda,t) = \sum_{j=0}^{N-1} c_j(\lambda)\tilde{f}_j(\lambda,t), \tag{4.18}$$

with c_j's as in (4.7).
Then there exists a mapping $\mathcal{F}$ defined on the set of functions of λ analytic and bounded in D^+ to a suitable space of functions defined on ∂D^+ such that

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x - w(\lambda)t}\hat{q}_0(\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} \mathrm{e}^{i\lambda x - w(\lambda)t}\mathcal{F}\left[\hat{q}_0(\cdot) - K(\cdot,T)\right](\lambda)d\lambda. \tag{4.19}$$

where $\hat{q}_0(\lambda)$ is given by (1.6) and ∂D^+ is given by (4.13).

In this form, the theorem is a rewording of the theorem stated in [56]. For the proof, see also [60].

Brief sketch of the proof:
The derivation of this result depends on the analysis of the global relation considered as an equation valid for $\lambda \in \mathbb{C}^-$.
Start with the global relation written explicitly in terms of all individual function $\tilde{f}_k(\lambda,t)$ defined by (4.11):

$$\mathrm{e}^{w(\lambda)t}\hat{q}(\lambda,t) - \hat{q}_0(\lambda) - \sum_{k=0}^{n-1} c_k\tilde{f}_k(\lambda,t) = 0,$$

where the coefficients c_k's are given in (4.10). Make the important observation that for all $k = 0, ..., n-1$, $\tilde{f}_k(\lambda,t)$ depends on λ only through $w(\lambda)$. Now denote by $\theta : \mathbb{C} \to \mathbb{C}$ any transformation of the complex plane such that

$$w(\theta(\lambda)) = w(\lambda), \quad \forall \lambda \in \mathbb{C},$$

where $w(\lambda)$ is given in (4.4). Such transformations also satisfy $f_k(\theta(\lambda),t) = f_k(\lambda,t)$.
Since $w(\lambda)$ is a polynomial, such θ's are given by the $n-1$ nontrivial (i.e. different from λ) roots of the polynomial (in θ)

$$w(\theta) - w(\lambda) = 0.$$

Any such transformation leaves invariant the domain

$$D = \{\lambda \in \mathbb{C} : \ Re\, w(\lambda) < 0\}, \qquad D^{\pm} = D \cap \mathbb{C}^{\pm}, \tag{4.20}$$

and it is not difficult to show that it effects a permutation of the connected components of D. The study of these transformations is the key ingredient in the proof of Proposition 4.4,

as well as in the derivation of a solution representation involving only the known boundary conditions.
The representation (4.12) involves *all* functions $\tilde{f}_k(\lambda,t)$, $k=0,..,n-1$, evaluated for $\lambda \in \partial D^+$. On the other hand, the global relation (4.15) is only valid in $\mathbb{C}^-$. The main idea involved in the derivation of (4.19) from (4.12) is to make use only of the transformations θ that map the connected components of D^+ into D. Indeed, it can be shown that while there are $n-1$ transformations $\theta_j(\lambda)$ leaving $w(\lambda)$ invariant (excluding the trivial one $\theta(\lambda)=\lambda$), for each connected component of D^+ only $n-N$ of these map the given component into C^-; namely (upon relabelling)

$$\lambda \in D^+ \Longrightarrow \theta_j(\lambda) \in C^-, \quad j=1,..,n-N.$$

Recalling that $\tilde{f}_k(\theta_j(\lambda)) = \tilde{f}_j(\lambda)$, by evaluating the global relation at the values $\theta_j(\lambda)$ one finds a system of $n-N$ equations for the $n-N$ unknown boundary values:

$$\begin{aligned}
\sum_{k=N}^{n-1} c_k(\theta_1(\lambda))\tilde{f}_k(\lambda,t) &= \hat{q}_0(\theta_1(\lambda)) - K(\theta_1(\lambda),t) - \mathrm{e}^{w(\lambda)t}\hat{q}(\theta_1(\lambda),t) \\
&\cdots \\
&\cdots \\
\sum_{k=N}^{n-1} c_k(\theta_{n-N}(\lambda))\tilde{f}_k(\lambda,t) &= \hat{q}_0(\theta_{n-N}(\lambda)) - K(\theta_{n-N}(\lambda),t) - \mathrm{e}^{w(\lambda)t}\hat{q}(\theta_{n-N}(\lambda),t)
\end{aligned}$$

The solution of this system can be given using Cramer's rule to write each of the functions $\tilde{f}_k(\lambda,t)$, $k=N,...n-1$, in terms of the known function on the right hand side and the determinant of the system, which can be shown to have no zeros. This solution defines the mapping $\mathcal{F}$ appearing in (4.19).
It would seem that this solution involves also the terms $\hat{q}(\theta_j(\lambda),t)$ that are *unknown*. However, it can be proved that the combination of these terms appearing in the solution is always *bounded and analytic* in D^+, in fact that these terms have decay of order $O\left(\frac{1}{\lambda}\right)$ as $\lambda \to \infty$, so that their integral along ∂D^+ vanishes. Hence these are *ghost terms* that do not contribute to the integral in (4.19), and can be ignored for the purpose of representing the solution in the form (4.19).
Finally, analyticity arguments can be used to substitute t with the final time T in the term $K(\lambda,t)$ appearing in the final integral representation.
QED

Remark 4.3 There is considerable flexibility in deforming the contour of integration in (4.12) and (4.19). In the expression (4.12), the contour can be deformed to any contour that is asymptotic to ∂D^+, as the integrand is analytic inside any bounded ball centred at the origin. In the expression (4.19), exploiting the analyticity properties of the specific $\hat{q}_0(\lambda)$, there may considerable scope to deform the integration contour, for example to achieve optimal efficiency in numerical evaluations [24, 61].

Example: the equation $q_t + q_{xxx} = 0$

As an illustration, consider this specific example, already discussed in the previous section.

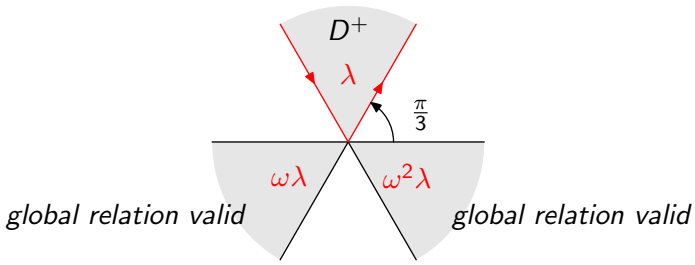

Figure 3: The contours ∂D^+, ∂D^- for the equation $q_t + q_{xxx} = 0$

Then $w(\lambda) = i\lambda^3$, and the invariant transformations are given by the roots of polynomial $\theta^3 = \lambda^3$, namely

$$\theta_0(\lambda) = \lambda, \quad \theta_1(\lambda) = \omega\lambda, \quad \theta_2(\lambda) = \omega^2\lambda, \qquad \omega = \mathrm{e}^{2\pi i/3}. \tag{4.21}$$

In this case, there are two unknown terms, involving the boundary values $q_x(0,t)$ and $q_{xx}(0,t)$ that cannot be prescribed arbitrarily. Starting from the analysis of the global relation, it is possible to determine an explicit expression for the transforms (4.11) of these terms, namely

$$\lambda\tilde{f}_1(\lambda) = \frac{1}{\omega - 1}[\omega\tilde{K}(\omega\lambda) - \tilde{K}(\omega^2\lambda)], \quad \tilde{f}_2(\lambda) = i\omega^2\frac{1}{1-\omega}[\tilde{K}(\omega^2\lambda) - \tilde{K}(\omega\lambda)], \tag{4.22}$$

where the dependence on $t = T$ has been dropped, and

$$\tilde{K}(\lambda) = \hat{q}_0(\lambda) + K(\lambda), \quad K(\lambda) = \lambda^2\tilde{f}_0(\lambda).$$

Indeed, for these functions of the spectral parameter λ the global relation reads:

$$\tilde{f}_2(\lambda) + i\lambda\tilde{f}_1(\lambda) = \hat{q}_0(\lambda) + \lambda^2\tilde{f}_0(\lambda) - \mathrm{e}^{i\lambda^3 T}\hat{q}(\lambda, T), \qquad \lambda \in \mathbb{C}^- \tag{4.23}$$

with $\hat{q}(\lambda, T)$ given by (4.14), and

$$\tilde{f}_i(\omega\lambda) = \tilde{f}_i(\omega^2\lambda) = \tilde{f}_i(\lambda), \qquad \omega = \mathrm{e}^{2\pi i/3}.$$

Note also that if $\lambda \in \partial D^+$, then $\omega\lambda, \omega^2\lambda \in \mathbb{C}^-$, see Figure 3. Hence since the global relation (4.23) holds in $\mathbb{C}^-$, by evaluating it at these two points for some $\lambda \in \partial D^+$ one finds

$$\begin{aligned}
\tilde{f}_2(\lambda) + i\omega\lambda\tilde{f}_1(\lambda) &= \omega^2\lambda^2\tilde{f}_0(\lambda) + \hat{q}_0(\omega\lambda) - \mathrm{e}^{i\lambda^3 T}\hat{q}(\omega\lambda, T), \\
\tilde{f}_2(\lambda) + i\omega^2\lambda\tilde{f}_1(\lambda) &= \omega\lambda^2\tilde{f}_0(\lambda) + \hat{q}_0(\omega^2\lambda) - \mathrm{e}^{i\lambda^3 T}\hat{q}(\omega\lambda, T).
\end{aligned} \tag{4.24}$$

I stress that what makes this approach successful is that, after solving these equations for the functions $\tilde{f}_1$ and $\tilde{f}_2$, the terms involving the unknown $\hat{q}(\cdot, T)$ are *guaranteed* to be analytic and bounded in D^+, hence by an application of Cauchy's theorem, or more specifically Jordan's lemma, the contribution of these terms to the integral representation of the solution disappears. For this example, one obtains the expressions (4.22) and the representation (1.4).

The mapping $\mathcal{F}$ of theorem 4.1 (assuming for convenience that the homogeneous boundary condition $q(0,t)=0$ is prescribed, hence that $\tilde{K}(\lambda)=\hat{q}_0(\lambda)$) is given for this example by

$$\mathcal{F}[\tilde{K}](\lambda)=\omega\hat{q}_0(\omega\lambda)+\omega^2\hat{q}_0(\omega^2\lambda)=\frac{1}{2\pi}\int_0^\infty\left[\omega\mathrm{e}^{-i\omega\lambda x}+\omega^2\mathrm{e}^{-i\omega^2\lambda x}\right]q_0(x)dx,\quad \lambda\in\partial D^+. \tag{4.25}$$

Remark 4.4 In the example above, it may seem important that the invariant transformations $\theta_k(\lambda)$ have the particularly simple and explicit form of rotations. However, even when it is not possible to find these transformations explicitly, the case above (or rather its generalisation for the n-th order PDE $q_t+(-i\partial_x)^n q=0$) is the essence of the general case. Indeed, as noted in Remark 4.3, analyticity implies that it is possible to deform the contour ∂D^+ to a contour which asymptotically, as $\lambda\to\infty$, approaches the rays defined by

$$Re(\lambda+\frac{\alpha_{n-1}}{\alpha_n})^n=0.$$

and the transformation between the domains defined by these rays are essentially rotations of an angle multiple of $2\pi/n$ [29, 53].

Boundary value problems posed on a finite interval

The general formal representation of the solution of a boundary value problem for the PDE (4.2) posed on a finite interval now takes the form

$$q(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{i\lambda x-w(\lambda)t}\hat{q}_0(\lambda)d\lambda-\frac{1}{2\pi}\int_{\partial D^+}\mathrm{e}^{i\lambda x-w(\lambda)t}\tilde{F}(\lambda,t)d\lambda, \tag{4.26}$$

$$-\frac{1}{2\pi}\int_{\partial D^-}\mathrm{e}^{i\lambda(x-L)-w(\lambda)t}\tilde{G}(\lambda,t)d\lambda,\quad 0<x<L,\ \ 0<t<T,$$

where $\tilde{F}$ is given by (4.10), $\tilde{G}(\lambda,t)$ is an analogous function but involving the boundary values at $x=L$:

$$\tilde{G}(\lambda,t)=\int_0^t\mathrm{e}^{w(\lambda)s}Q(L,s,\lambda)ds=\sum_{k=0}^{n-1}c_k(\lambda)\tilde{g}_k(\lambda,t)\qquad\lambda\in\mathbb{C}. \tag{4.27}$$

$$\tilde{g}_k(\lambda,t)=\int_0^t\mathrm{e}^{w(\lambda)s}\partial_x^{k-1}q(L,s)ds, \tag{4.28}$$

and $D^-=D\cap\mathbb{C}^+$ for D given by (4.20).

This integral representation is seemingly at odds with the usual series representation for two-point boundary value problems. I shall discuss below how this is not the case. Indeed, as for the case of the half-line, the system obtained for the unknown boundary values contains always terms involving the unknown function $q(x,t)$, namely the value of the solution one is trying to represent. The significant difference between the half-line and the finite interval case is that in the latter case, these terms *do not always disappear*. Hence the resulting terms in the solution of the system may be meromorphic rather than analytic functions in

the domain D. In this case, the evaluation of the residue of these functions at the poles gives rise to the usual series representation.

The global relation is now the following equation, valid for all $\lambda \in \mathbb{C}$:

$$\tilde{F}(\lambda,t) - \mathrm{e}^{-i\lambda L}\tilde{G}(\lambda,t) = \hat{q}_0(\lambda) - \mathrm{e}^{w(\lambda)t}\hat{q}(\lambda,t), \quad \lambda \in \mathbb{C},\ 0 < t \leq T. \tag{4.29}$$

Using the analysis of the global relation just as for the half-line case, it is possible to determine the value of $\tilde{F}(\lambda,t)$ and $\tilde{G}(\lambda,t)$ in terms only of the given initial and boundary conditions.

Proposition 4.5 *A boundary value problem for the equation (4.2) posed for $0 < x < L$, $0 < t < T$ satisfying the initial condition $q(x,0) = q_0(x) \in C^\infty(0,L)$, admits a unique solution if and only if N boundary conditions $\partial_x^j q(x,t)|_{x=0} = f_j(t)$, $j = 0,...,N-1$ are prescribed at $x = 0$ and $n-N$ conditions $\partial_x^k q(x,t)|_{x=L} = g_k(t)$, $j = 0,...,n-N-1$ are prescribed at $x = L$, where N is given by (4.16).*
Consider the global relation (4.15), and write it in the form

$$\sum_{k=N}^{n-1} c_k(\lambda)\tilde{f}_k(\lambda,t) - \mathrm{e}^{-i\lambda L}\sum_{k=n-N}^{n-1} c_k(\lambda)\tilde{g}_k(\lambda,t) = \hat{q}_0(\lambda) - K(\lambda,t) + \mathrm{e}^{-i\lambda L}H(\lambda,t) - \mathrm{e}^{w(\lambda)t}\hat{q}(\lambda,t), \tag{4.30}$$

where $\tilde{f}_k$, $\tilde{g}_k$ and $K(\lambda,t)$ are given by (4.11), (4.28) and (4.18) respectively, and $H(\lambda,t)$ is the known function

$$H(\lambda,t) = \sum_{j=0}^{n-N-1} c_j(\lambda)\tilde{g}_j(\lambda,t). \tag{4.31}$$

Then there exists mappings $\mathcal{F}$, $\mathcal{G}$ defined on the set of functions of λ analytic and bounded in $D^\pm$ to a suitable space of functions defined on $\partial D^\pm$ respectively, such that

$$\begin{aligned} q(x,t) \;=\; & \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x - w(\lambda)t}\hat{q}_0(\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} \mathrm{e}^{i\lambda x - w(\lambda)t}\mathcal{F}\left[\hat{q}_0(\cdot) - K(\cdot,T) - H(\cdot,T)\right](\lambda)d\lambda \\ & -\frac{1}{2\pi}\int_{\partial D^-} \mathrm{e}^{i\lambda(x-L) - w(\lambda)t}\mathcal{G}\left[\hat{q}_0(\cdot) - K(\cdot,T) - H(\cdot,T)\right](\lambda)d\lambda. \end{aligned} \tag{4.32}$$

where $\hat{q}_0(\lambda)$ is given (1.6) and $D^\pm = D \cap \mathbb{C}^\pm$ are given by (4.20).

Sketch of the proof: In this case, the functions $\tilde{F}(\lambda,t)$, $\tilde{G}(\lambda,t)$, hence the mappings $\mathcal{F}$, $\mathcal{G}$, can be determined in terms of the given data by solving the system of n equations for the n unknown boundary values obtained by evaluating the global condition at λ and at the $n-1$ nontrivial roots of the polynomial

$$w(\theta_k(\lambda)) = w(\lambda),\ k = 1,...,n-1 \qquad \lambda \in \mathbb{C}. \tag{4.33}$$

Namely, the system is

$$\begin{aligned}
&\sum_{k=N}^{n-1} c_k(\lambda)\tilde{f}_k(\lambda,t) - \mathrm{e}^{-i\lambda L}\sum_{k=n-N}^{n-1} c_k(\lambda)\tilde{g}_k(\lambda,t) = \hat{q}_0(\lambda) - K(\lambda,t) - H(\lambda,t) \\
&\qquad\qquad -\mathrm{e}^{w(\lambda)t}\hat{q}(\lambda,t), \\
&\sum_{k=N}^{n-1} c_k(\theta_1)\tilde{f}_k(\lambda,t) - \mathrm{e}^{-i\theta_1 L}\sum_{k=n-N}^{n-1} c_k(\theta_1)\tilde{g}_k(\lambda,t) = \hat{q}_0(\theta_1) - K(\theta_1,t) - H(\theta_1,t) \\
&\qquad\qquad -\mathrm{e}^{w(\lambda)t}\hat{q}(\theta_1,t), \\
&.... \\
&.... \\
&\sum_{k=N}^{n-1} c_k(\theta_{n-1})\tilde{f}_k(\lambda,t) - \mathrm{e}^{-i\theta_{n-1} L}\sum_{k=n-N}^{n-1} c_k(\theta_{n-1})\tilde{g}_k(\lambda,t) = \hat{q}_0(\theta_{n-1}) - K(\theta_{n-1},t) - H(\theta_{n-1},t) \\
&\qquad\qquad -\mathrm{e}^{w(\lambda)t}\hat{q}(\theta_{n-1},t),
\end{aligned} \tag{4.34}$$

Let $\Delta(\lambda)$ denote the determinant of the system above. The the solution of the system (4.34) involves terms of the form $\frac{\mathrm{e}^{w(\lambda)t}}{\Delta(\lambda)}\Lambda[\hat{q}(\lambda,t),\hat{q}(\theta_1,t),..,\hat{q}(\theta_{n-1},t)]$, Λ a polynomial function.

There are two cases:

(case 1): $\Delta(\lambda)$ has no zeros in $\bar{D}$

In this case, the unknown terms involved in the solution of the system (4.34) are analytic in D, and therefore they do not contribute to the integral representation (4.32) (just as in the case of problem posed on the half line).

(case 2): $\Delta(\lambda)$ has zeros in $\bar{D}$

In this case, *these terms are meromorphic rather than bounded analytic functions*, hence in this case the residue at the zeros of Δ must be computed. These residues can be computed explicitly in terms of the values of the two function $K(\lambda,t)$ and $H(\lambda,t)$. It turns out that all other terms cancel and the series obtained from the residues is the only contribution remaining.

QED

The integral representation (4.32), for the well-known cases of two-point boundary value problems such as the heat equation, posed on a finite interval, yields the solution in terms of Fourier series via the residue computation described in (case 2 of) the sketch of the proof just given - at least for sufficiently smooth and separable boundary conditions. However, the above theorem hints that this is not the general case. Namely, *there exist two-point boundary value problems for linear evolution PDE whose solution does not admit a series representation.*

The crucial quantity involved in the solution, and characterising the two cases, is the determinant $\Delta(\lambda)$ of the system (4.34).

Proposition 4.6 *The determinant $\Delta(\lambda)$ of the system (4.34) is a polynomial of exponential type of the form*

$$\Delta(\lambda) = a_0\mathrm{e}^{i\lambda L} + a_1\mathrm{e}^{i\theta_1(\lambda)L} + ... + a_{n-1}\mathrm{e}^{i\theta_{n-1}(\lambda)L} + b_0\mathrm{e}^{-i\lambda L} + b_1\mathrm{e}^{-i\theta_1(\lambda)L} + ... + b_{n-1}\mathrm{e}^{-i\theta_{n-1}(\lambda)L} \tag{4.35}$$

where $a_k, b_k \in \mathbb{C}$, $k = 0, ..., n-1$.

The above discussion makes it clear that it is of crucial importance to determine the location of the zeros of the function $\Delta(\lambda)$. General results in complex analysis [49] allow one to determine the asymptotic location of the zeros of $\Delta(\lambda)$ for large λ and this is sufficient to give a complete characterisation of the solution of these boundary value problems.

The approach discussed here provides a constructive criterion to distinguish between boundary conditions that yield a self-adjoint spectral problem, and those that do not. In the latter case, the integral representation of the solution is not equivalent to a series representation. The first results on this phenomenon were presented in [35, 36, 53], but recently there has been a rigorous treatment of this issue from the point of view of classical spectral theory [56].

Series versus integral representation - an example

For the illustrative case of the PDE (1.1), two cases of boundary conditions exemplify the situation. Suppose that one has

$$q_t + q_{xxx} = 0, \;\; 0 < x < 1, \;\; 0 < t < T, \qquad q_0(x) = q_0(x), \;\; 0 < x < 1, \tag{4.36}$$

and either of the following two sets of boundary conditions:

$$(A) \qquad q(0,t), \; q(1,t), \; q_x(0,t) \;\; given \tag{4.37}$$

$$(B) \qquad q(0,t) \; q(1,t) \;\; given, \quad q_x(0,t) = \beta q_x(1,t), \; \beta \in \mathbb{R} \setminus \{0\}. \tag{4.38}$$

In both cases, since $w(\lambda) = \lambda^3$ for this PDE, the solution is obtained as an integral along the boundary of the region

$$D = \{\lambda \in \mathbb{C} : Im(i\lambda^3) < 0\}. \tag{4.39}$$

The global relation is now (dropping the dependence on t)

$$\tilde{f}_2(\lambda) + i\lambda\tilde{f}_1(\lambda) - \lambda^2\tilde{f}_0(\lambda) - \mathrm{e}^{-i\lambda}\left[\tilde{g}_2(\lambda) + i\lambda\tilde{g}_1(\lambda) - \lambda^2\tilde{g}_0(\lambda)\right] = \hat{q}_0(\lambda) - \mathrm{e}^{i\lambda^3 t}\hat{q}(\lambda, t), \quad \lambda \in \mathbb{C} \tag{4.40}$$

where $\tilde{f}_k$ represent transforms of the solution evaluated at $x = 0$ and $\tilde{g}_k$ represent transforms of the solution evaluated at $x = 1$, $k = 0, 1, 2$. Note that now the global relation is valid for all $\lambda \in \mathbb{C}$, as all functions involved are entire functions of λ.

The unknown boundary values can be obtained as in the half-line case by solving a system, for each fixed $\lambda \in \mathbb{C}$, of three equations for three unknowns obtained by evaluating the

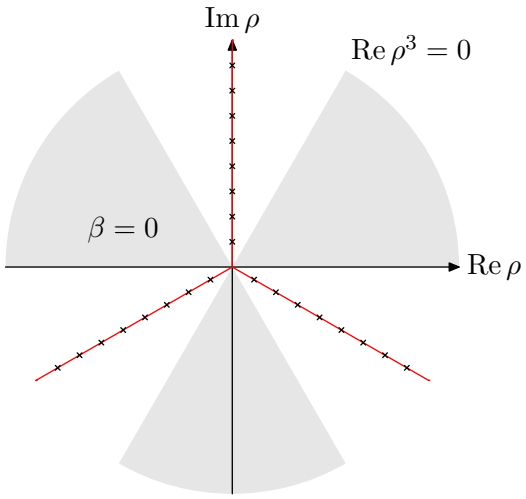

Figure 4: The location of the zeros of $\Delta(\lambda)$ for case (A)

global relation at the three roots given by (4.21):

(A) :

$$\left\{\begin{array}{lcl} \tilde{f}_2(\lambda) + i\lambda\tilde{f}_1(\lambda) - \mathrm{e}^{-i\lambda}\tilde{g}_2(\lambda) & = & \hat{q}_0(\lambda) + K(\lambda) - \mathrm{e}^{i\lambda^3 t}\hat{q}(\lambda,t), \\ \tilde{f}_2(\lambda) + i\omega\lambda\tilde{f}_1(\lambda) - \mathrm{e}^{-i\omega\lambda}\tilde{g}_2(\lambda) & = & \hat{q}_0(\omega\lambda) + K(\omega\lambda) - \mathrm{e}^{i\lambda^3 t}\hat{q}(\omega\lambda,t), \\ \tilde{f}_2(\lambda) + i\omega^2\lambda\tilde{f}_1(\lambda) - \mathrm{e}^{-i\omega^2\lambda}\tilde{g}_2(\lambda) & = & \hat{q}_0(\omega^2\lambda) + K(\omega^2\lambda) - \mathrm{e}^{i\lambda^3 t}\hat{q}(\omega^2\lambda,t), \end{array}\right.$$

$$\Big(with \;\; K(\lambda) = \lambda^2\tilde{f}_0(\lambda) + \mathrm{e}^{-i\lambda}[i\lambda\tilde{g}_1(\lambda) - \lambda^2\tilde{g}_0(\lambda)]\Big)$$

(B) :

$$\left\{\begin{array}{lcl} \tilde{f}_2(\lambda) + \mathrm{e}^{-i\lambda}\left[i\lambda(\beta-1)\tilde{g}_1(\lambda) - \tilde{g}_2(\lambda)\right] & = & \hat{q}_0(\lambda) + K(\lambda) - \mathrm{e}^{i\lambda^3 t}\hat{q}(\lambda,t), \\ \tilde{f}_2(\lambda) + \mathrm{e}^{-i\omega\lambda}\left[i\omega\lambda(\beta-1)\tilde{g}_1(\lambda) - \tilde{g}_2(\lambda)\right] & = & \hat{q}_0(\omega\lambda) + K(\omega\lambda) - \mathrm{e}^{i\lambda^3 T}\hat{q}(\omega\lambda,t), \\ \tilde{f}_2(\lambda) + \mathrm{e}^{-}i\omega^2\lambda\left[i\omega^2\lambda(\beta-1)\tilde{g}_1(\lambda) - \tilde{g}_2(\lambda)\right] & = & \hat{q}_0(\omega^2\lambda) + K(\omega^2\lambda) - \mathrm{e}^{i\lambda^3 t}\hat{q}(\omega^2\lambda,t), \end{array}\right.$$

$$\Big(with \;\; K(\lambda) = \lambda^2\tilde{f}_0(\lambda) + \mathrm{e}^{-i\lambda}\lambda^2\tilde{g}_0(\lambda)\Big)$$

The determination of the spectral functions $\tilde{F}$ and $\tilde{G}$ in terms of given initial and boundary data only is obtained by solving the system - in this example, the explicit system in (A) or (B).

Case (A) The determinant of this linear system is

$$\Delta(\lambda) = (\omega - \omega^2)[\mathrm{e}^{-i\lambda} + \omega\mathrm{e}^{-i\omega\lambda} + \omega^2\mathrm{e}^{-i\omega^2\lambda}]$$

The zeros lie asymptotically on the rays bisecting the three connected components of D^c, see Figure 4.

These zeros are outside D and in fact it is not possible to deform the integration contour to include them - the integral representation on this case is *not equivalent to a series representation.*

Case (B) The determinant of this linear system is

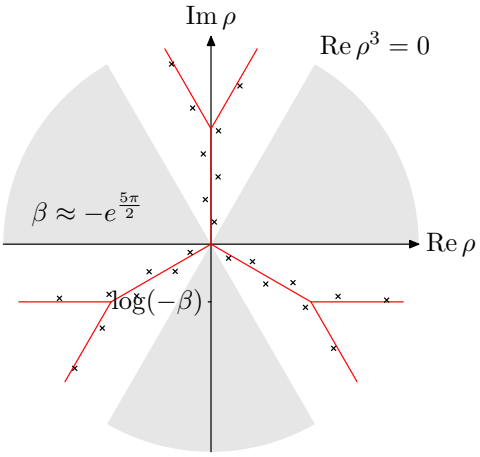

Figure 5: The location of the zeros of $\Delta(\lambda)$ for case (B) - $\beta \sim -\mathrm{e}^{5\pi/2}$

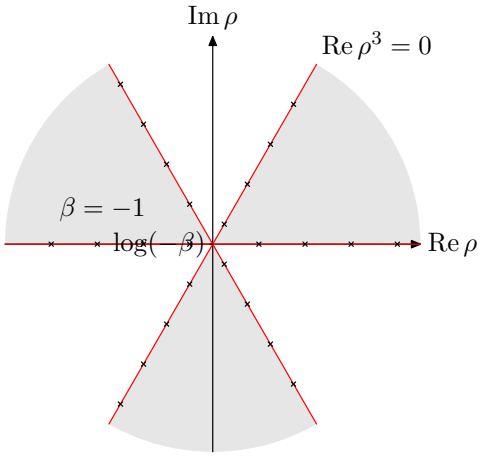

Figure 6: The location of the zeros of $\Delta(\lambda)$ for case (B) - $\beta = -1$

$$\Delta(\lambda) = (\omega - \omega^2)[\mathrm{e}^{-i\lambda} + \omega \mathrm{e}^{-i\omega\lambda} + \omega^2 \mathrm{e}^{-i\omega^2\lambda} + \beta(\mathrm{e}^{i\lambda} + \omega \mathrm{e}^{i\omega\lambda} + \omega^2 \mathrm{e}^{i\omega^2\lambda})]$$

The zeros lie asymptotically on ∂D, see Figures 5 and 6 for the example of two particular values of β. Therefore the residues at these zeros must be computed and yield the series representation.

5 An integral transform for nonlinear boundary value problems

The Unified Transform to solve boundary value problems for linear equations, outlined in the previous section, is based on deriving an integral transform through a Riemann-Hilbert problem associated with both ODE in the Lax pair. This transform yields (1) an integral representation for the solution and (2) a global relation among certain transforms of the boundary values.

Since the starting point for the approach above is the Lax pair formulation of the PDE, it seems natural to expect that this construction can be generalised to the case of nonlinear integrable equations, that are precisely those characterised by a Lax pair formulation.

I will consider the case of evolution equations posed on a half-line, as this case gives the full flavour of the techniques and results.

Consider an integrable nonlinear PDE in the variables $x \in \mathbb{R}$ and $t \in \mathbb{R}$, posed in the domain

$$\Omega = \{(x,t) : 0 < x < \infty,\ 0 < t < T\} \subset \mathbb{R}^2. \tag{5.1}$$

Example of such equations are the NLS, KdV and mKdV equations given by (3.2), (1.7), (1.8) respectively, as well as many other important equations of mathematical physics.
As for the linear case, it is indeed possible to derive, starting from the Lax pair, a global relation and a formal integral representation, that is now implicitly characterised through a singular linear integral equation.

I have discussed how the idea of the simultaneous spectral analysis of the Lax pair is implemented for the case of linear evolution PDEs, namely by formulating and solving a Riemann-Hilbert problem. The approach can be generalised and remains conceptually the same for the nonlinear case. However, there is an important technical difference: the associated Riemann-Hilbert problem in the nonlinear case is matrix-valued, hence non commutative, rather than scalar as for the linear case. The lack of commutativity of the Riemann-Hilbert problem implies that it is not possible to write down explicit formulas.

To illustrate the difference, I sketch the case of the defocusing NLS equation, namely equation (3.2) with $\nu = 1$ (more details on the construction of the solution representation are given below in section 5.1). Then, in analogy with the case of the full line in section 3.1, the role of the Fourier transform of the initial condition $\hat{q}_0(\lambda)$, where $q_0(x) = q(x,0)$ is played by the spectral data defined in terms of the initial information. These spectral data are the pair of functions $a(\lambda)$, $b(\lambda)$ such that

$$\begin{pmatrix} b(\lambda) \\ a(\lambda) \end{pmatrix} = \begin{pmatrix} M_{21}(x,0,\lambda) \\ M_{22}(x,0,\lambda) \end{pmatrix}.$$

where $M(x,0,\lambda)$ is the solution of the x part of the Lax pair, i.e. the first ODE in (3.6), evaluated at $t = 0$. It is useful to express more transparently the relation between $q_0(x)$ and $a(\lambda), b(\lambda)$. To conform with the notation in the literature cited, set

$$\Psi(x,\lambda) = M(x,0,\lambda).$$

The function $\Psi(x,\lambda)$ has symmetries inherited from the matrices Q, $\tilde{Q}$, so it is enough to consider one of its columns. For example, the second column of $\Psi(x,\lambda)$ satisfies, for $0 < x < \infty$, the following ODE:

$$\begin{aligned} &\partial_x \begin{pmatrix} \Psi_{21}(x,\lambda) \\ \Psi_{22}(x,\lambda) \end{pmatrix} + 2i\lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \psi(x,\lambda) = \begin{pmatrix} 0 & q_0(x) \\ \overline{q_0}(x) & 0 \end{pmatrix} \begin{pmatrix} \Psi_{21}(x,\lambda) \\ \Psi_{22}(x,\lambda) \end{pmatrix}, \quad \lambda \in \mathbb{C}^+ \\ &\lim_{x\to\infty} \begin{pmatrix} \Psi_{21}(x,\lambda) \\ \Psi_{22}(x,\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \lambda \in \mathbb{C}^+. \end{aligned} \tag{5.2}$$

The solution of these ODEs is equivalent to a linear Volterra integral equation, hence well defined.

Similarly, the boundary conditions $q(0,t), q_x(0,t)$ determine a pair of functions $A(\lambda)$, $B(\lambda)$ such that

$$\begin{pmatrix} B(\lambda) \\ A(\lambda) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{2i\lambda^2 T} M_{21}(0,T,\lambda) \\ M_{22}(0,T,\lambda) \end{pmatrix},$$

where $M(0,t,\lambda)$ is the solution of the t part of the Lax pair, i.e. the second ODE in (3.6), evaluated at $x = 0$. It is desirable to express the relation between boundary values and these spectral functions, and to conform with the usual notation, set

$$\Phi(t,\lambda) = M(0,t,\lambda).$$

Again, symmetry considerations imply that it is enough to consider one column of this function. For example, the second column of $\Phi(t,\lambda)$ satisfies, for $0<t<T$ and $\lambda\in\mathbb{C}$, the following ODE:

$$\partial_t \begin{pmatrix} \Phi_{21}(t,\lambda) \\ \Phi_{22}(t,\lambda) \end{pmatrix} + 4i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi_{21}(t,\lambda) \\ \Phi_{22}(t,\lambda) \end{pmatrix} = \\ \begin{pmatrix} -i|q(0,t)|^2 & -iq_x(0,t)+2\lambda q(0,t) \\ -i\overline{q_x}(0,t)+2\lambda\overline{q}(0,t) & i|q(0,t)|^2 \end{pmatrix} \begin{pmatrix} \Phi_{21}(t,\lambda) \\ \Phi_{22}(t,\lambda) \end{pmatrix},$$

$$\begin{pmatrix} \Phi_{21}(0,\lambda) \\ \Phi_{22}(0,\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \lambda\in\mathbb{C}, \tag{5.3}$$

which again is equivalent to a linear Volterra integral equation, hence well defined.
Note that since in general only one boundary condition involving the two boundary values $q(0,t)$ and $q_x(0,t)$ can be prescribed, the boundary spectral functions $A(\lambda)$, $B(\lambda)$ are not fully characterised by the above ODE.

Nevertheless the solution $q(x,t)$ has a formal representation in terms of the solution $M(x,t,\lambda)$ of a Riemann-Hilbert problem defined on the real and imaginary axes, whose jump is defined in terms of the spectral functions $a(\lambda)$, $b(\lambda)$, $A(\lambda)$ and $B(\lambda)$. The function $q(x,t)$ exists uniquely, has explicit x and t dependence, and it represents a solution of the PDE satisfying the initial condition. However, in general, it will not satisfy prescribed boundary conditions. However, if the full set of boundary values is assumed *a priori* to satisfy the additional constraint given by the global relation, then the function $q(x,t)$ satisfies these boundary values.

The general picture is similar. For many interesting integrable PDEs of mathematical physics (for example NLS, KdV, mKdV, sine-Gordon), the representation of the solution $q(x,t)$ of the PDE is based on the unique solvability of the associated Riemann-Hilbert problem, which in turn is based on the distinctive form of the matrix describing the jump of the Riemann-Hilbert problem across a contour that, in general, is not simply the real axis anymore, but it also involves the curve $Im\, f_2(\lambda)=0$. The jump matrix has explicit x and t dependence, in the form of $exp[i(f_1(\lambda)x+f_2(\lambda)t)]$, where $f_1(\lambda)$, $f_2(\lambda)$ are given in (3.1), and involves the four spectral functions $a(\lambda), b(\lambda)$, defined in terms of the initial condition, and $A(\lambda), B(\lambda)$, defined in terms of all boundary values of the solution.
Indeed, using certain symmetry properties of the spectral functions, it can be shown that, for all equations mentioned, the associated homogeneous Riemann-Hilbert problem has only the trivial solution (i.e. there exists a so-called *vanishing lemma*) and hence the Riemann-Hilbert problem admits a unique solution. Hence it is possible to characterise $q(x,t)$ uniquely in terms of the spectral functions.
The solution thus represented satisfies the given initial condition at $t=0$. It also satisfies, at $x=0$, any full set of prescribed boundary values but only if it is *assumed a-priori that such set of boundary values satisfies the additional constraint imposed by the global relation.*

For boundary value problems, two additional questions need to be addressed:

- determine how many boundary conditions should be prescribed at the boundary $x=0$ to guarantee the existence of a unique solution;

- derive a solution representation that involves only the prescribed boundary conditions.

I will assume that the answer to the first question is the same as for the linearised version of the PDE. This assumption can be verified a posteriori by using the representation of the solution to *prove* existence and uniqueness for the given boundary value problem.

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Thus, given $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$ and a subset of set of all boundary values $\{f_k(t) = \partial_k^x q(0,t)\}_{k=0}^{n-1}$, the main problem becomes to show that the global relation characterises all other unknown boundary values. Namely, the last step in the full solution of a given, well -posed boundary value problem is *the analysis of the invariance of the global relation in the complex λ plane to determine a representation depending only on the prescribed initial and boundary conditions.* As discussed below, this step is fully successful in the nonlinear case *only* for certain special types of boundary conditions, called *linearisable boundary condition* in the literature.
For generic boundary conditions, the characterisation of the unknown boundary values via the global relation is itself a nonlinear problem, as it can be shown to be equivalent to solving a nonlinear system of equations [28].

5.1 The integral representation of the solution

In this section I summarise the steps to derive the main formal statement regarding integrable evolution PDE in the two independent variables $(x,t) \in \Omega$, where Ω is given by (5.1). Rather than specifying a set of boundary conditions, one assumes a-priori that the initial condition and the full set of boundary values satisfy the global relation. See [41] for the details.

For several of the most physically relevant PDEs in this class, the Lax pair takes the form (3.1). In this form, $M(x,t,\lambda)$ is a 2×2 matrix-valued function, while $f_1(\lambda)$, $f_2(\lambda)$ are given analytic (usually polynomial) functions of λ, encoding the dispersion relation of the PDE. The Lax pair (3.1) can be written in terms of a differential form $W(x,t,\lambda)$ as

$$d[\mathrm{e}^{(if_1(\lambda)x+if_2(\lambda)t)\widehat{\sigma_3}}M(x,t,\lambda)] = \mathrm{e}^{(if_1(\lambda)x+f_2(\lambda)t)\widehat{\sigma_3}}W(x,t,\lambda), \tag{5.4}$$

where the meaning of the notation $e^{\widehat{\sigma_3}}$ is given in (2.10) and

$$W(x,t,\lambda) = \Big[Q(x,t,\lambda)dx + \tilde{Q}(x,t,\lambda)dt\Big]\, M(x,t,\lambda). \tag{5.5}$$

The direct problem

As for the linear case, the first step is constructing simultaneous solutions of the two ODEs in the Lax pair, in such a way that for each $\lambda \in \mathbb{C}$ there is only one solution bounded and analytic in a neighbourhood of λ. These basic eigenfunctions are given by

$$M_j(x,t,\lambda) = I + \int_{(x_j,t_j)}^{(x,t)} \mathrm{e}^{(-if_1(\lambda)(x-\xi)-if_2(\lambda)(t-\tau))\widehat{\sigma_3}}W_j(\xi,\tau,\lambda), \quad (x,t),(x_j,t_j)\in\Omega. \tag{5.6}$$

In order to define a solution $M(x,t,\lambda)$ defined and analytic everywhere except on a contour, it is sufficient to consider the points (x_j,t_j) as the the vertices of the unbounded polygon Ω, namely

$$(x_1,t_1) = (0,T), \quad (x_2,t_2) = (0,0), \quad (x_3,t_3) = (\infty,t),$$

Thus one obtains three *sectionally analytic* basic eigenfunctions, $M_1,\ M_2$ and M_3. Their definition is independently of the path of integration,and the column vectors are bounded and analytic in certain domains. On the common boundary of these domains, the eigenfunctions satisfy the following jump conditions:

$$M_3(x,t,\lambda) = M_2(x,t,\lambda)\mathrm{e}^{(-if_1(\lambda)x - if_2(\lambda)t)\widehat{\sigma_3}}s(\lambda), \quad \lambda \in (\mathbb{C}^-, \mathbb{C}^+), \tag{5.7}$$

$$M_1(x,t,\lambda) = M_2(x,t,\lambda)\mathrm{e}^{(-if_1(\lambda)x - if_2(\lambda)t)\widehat{\sigma_3}}S(\lambda), \quad \lambda \in (D_2, D_3), \tag{5.8}$$

where $\lambda \in (D, \tilde{D})$ means the matrix identity is valid for the first column in the domain D and for the second column in the domain $\tilde{D}$,

$$\begin{aligned} &D_1 = \{\lambda : Imf_1(\lambda) > 0 \ and \ Imf_2(\lambda) > 0\}, \quad D_2 = \{\lambda : Imf_1(\lambda) > 0 \ and \ Imf_2(\lambda) < 0\}, \\ &D_3 = \{\lambda : Imf_1(\lambda) < 0 \ and \ Imf_2(\lambda) > 0\}, \quad D_4 = \{\lambda : Imf_1(\lambda) < 0 \ and \ Imf_2(\lambda) < 0\}, \end{aligned} \tag{5.9}$$

and

$$s(\lambda) = M_3(0,0,\lambda); \qquad S(\lambda) = [\mathrm{e}^{if_2(\lambda)T\widehat{\sigma_3}}M_2(0,T,\lambda)]^{-1}. \tag{5.10}$$

The spectral functions

Letting

$$\Psi(x,\lambda) = M_3(x,0,\lambda),\ \lambda \in (\mathbb{C}^-, \mathbb{C}^+), \qquad \Phi(t,\lambda) = M_2(0,t,\lambda),\ \lambda \in \mathbb{C}. \tag{5.11}$$

one can write the matrices in (5.10) as

$$s(\lambda) = \Psi(0,\lambda), \qquad \lambda \in (\mathbb{C}^-, \mathbb{C}^+), \tag{5.12}$$

$$S(\lambda) = \left[\mathrm{e}^{if_2(\lambda)T\widehat{\sigma_3}}\Phi(T,\lambda)\right]^{-1}, \qquad \lambda \in \mathbb{C}. \tag{5.13}$$

Since they solve the two ODEs in the Lax pair, these functions are the solutions of the following linear Volterra integral equations:

$$\Psi(x,\lambda) = I - \int_x^\infty \mathrm{e}^{-if_1(\lambda)(x-\xi)\widehat{\sigma_3}}Q(\xi,0,\lambda)\Psi(\xi,\lambda)d\xi, \quad x \in (0,\infty);\ \lambda \in (\mathbb{C}^-, \mathbb{C}^+), \tag{5.14}$$

$$\Phi(t,\lambda) = I + \int_0^t \mathrm{e}^{-if_2(\lambda)(t-\tau)\widehat{\sigma_3}}\tilde{Q}(0,\tau,\lambda)\Phi(\tau,\lambda)d\tau, \quad t \in (0,T);\ \lambda \in \mathbb{C}, \tag{5.15}$$

which are respectively equivalent to ODE (5.2) and to the following analogue of (5.3):

$$\begin{aligned} &\partial_t \begin{pmatrix} \Phi_{21}(t,\lambda) \\ \Phi_{22}(t,\lambda) \end{pmatrix} + 2if_2(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi_{21}(t,\lambda) \\ \Phi_{22}(t,\lambda) \end{pmatrix} = \tilde{Q}(0,t,\lambda) \begin{pmatrix} \Phi_{21}(t,\lambda) \\ \Phi_{22}(t,\lambda) \end{pmatrix}, \\ &\begin{pmatrix} \Phi_{21}(0,\lambda) \\ \Phi_{22}(0,\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \lambda \in \mathbb{C}. \end{aligned} \tag{5.16}$$

In particular, the spectral functions satisfy the following:

- $s(\lambda)$ is defined by the values of the solution at $t = 0$ - the initial condition;
- $S(\lambda)$ is defined by the values of the solution at $x = 0$ - the boundary values.

The matrices Q, $\tilde{Q}$ for the integrable PDE considered have symmetry properties that imply that $s(\lambda)$, $S(\lambda)$ can be written as

$$s(\lambda) = \begin{pmatrix} \overline{a(\bar{\lambda})} & b(\lambda) \\ \pm\overline{b(\bar{\lambda})} & a(\lambda) \end{pmatrix}, \qquad S(\lambda) = \begin{pmatrix} \overline{A(\bar{\lambda})} & B(\lambda) \\ \pm\overline{B(\bar{\lambda})} & A(\lambda) \end{pmatrix}. \tag{5.17}$$

Hence the jump matrices depend on the four distinct functions of the spectral parameter λ defined by (5.17).

The global relation

To obtain an additional relation involving the spectral functions $s(\lambda)$ and $S(\lambda)$, one observes that the function Φ defined by (5.11) and the function $M_3(0, t, \lambda)$ satisfy the same differential equation, namely the t-part of the Lax pair at $x = 0$. Hence they must be related, and the explicit identity that this induces yields the global relation:

$$S^{-1}(\lambda)s(\lambda) = \mathrm{e}^{if_2(\lambda)T\widehat{\sigma_3}}M_3(0, T, \lambda), \qquad \lambda \in (\mathbb{C}^-, \mathbb{C}^+). \tag{5.18}$$

This relation can be written explicitly in terms of the component of the matrices $s(\lambda)$ and $S(\lambda)$, namely the spectral functions. The $(1, 2)$ element of the relation then yields

$$a(\lambda)B(\lambda) - A(\lambda)b(\lambda) = \mathrm{e}^{2if_2(\lambda)T}C(\lambda), \quad \lambda \in \mathbb{C}^+ \tag{5.19}$$

where $C(\lambda)$ is a function analytic in $\mathbb{C}^+$, defined in terms of the solution at the final time $t = T$, and such that $C(\lambda) = O\left(\frac{1}{\lambda}\right)$ as $\lambda \to \infty$. This function plays the role of the ghost terms already described in the linear case, and does not contribute to the final solution representation.

The inverse problem

Assume that a suitable initial condition $q(x, 0) = q_0(x)$ is specified, and that the functions $a(\lambda)$, $b(\lambda)$ are defined in terms of $q_0(x)$ as solutions of the ODE (5.2).
Assume also that there exists a set of boundary values $q(0, t), q_x(0, t)$ (and $q_{xx}(0, t)$ for mKdV) such that the spectral functions $A(\lambda)$, $B(\lambda)$ given as solutions of ODE (5.16), together with $(a(\lambda), b(\lambda))$, satisfy the global relation (5.19).
Equations (5.7)-(5.8) can be rewritten in the form

$$M_-(x, t, \lambda) = M_+(x, t, \lambda)J(x, t, \lambda), \quad \lambda \in \mathcal{L} \tag{5.20}$$

where $\mathcal{L} = \mathcal{L}_1 \cup .. \cup \mathcal{L}_4$ with

$$\mathcal{L}_k = D_k \cap D_{k+1},\ k = 1, 2, 3, \quad \mathcal{L}_4 = D_4 \cap D_1. \tag{5.21}$$

The matrix J can be computed explicitly in terms of the entries of the spectral functions $s(\lambda)$, $S(\lambda)$ of (5.17), and has the form $J = \mathrm{e}^{(if_1(\lambda)x + if_2(\lambda)t)\widehat{\sigma}_3}\tilde{J}$ with

$$\tilde{J}(\lambda) = J_k(\lambda), \quad \lambda \in \mathcal{L}_k, \ \ k = 1, .., 4, \tag{5.22}$$

$$J_1 = \begin{pmatrix} 1 & 0 \\ \Gamma(\lambda) & 1 \end{pmatrix}; \quad J_3 = \begin{pmatrix} 1 & -\nu\overline{\Gamma(\lambda)} \\ 0 & 1 \end{pmatrix};$$

$$J_4 = \begin{pmatrix} 1 & -\gamma(\lambda) \\ \nu\overline{\gamma(\lambda)} & 1-\nu|\gamma(\lambda)|^2 \end{pmatrix}, \quad J_2 = J_3 J_4^{-1} J_1.$$

where

$$\gamma(\lambda) = \frac{b(\lambda)}{\overline{a(\bar{\lambda})}}, \quad \Gamma(\lambda) = \frac{\nu\overline{\frac{B(\bar{\lambda})}{A(\bar{\lambda})}}}{a(\lambda)[a(\lambda) - \nu b(\lambda)\overline{\frac{B(\bar{\lambda})}{A(\bar{\lambda})}}]}. \tag{5.23}$$

For the case of NLS, $\mathcal{L}_1, .., \mathcal{L}_4$ are the four semiaxes issuing from $\lambda = 0$, so M is well defined everywhere except the real and imaginary axes. For the case of mKdV, M is well defined off the real axis and the rays that make angles of $\pi/3$, $2\pi/3$, $4\pi/3$ and $5\pi/3$ with the positive real axis.
Assuming in addition that $M = I + O\left(\frac{1}{\lambda}\right)$, as $|\lambda| \to \infty$, the above Riemann-Hilbert problem admits a unique solution $M(x,t,\lambda)$. From this solution, the solution $q(x,t)$ of the PDE is determined through the relation

$$q(x,t) = \pm 2i \lim_{\lambda\to\infty} (\lambda M(x,t,\lambda)_{12} \quad (+ : NLS, \ - : mKdV). \tag{5.24}$$

The main theorem stated below summarises these results, as well as formalising the fact that the function $q(x,t)$ given by (5.24) also satifies $q(x,0) = q_0(x)$ and the given set of boundary values.

The main theorem

The theorem below was first proven in the context of the NLS equation, [32]. This result justifies the construction of the solution outlined above for any integrable PDE with a Lax pair of the form (3.1) and satisfying symmetry conditions, though to avoid technicalities I restrict attention to PDE with spatial derivatives up to third order, and really have in mind the two examples of the NLS and mKdV equations. I state it for the case of these two equations, without giving its proof, that can be found in [29].
The result justifies the main intuition behind the Unified Transform approach: combining the main idea of the Inverse Scattering Transform of starting with the Lax pair formulation with the idea of solving boundary value problem by the simultaneous solution of both ODEs in the Lax pair.

Theorem 5.1 *Let a function $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$ be given, and define $a(\lambda)$, $b(\lambda)$ through the solution of ODE (5.2), where Q is given by (2.5), and assume additionally that $a(\lambda)$ has no zeros.*
Assume that a set of functions $f_0(t)$, $f_1(t)$, $(f_2(t))$ can be given in such a way that the spectral functions $A(\lambda)$, $B(\lambda)$ defined in terms of the boundary values $\partial_x^k q(0,t) = f_k(t)$, $k = 0, 1, (2)$, through the solution of ODE (5.16), with $\tilde{Q}$ as in (3.2) or (3.3), satisfy the global relation (5.19).
Define $M(x,t,\lambda)$ as the solution the Riemann-Hilbert problem defined on the contour $\mathcal{L}$ of (5.21), with jump matrix J given by (5.22) in terms of the spectral functions $\{a(\lambda), b(\lambda), A(\lambda), B(\lambda)\}$, and such that $M(x,t,\lambda) = I + O\left(\frac{1}{\lambda}\right)$.

This Riemann-Hilbert problem admits a unique solution $M(x,t,\lambda)$. In addition the function $q(x,t)$ defined in terms of of M by (5.24) satisfies either the NLS equation or mKdV equations, as well as

$$q(x,0)=q_0(x); \quad q(0,t)=f_0(t),\ q_x(0,t)=f_1(t),\ (q_{xx}(0,t)=f_2(t)\ for\, mKdV).$$

Remark 5.1 The proof that $q(x,t)$ solves the given nonlinear PDE uses the standard arguments of the dressing method. The proof that $q(x,0)=q_0(x)$ is based on the fact that the Riemann-Hilbert problem satisfied by $M(x,0,\lambda)$ is equivalent to the Riemann-Hilbert problem defined by $s(\lambda)$, namely the Riemann-Hilbert problem which characterises $q_0(x)$. The proof that $\partial_x^k q(0,t)$, $k=0,...,n-1$ are the boundary values of the solution makes crucial use of the global relation [32]. Indeed, the Riemann-Hilbert problem satisfied by $M(0,t,\lambda)$ is equivalent to the Riemann-Hilbert problem defined by $S(\lambda)$, which characterises the boundary values, if and only if the spectral functions satisfy this global relation, hence this relation is a *necessary and sufficient condition* for the existence of a solution.

Remark 5.2 To simplify the exposition and stress the points that are of specific relevance for the case of boundary value problems, I have assumed the spectral functions have no isolated singularities. However, any zeros in the denominator of the expressions given in (5.23), namely any zero of the function $a(\lambda)$ or $d(\lambda)=a(\lambda)-\nu b(\lambda)\frac{\overline{B(\bar\lambda)}}{\overline{A(\bar\lambda)}}$, will result in such a singularity. The singularities generated by the zeros of $a(\lambda)$ are well understood, in particular it is known that the residue conditions at these singularities, for initial as well as for boundary value problems, describe the soliton part of the solution [1]. The question of whether $d(\lambda)$ has zeros remains open.

5.2 Linearisable boundary conditions

The most difficult step in solving a boundary vlue problem by any method, including the Unified Transform method, is the characterisation of the two spectral functions $A(\lambda)$, $B(\lambda)$ in terms of the given initial and boundary data, i.e. the characterisation of the unknown boundary values. For certain boundary conditions, called linearisable, this can be achieved simply using algebraic manipulations analogous to what can be done in the linear evolution case. It should be noticed that all boundary conditions that are linearisable for integrable evolution PDEs had been known before: for the second order case (NLS) these are conditions that allow the problem to be solved essentially by restriction of a problem posed on the full line, see the review in [14]; for the sine-Gordon equation, they had been found in an important work of Sklyanin based on physical considerations,[58, 59]; and, for KdV, they are conditions for which the Bäcklund transformation can be linearised, [46]. Nevertheless, as I explain below, in the approach described here these conditions all follow from the requirement that global invariant transformation of the spectral functions can be defined.
The strategy of evaluating the global relation at all transformations that leave $f_2(\lambda)$ invariant, which is fully successful in the linear case, fails in general. This failure is due to the fact that the two relevant spectral functions, namely $A(\lambda)$ and $B(\lambda)$, involve not only $\mathrm{e}^{if_2(\lambda)t}$ but also the components of the function $\Phi(t,\lambda)$ given by (5.11), that in general are not invariant under transformations that leave $f_2(\lambda)$ invariant.

Linearisable boundary conditions are precisely the conditions such that the components of $\Phi(t,\lambda)$ admit this additional invariance property. A more precise statement is given in the following proposition, see [29].

Proposition 5.1 *Suppose that the t part of the Lax pair of an integrable nonlinear PDE is characterised by the scalar function $f_2(\lambda)$ and by the 2×2 matrix-valued function $\tilde{Q}(x,t,\lambda)$ given in (2.5). Let $\theta(\lambda)$ be the transformations of complex (λ)-plane which leave $f_2(\lambda)$ invariant.*
Define $U(t,\lambda)$ by

$$U(t,\lambda) = if_2(\lambda)\sigma_3 - \tilde{Q}(0,t,\lambda). \tag{5.25}$$

If it is possible to define a matrix-valued function $N(\lambda)$, in terms only of the prescribed boundary conditions, such that

$$U(t,\theta(\lambda))N(\lambda) = N(\lambda)U(t,\lambda) \tag{5.26}$$

then the boundary spectral function $A(\lambda)$, $B(\lambda)$ defined in (5.17) possess explicit symmetry properties of the form

$$A(\omega(\lambda)) = L_1(A(\lambda),B(\lambda)), \qquad B(\omega(\lambda)) = L_2(A(\lambda),B(\lambda))$$

where L_1, L_2 are linear functions of $A(\lambda)$, $B(\lambda)$, $\overline{A(\bar{\lambda})}$, $\overline{B(\bar{\lambda})}$ with coefficients depending only on the entries of the matrix $N(\lambda)$.

When the condition of this proposition is satisfied, the functions $A(\lambda)$, $B(\lambda)$ can be computed as effectively as in the linear case in terms of $a(\lambda)$, $b(\lambda)$ and the prescribed boundary conditions.
It follows from this proposition that a *necessary* condition for the existence of linearisable boundary conditions is that the determinant of the matrix $U(t,\lambda)$ defined by (5.25) is a function of λ only through $f_2(\lambda)$. However, this condition is *not sufficient*. In particular, since the function U depends on the particular choice of Lax pair, it follows that different Lax pairs allow one to uncover different linearisable conditions. An explicit example is given by the case of the sine-Gordon equation, derived following this strategy in [27]. Indeed, using the usual Lax pair for this equation, an invariance property can be established for a constant boundary condition, while an alternative Lax pair leads to the boundary condition (b) in (5.30) below, originally discovered by Sklyanin [59].

Particular conditions that are linearisable for some of the important integrable equations of mathematical physics are listed below:

NLS In this case, there are three linearisable boundary conditions satisfying the necessary condition on the determinant of $U(t,\lambda)$ with $\tilde{Q}$ defined by (2.5):

$$(a)\ q(0,t)=0; \quad (b)\ q_x(0,t)=0; \quad (c)\ q_x(0,t)-\chi q(0,t)=0,\ \chi\in\mathbb{R}^+; \tag{5.27}$$

KdV- This refers to the *KdV equation with dominant surface tension*, hence with a negative sign in front of the third derivative term:

$$q_t + q_x - q_{xxx} + 6qq_x = 0. \tag{5.28}$$

In this case, $N = 2$ so two boundary conditions must be prescribed at $x = 0$.

$$(a)\ q(0,t) = \chi,\ q_{xx}(0,t) = \chi + 3\chi^2,\ \ \chi \in \mathbb{R}; \tag{5.29}$$

sG I also mention the case of the *sine-Gordon equation* $q_{tt} - q_{xx} + \sin q = 0$, as in this case the two cases of linearisable conditions are obtained by considering the invariance (5.26) with respect to two distinct Lax pairs, [27, 29].

$$(a)\ q(0,t) = \chi,\ \chi \in \mathbb{R}; \quad (b)\ q_x(0,t) + \chi_1 \cos(\frac{q(0,t)}{2}) + \chi_2 \sin(\frac{q(0,t)}{2}) = 0,\ \ \chi_1, \chi_2 \in \mathbb{R}. \tag{5.30}$$

5.3 General boundary conditions

For general boundary conditions, not necessarily linearisable, the invariance analysis of the global relation is not sufficient to characterise the solution of the problem without involving the unknown boundary values. However, for general boundary conditions that *decay for large t*, the representation obtained through the Unified Transform yields useful asymptotic information even without the explicit characterisation of the spectral functions.
In addition, two different approaches for analysing the *generalised Dirichlet to Neumann map* for the case of the NLS equation, i.e. to express $q_x(0,t)$ in terms of the given boundary condition $f(t)$ and initial condition $q_0(x)$, have been presented recently in [41, 42].

For non-decaying boundary conditions, the computation of the large t behaviour of the solution and of its boundary values requires new ideas. The most significant example of this situation is the case of a *time-periodic* given boundary condition, an important condition in practice. For example, the KdV equation with given zero initial condition $q(x,0) = 0$ and a periodic boundary condition such as $q(0,t) = a\sin(\omega t)$, corresponds to the very realistic situation of shallow water waves in a tank, excited by a periodic wavemaker. The linear case of this model is studied in [15].
The first results on the analysis of a periodic boundary value problem of this type, for the NLS equation, were obtained in [16, 17, 18] for the particular case that $f(t) = ae^{i\omega t}$. More recently, using the general approach described in this review, coupled with perturbation techniques, significant progress has been achieved for the physically significant case of the NLS, and mKdV equations given the boundary condition $f(t) = a\sin t$, $a \in \mathbb{R}$, using a perturbation scheme. However this perturbative approach becomes cumbersome and only a few terms in the perturbative expansion can be computed.
The most recent result, presented in [43], uses a new perturbative approach to compute the asymptotic behaviour of $q_x(0,t)$ given a periodic Dirichlet datum of the form $f(t) = \omega e^{i\omega t} + \beta e^{-i\omega t}$. By carrying out the analysis directly in the large t limit, the algorithm for characterisation the perturbative expansion of $q_x(0,t)$ is greatly simplified, and used to prove that this function is asymptotically periodic of the same period as the given datum.

Nonlinearisable problems for NLS on the half-line

Recall that the unknown boundary values enter the solution representation through the spectral functions $A(\lambda)$, $B(\lambda)$ given by (5.17) as particular value of the function Φ solution of (5.15). The functions $A(\lambda)$ and $B(\lambda)$ are in general characterised by a system of integral

equations. Indeed, for the particular example of NLS, this system is given explicitly as follows

$$A(\lambda) = \overline{\varphi_2(T, \bar{\lambda})}, \quad B(\lambda) = -\mathrm{e}^{4i\lambda^2 T}\varphi_1(T, \lambda),$$

where $\varphi_1(t, \lambda)$, $\varphi_2(t, \lambda)$ are solutions of

$$\varphi_1(t, \lambda) = \int_0^t \mathrm{e}^{4i\lambda^2(s-t)}[\nu|f_0(t)|^2\varphi_1 + (2\lambda f_0(t) + if_1(t))\varphi_2](s, \lambda)ds \tag{5.31}$$

$$\varphi_2(t, \lambda) = 1 \pm \int_0^t [(2\lambda \bar{f}_0(t) - i\bar{f}_1(t))\varphi_1 + i|f_0(t)|^2\varphi_2](s, \lambda)ds \qquad 0 < t < T, \;\; \lambda \in \mathbb{C}.$$

where I use the notation

$$f_0(t) = q(0, t), \qquad f_1(t) = q_x(0, t).$$

By substituting into the equations above the expression for f_0 and f_1 given below in equations (5.32)-(5.33), it becomes apparent that this system is itself nonlinear.
Indeed the following result summarises the situation for the general (non-homogeneous) Dirichlet or Neumann case, directly in terms of boundary functions in physical variables.

Proposition 5.2 *Let $T < \infty$. Consider the NLS equation on the positive half-line*

$$iq_t + q_{xx} - 2\nu q|q|^2 = 0, \quad x \in \mathbb{R}^+, 0 < t < T.$$

Let $q_0(x) \in \mathcal{S}(\mathbb{R}^+)$ be a given function, and consider one of the following two cases of boundary conditions (BC):

(a) Dirichlet BC

Let $q(0, t) = f_0(t)$ be given, smooth and compatible with $q_0(x)$ for $t = 0$.

(b) Neumann BC

Let $q_x(0, t) = f_1(t)$ be given, smooth and compatible with $q_0(x)$ for $t = 0$.

Suppose that the spectral function $a(\lambda)$ defined in terms of $q_0(x)$ has a finite set of simple zeros $\{\lambda_j\}$, none on which is on the real or imaginary axis.
Let

$$\chi_j(t, \lambda) = \varphi_j(t, \lambda) - \varphi_j(t, -\lambda); \quad \tilde{\chi}_j = \varphi_j(t, \lambda) + \varphi_j(t, -\lambda), \qquad j = 1, 2; \;\; 0 < t < T, \;\; \lambda \in \mathbb{C}.$$

Then

(a) For $f_0(t)$ given,

$$\begin{aligned} f_1(t) &= \frac{2}{\pi i}\int_{\partial D_3}(\lambda\chi_1(t, \lambda) + if_0(t))d\lambda + \frac{2f_0(t)}{\pi}\int_{\partial D_3}\chi_2(t, \lambda)d\lambda \\ &- \frac{4}{\pi i}\int_{\partial D_3}\lambda \mathrm{e}^{-4i\lambda^2 t}\frac{b(-\lambda)}{a(-\lambda)}\overline{\varphi_2(t, -\bar{\lambda})}d\lambda + 8\sum_{\lambda_j \in D_1}\lambda_j \mathrm{e}^{-4i\lambda_j^2 t}\frac{b(\lambda_j)}{\dot{a}(\lambda_j)}\overline{\varphi_2(t, \bar{\lambda}_j)}, \end{aligned} \tag{5.32}$$

(b) For $f_1(t)$ given,

$$\begin{aligned} f_0(t) &= \frac{1}{\pi}\int_{\partial D_3}\tilde{\chi}_1(t,\lambda)d\lambda + \frac{2}{\pi}\int_{\partial D_3} \mathrm{e}^{-4i\lambda^2 t}\frac{b(-\lambda)}{a(-\lambda)}\overline{\varphi_2(t,-\bar{\lambda})}d\lambda \\ &+ 4i\sum_{\lambda_j\in D_1}\lambda_j \mathrm{e}^{-4i\lambda_j^2 t}\frac{b(\lambda_j)}{\dot{a}(\lambda_j)}\overline{\varphi_2(t,\bar{\lambda}_j)}. \end{aligned} \tag{5.33}$$

It remains to show that the resulting nonlinear systems for φ_1, φ_2, obtained after substituting expressions (5.32) or (5.33) in the system (5.31), provide *effective* characterisation of the spectral functions $A(\lambda)$, $B(\lambda)$.
Here, effective means that:
(a) the linear limit yields the effective solution of the linearised boundary value problem, i.e. a representation involving only the known boundary data;
(b) for sufficiently small boundary data, the characterisation yields a perturbative scheme in which all terms can be computed uniquely via a well defined recursive scheme.

The general approach of [41] uses three ingredients.

1. The large λ asymptotics of the matrix-valued function $\Phi(t,\lambda)$ solution of (5.15), which defines $A(\lambda)$, $B(\lambda)$.

2. The global relation and the equations obtained under the transformations that fix the linearised dispersion relation

3. A perturbative scheme to show that the methodology is effective.

The analysis of large λ asymptotics to obtain additional conditions on the spectral functions was first employed in [20] for the case of the NLS equation, but it is only the combination of all three ingredients above that yields a general result that generalises also, for example, to equations such as mKdV involving third order derivatives .

Periodic boundary conditions

As already mentioned, even before the explicit characterisation of the unknown boundary data, the representation of the solution obtained via the Riemann-Hilbert approach described allows one to obtain precise asymptotic information on the behaviour of the solution for large times. However, this is only possible when the solution, and the given boundary conditions, decays when $t \to \infty$. For other cases, such as the case of given boundary conditions periodic in time, it is very hard to extract asymptotic information from the representation formulas directly.
After the pioneering results of [18], recent progress on this problem has been reported in [43]. In this paper, Fokas and Lenells use the ideas of the perturbative approach described in the previous section to show that, given a t-periodic Dirichlet boundary condition, for either the NLS or the modified KdV equation, the coefficient in the perturbative expansion of the Neumann datum, to any given order, is periodic in the limit as $t \to \infty$. This result is shown by proving the following two properties:
(a) The perturbative approach allows one to show that given a t-periodic Dirichlet datum the Neumann datum also becomes periodic as $t \to \infty$. This analysis is based on analyticity

considerations, and does not use the global relation. It is carried out directly in the asymptotic limit, and this simplifies the procedure to the extent that it is possibly to compute the coefficient of the perturbative expansion to all order.
(b) Assuming that the that the Neumann datum is periodic the coefficients of the Fourier series of this periodic function can be characterised uniquely. Since this step is constructive, using the fact that the there exists a unique solution one can justify the a-priori periodicity assumption.
The idea of the construction is as follows. Starting from the expression (5.32), assuming a zero initial condition and additionally *assuming*

$$\begin{aligned} \varphi_1 &= a\varphi_{11} + a^2\varphi_{12} + O(a^3), \quad a \to 0, && (5.34)\\ \varphi_2 &= 1 + a\varphi_{21} + a^2\varphi_{22} + O(a^3), \quad a \to 0, && (5.35)\\ f_0(t) &= af_{01}(t), \quad f_1(t) = af_{11}(t) + a^2 f_{12} + a^3 f_{13} + O(a^4), \quad a \to 0. && (5.36) \end{aligned}$$

one can construct a recursive scheme for the coefficients of $\varphi_1(t,\lambda)$, $\varphi_2(t,\lambda)$ in terms of those of $f_0(t)$ and $f_1(t)$. If the latter two are arbitrary functions, then all one knows is that $\varphi_1(t,\lambda)$, $\varphi_2(t,\lambda)$ are entire functions of λ bounded as $\lambda \to \infty$ in $D_2 \cup D_4$. However, if $f_0(t)$ and $f_1(t)$ are the boundary values of a solution of the boundary value problem, with $q_0(x) = 0$, then this boundedness must also hold in D_1. Using this additional analyticity constraint it is possible and indeed rather straightforward to compute the coefficients in the expansion of $f_1(t)$ in terms of the given $f_0(t)$.

6 Integrable nonlinear equations of elliptic type

The foremost example of integrable equation of elliptic type is the so-called *elliptic sine-Gordon equation*, given by (3.4). To solve this equation in any domain that is not the whole plane, it is necessary to determine missing boundary values - therefore the classical inverse scattering transform approach is not sufficient, and the successful analysis of this equation remained until recently an open problem. Indeed, only one paper treated this equation, in the half plane $x \in \mathbb{R}, y > 0$, with a given condition at $y = 0$, [45], under the assumption of validity of a certain nonlinear relation between the boundary values, deduced heuristically by analogy with the linearised case.
The Unified Transform, starting from the Lax pair (3.5), allows one to construct a formal integral representation of the solution to this equation analogous to the one obtained in the case of evolution equation. The difficulty is that there is now an unknown boundary value on each side of the boundary (assumed to be a convex polygon). This should be compared with the evolution case, for which all data are known on the boundary $t = 0$.

The problem has been analysed for the case of the half plane in [37, 55], and for the case of the semistrip $\{x > 0, 0 < y < L\}$ in [33].
For the half plane, the Dirichlet to Neumann map is characterised in general. Although the relevant characterisation shares conceptual similarities with the method used for evolution PDEs, it does involve novel elements. In particular, while for evolution PDEs it suffices to analyse a single eigenfunction, in the case of the elliptic sine Gordon equation formulated in the upper half plane it is necessary to analyse two different eigenfunctions and to combine the resulting expressions.

For the semistrip, the analysis is fully carried out in [33] but only for a simple example of linearisable boundary conditions.

The half-plane problem

Consider the Dirichlet problem for the elliptic sine-Gordon equation in the half plane $\{(x,y) \in \mathbb{R}^2 : y > 0\}$. In [37], it is shown that the solution $q(x,y)$ can be expressed in terms of a Riemann-Hilbert problem whose jump matrix is uniquely defined by a certain function $b(\lambda)$, $\lambda \in \mathbb{R}$, explicitly expressed in terms of the given Dirichlet datum $g_0(x) = q(x,0)$ and the unknown Neumann boundary value $g_1(x) = q_y(x,0)$, where $g_0(x)$ and $g_1(x)$ are related via the global relation, which in this case is the following constraint:

$$b(\lambda) = 0 \quad \text{for} \quad \lambda \geq 0. \tag{6.1}$$

Furthermore, it is shown that the latter relation can be used to characterise the Dirichlet to Neumann map, i.e. to express $g_1(x)$ in terms of $g_0(x)$. It appears that this provides the first case that such a map is explicitly characterised for a nonlinear integrable *elliptic* PDE, as opposed to an *evolution* PDE.
I give first the main theorem on the representation of the solution under the assumption that the global relation holds. This theorem, analogous to Theorem 5.1 for the evolution case, is based on the analysis of the Lax pair (3.5).

Theorem 6.1 *Let the functions $g_0(x)$, $g_1(x)$ be such that $g_0 - 2\pi m \in \mathbf{H}^1(\mathbb{R})$, $m \in \mathbb{Z}$, and $g_1(x) \in \mathbf{H}^1(\mathbb{R})$.*
Let

$$\omega(\lambda) = \frac{1}{4i}\left(\lambda - \frac{1}{\lambda}\right), \quad \Omega(\lambda) = \frac{1}{4}\left(\lambda + \frac{1}{\lambda}\right).$$

Define $a(\lambda)$ and $b(\lambda)$ by

$$a(\lambda) = 1 - \frac{1}{4}\int_{-\infty}^{\infty} \left\{ \frac{i}{\lambda}(1-\cos g_0(\xi))m_1(\xi,\lambda) + \left[-\frac{1}{\lambda}\sin g_0(\xi) + i\dot{g}_0(\xi) + g_1(\xi)\right] m_2(\xi,\lambda) \right\} d\xi, \quad Im\lambda \geq 0, \tag{6.2}$$

$$b(\lambda) = -\frac{1}{4}\int_{-\infty}^{\infty} \mathrm{e}^{-2\omega(\lambda)\xi} \left\{ \frac{i}{\lambda}(\cos g_0(\xi) - 1)m_2(\xi,\lambda) + \left[\frac{1}{\lambda}\sin g_0(\xi) + i\dot{g}_0(\xi) + g_1(\xi)\right] m_1(\xi,\lambda) \right\} d\xi, \quad \lambda \in \mathbb{R}, \tag{6.3}$$

where $(m_1(x,\lambda),\ m_2(x,\lambda))$ denotes the solution of the following system of ODEs:

$$\begin{cases} (m_1)_x = \frac{i}{\lambda}[1-\cos g_0(x)]m_1 - [\frac{1}{\lambda}\sin g_0(x) - i\dot{g}_0(x) - g_1(x)]m_2, \\ (m_2)_x + 2\omega(\lambda)m_2 = [\frac{1}{\lambda}\sin g_0(x) + i\dot{g}_0(x) + g_1(x)]m_1 - \frac{i}{\lambda}[1-\cos g_0(x)]m_2, \\ \lim_{x\to\infty}(m_1,m_2) = (1,0), \qquad x \in \mathbb{R},\ \lambda \in \mathbb{C}^+. \end{cases} \tag{6.4}$$

Assume that, given $g_0(x)$, there exists a function $g_1(x)$ such that $a(\lambda)$, $b(\lambda)$ satisfy the following constraints:

$$a(\lambda) = 1,\ 0 \leq arg(\lambda) \leq \pi, \qquad b(\lambda) = 0,\ \lambda \geq 0. \tag{6.5}$$

Define the following Riemann-Hilbert problem in terms of $b(\lambda)$:

$$\Psi^-(x,y,\lambda) = \Psi^+(x,y,\lambda)J(x,y,\lambda), \qquad \lambda \in \mathbb{R}, \quad \Psi = I + O\left(\frac{1}{\lambda}\right),\ \lambda \to \infty, \tag{6.6}$$

where

$$J = \begin{pmatrix} 1 & b(-\lambda)\mathrm{e}^{-\theta(x,y,\lambda)} \\ -b(\lambda)\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix} \qquad \theta(x,y,\lambda) = 2(\omega(\lambda)x + \Omega(\lambda)y. \tag{6.7}$$

If the $\mathbf{H}^1$ norm of the data $g_0(x)$, $g_1(x)$ is sufficiently small, the above Riemann-Hilbert problem admits a unique solution $\Psi(x,y,\lambda)$.
Let the function $q(x,y)$, $x \in \mathbb{R}$, $0 < y < \infty$, be defined in terms of this unique solution by

$$iq_x + q_y = -\lim_{\lambda\to\infty}(i\lambda\Psi)_{12}\,, \quad \cos q(x,y) = 1 - \lim_{\lambda\to\infty} 4i\lambda\left(\frac{\partial\Psi}{\partial x}\right)_{22} - 2\lim_{\lambda\to\infty}(\lambda\Psi)^2_{12}\,. \tag{6.8}$$

Then $q(x,y)$ solves the elliptic sine-Gordon equation (3.4) in the half plane $y > 0$, and furthermore

$$q(x,0) = g_0(x), \qquad q_y(x,0) = g_1(x), \quad x \in \mathbb{R}. \tag{6.9}$$

It is evident that the structure of this result mirrors precisely the structure of the main representation Theorem 5.1 for evolution equations: provided the set of boundary values satisfies the global relation, the solution can be represented via the solution of the Riemann-Hilbert problem determined by the analysis of the Lax pair.
The issue is thus reduced again to the characterisation of the unknown boundary conditions, i.e. to the determination of the Dirichlet to Neumann map. This can be done in great generality for this case, as stated in the following result, where I use the following somewhat imprecise notation:

$$\chi(x,y) = iq_x(x,y) + q_y(x,y), \qquad \chi(x) = ig_0'(x) + q_y(x,0). \tag{6.10}$$

Theorem 6.2 *Let $q(x,y)$ satisfy the elliptic sine-Gordon equation (3.4), for $-\infty < x < \infty$, $0 < y < \infty$, with prescribed Dirichlet boundary condition (within 2π multiples)*

$$q(x,0) = g_0(x), \quad g_0(x) - 2\pi m \in \mathbf{H}^1(\mathbb{R}) \text{ (some } m \in \mathbb{Z}). \tag{6.11}$$

The Neumann boundary value $q_y(x,0)$ is characterised by

$$\begin{aligned} & q_y(x,0)\cos\frac{g_0(x)}{2} = -ig_0'(x)\cos\frac{g_0(x)}{2} \qquad\qquad x \in \mathbb{R} \\ + \quad & \frac{1}{\pi}\int_{\partial\mathbb{R}} \Omega(l)\left\{\int_x^\infty \left[\sin g_0(\xi)m_1(\xi,-\frac{1}{l}) + i(\cos g_0(\xi)-1)m_2(\xi,-\frac{1}{l})\right]\mathrm{e}^{-2\omega(l)(\xi-x)}d\xi\right. \\ + \quad & \left.\int_{-\infty}^x \left[\sin g_0(\xi)n_1(\xi,-\frac{1}{l}) + i(\cos g_0(\xi)-1)n_2(\xi,-\frac{1}{l})\right]\mathrm{e}^{-2\omega(l)(\xi-x)}d\xi\right\}d\lambda, \end{aligned} \tag{6.12}$$

where the vectors $(m_1(x,\lambda), m_2(x,\lambda))$ *and* $(n_1(x,\lambda), n_2(x,\lambda))$ *satisfy the ODEs*

$$\begin{cases} (m_1)_x = \frac{i}{\lambda}(1-\cos g_0(x))m_1 - [\frac{1}{\lambda}\sin g_0(x) - \chi(x)]m_2 \\ (m_2)_x + 2\omega(\lambda)m_2 = [\frac{1}{\lambda}\sin g_0(x) + \chi(x)]m_1 - \frac{i}{\lambda}(1-\cos g_0(x))m_2 \\ \qquad x \in \mathbb{R},\ \lambda \in \mathbb{C}^+, \end{cases} \quad \lim_{x\to\infty}(m_1, m_2) = (1,0)$$

$$\begin{cases} (n_1)_x = \frac{i}{\lambda}(1-\cos g_0(x))n_1 - [\frac{1}{\lambda}\sin g_0(x) - \chi(x)]n_2 \\ (n_2)_x + 2\omega(\lambda)n_2 = [\frac{1}{\lambda}\sin g_0(x) + \chi(x)]n_1 - \frac{i}{\lambda}(1-\cos g_0(x))n_2 \\ \qquad x \in \mathbb{R},\ \lambda \in \mathbb{C}^-. \end{cases} \quad \lim_{x\to-\infty}(n_1, n_2) = (1,0)$$

The semistrip problem

In the case of the sine-Gordon equation posed in a semistrip, there are three unknown boundary values to be determined, one on each of the three boundaries $y = 0$, $y = L$ and $x = 0$. In general, the complexity of this problem appears out of reach of the current techniques.
In [33], the problem is analysed for one particularly simple example of linearisable boundary conditions, namely the case that the prescribed boundary conditions are zero along the unbounded sides of a semistrip and constant along the bounded side. A major difficulty for this problem is the existence of *non-integrable* singularities of the function q_y at the two corners of the semistrip; these singularities are generated by the discontinuities of the boundary condition at these corners. Following the spirit of the recent solution of the analogous problem for the modified Helmholtz equation [9], it is possible to introduce an appropriate regularisation which overcomes this difficulty. Furthermore, by mapping the basic Riemann-Hilbert problem to an equivalent modified Riemann-Hilbert problem, it can be shown that the solution can be expressed in terms of a 2×2 matrix Riemann-Hilbert problem whose jump matrix depends explicitly on the width of the semistrip L, on the constant value d of the solution along the bounded side, and on the residues at the given poles of a certain spectral function denoted by $h(\lambda)$. The explicit determination of the function h remains open, even for this simplest case of boundary conditions.

7 Conclusions

This review is intended as a summary of the most recent results obtained by the Unified Transform, or Fokas Transform, for solving boundary value problems for linear evolution and integrable nonlinear PDEs in two variables.
This method is truly unifying, in the sense that the construction of the formal solution representation follows the same steps in all cases, and is based on solving simultaneously the system given by the Lax pair via a Riemann-Hilbert problem. This leads not only to a formal solution representation involving an integral in the complex plane, but also to the formulation of a global constraint among the boundary values. This constraint, although

elementary (it can be derived by a straightforward argument appealing to Green's Theorem in the plane), holds the key to the effective characterisation of the problem in terms only of the prescribed boundary data. The important conceptual step to unlock the potential of the global relation is considering it as a constraint in the complex spectral plane.
Each particular problem differs is in the way the global relation can be used to determine the unknown boundary values. I have discussed the solution in the following cases:

- Boundary value problems posed on the half line $x > 0$ for linear, constant-coefficient evolution PDEs.
- Boundary value problems posed on a finite interval $0 < x < L$ for linear, constant-coefficient evolution PDEs.
- Boundary value problems posed on the half line $x > 0$ for integrable nonlinear evolution PDEs;
- An example of integrable PDE of elliptic type.

These cases illustrate most of the techniques and ideas that have been put forth in the last decade in the application of the Unified Transform to a wider variety of problems, and I hope that they suffice to give an idea of the general applicability, conceptual efficiency and aesthetic appeal of this approach.

I have not discussed the case of linear elliptic PDEs (Laplace or Helmholtz equations), for which there is also a rich literature, as well as interesting applications for numerical schemes, see [8, 29, 38, 39] and the references therein.
I have also neglected to include a discussion of singularities in the Riemann-Hilbert problem, though these singularities are the mechanism that gives rise to the distinguished localised solutions known as *solitons*. Since this mechanism has been well understood for a few decades now, and it is not substantially different in the boundary value case, I did not include it in this review.
Finally, I have mentioned only briefly the possibility of using the relevant representation given by the Fokas Transform for deriving rigorous results on the asymptotic behaviour of the solution for large times. This is obtained by applying the nonlinear steepest descent method of Deift-Zhou, and a detailed discussion is beyond the scope of this review.

References

[1] M. J. Ablowitz and P.A. Clarkson. *Solitons, nonlinear evolution equations and inverse scattering*. Cambridge University Press, 1991.

[2] M. J. Ablowitz and A. S. Fokas. *Complex Variables*. Cambridge University Press, 1997.

[3] Mark J Ablowitz and Harvey Segur. *Solitons and the inverse scattering transform*, volume 4. SIAM, 1981.

[4] M.J. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur. Method for solving the sine-gordon equation. *Physical Review Letters*, 30(25):1262–1264, 1973.

[5] M.J. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur. The inverse scattering transform-fourier analysis for nonlinear problems. *Studies in Applied Mathematics*, 53:249–315, 1974.

[6] Tuncay Aktosun. Inverse scattering transform and the theory of solitons. In *Encyclopedia of Complexity and Systems Science*, pages 4960–4971. Springer, 2009.

[7] A.C.L. Ashton. Regularity theorems for elliptic and hypoelliptic operators via the global relation. *Journal of Partial Differential Equations*, 24(1):83–96, 2011.

[8] A.C.L. Ashton. On the rigorous foundations of the Fokas method for linear elliptic partial differential equations. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 468(2141):1325–1331, 2012.

[9] A.C.L. Ashton and A.S. Fokas. Elliptic boundary value problems in convex polygons with low regularity boundary data via the unified method. *arXiv preprint arXiv:1301.1490*, 2013.

[10] R. Beals and R.R. Coifman. Scattering and inverse scattering for first order systems. *Comm. Pure Appl. Math.*, 37:39–90, 1984.

[11] R. Beals and R.R. Coifman. Scattering and inverse scattering for first order systems: Ii. *Inv. Prob.*, 3:577–593, 1987.

[12] R. Beals and R.R. Coifman. Linear spectral problems, non-linear equations and the $\bar{\partial}$-method. *Inv. Prob.*, 5:87–130, 1989.

[13] ED Belokolos, AI Bobenko, VZ Enol'Skii, AR Its, and VB Matveev. *Algebro-geometric approach to nonlinear integrable equations*, volume 1994. Springer-Verlag Berlin, 1994.

[14] G. Biondini, A.S. Fokas, and D. Shepelsky. Comparison of two approaches to the initial-boundary value problem for the nonlinear Schrödinger equation on the half line with robin boundary conditions. (submitted), 2013.

[15] J. L. Bona and A. S. Fokas. Initial-boundary-value problems for linear and integrable nonlinear dispersive partial differential equations. *Nonlinearity*, 21:195–203, 2008.

[16] A. Boutet de Monvel and V. Kotlyarov. Generation of asymptotic solitons of the nonlinear Schrödinger equation by boundary data. *J. Math. Phys.*, 44:3185–3215, 2003.

[17] A. Boutet de Monvel and V. Kotlyarov. Characteristic properties of the scattering data for the mKdV equation on the half-line. *Comm. Math. Phys.*, 253(1):51–79, 2005.

[18] A. Boutet de Monvel and V. Kotlyarov. The focusing nonlinear Schrödinger equation on the quarter plane with time-periodic boundary condition: a Riemann-Hilbert approach. *J. Inst. Math. Jussieu*, 6:579–611, 2007.

[19] Samuel Butler and Mike Hay. Simple identification of fake lax pairs. *arXiv preprint arXiv:1311.2406*, 2013.

[20] A. Degasperis, S. Manakov, and P. Santini. On the initial-boundary value problems for soliton equations. *JETP Letters*, 74:481–485, 2001.

[21] P. Deift. Universality for mathematical and physical systems. *arXiv preprint math-ph/0603038*, 2006.

[22] Percy Deift and Eugene Trubowitz. Inverse scattering on the line. *Communications on Pure and Applied Mathematics*, 32:121–251, 1979.

[23] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov. Non-linear equations of kortewegde vries type, finite-zone linear operators, and abelian varieties. *Russian mathematical surveys*, 31(1):59, 1976.

[24] N. Flyer and A. S. Fokas. A hybrid analytical numerical method for solving evolution partial differential equations. I. the half-line. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 464(2095):1823–1849, 2008.

[25] N. Flyer and B. Fornberg. A numerical implementation of Fokas boundary integral approach: Laplace's equation on a polygonal domain. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 2011. (not yet in print).

[26] A. S. Fokas. Integrable nonlinear evolution equations on the half-line. *Comm. Math. Phys.*, 230(1):1–39, 2002.

[27] A. S. Fokas. Linearizable initial boundary value problems for the sine-Gordon equation on the half-line. *Nonlinearity*, 17(4):1521–1534, 2004.

[28] A. S. Fokas. The generalized Dirichlet-to-Neumann map for certain nonlinear evolution PDEs. *Comm. Pure Appl. Math.*, 58:639–670, 2005.

[29] A. S. Fokas. *A Unified Approach to Boundary Value Problems*. CBMS-SIAM, 2008.

[30] A. S. Fokas and Novikov R. G. Discrete analogues of -equations and of radon transform. *CR Acad Sci PAris Ser I Math*, 313:75, 1991.

[31] A. S. Fokas and I. M. Gelfand. Integrability of linear and nonlinear evolution equations and the associated nonlinear fourier transforms. *Lett. Math. Phys.*, 32:189–210, 1994.

[32] A. S. Fokas, A. R. Its, and L. Y. Sung. The nonlinear Schrödinger equation on the half-line. *Nonlinearity*, 18(4):1771–1822, 2005.

[33] A. S. Fokas, J. Lenells, and B. Pelloni. Explicit solutions for the elliptic sine-Gordon equation in a semistrip. *J Nonlin Science*, 23:241–282, 2013.

[34] A. S. Fokas and B. Pelloni. Integral transforms, spectral representations and the d-bar problem. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 456:805–833, 2000.

[35] A. S. Fokas and B. Pelloni. Two-point boundary value problems for linear evolution equations. *Math. Proc. Cambridge Philos. Soc.*, 131:521–543, 2001.

[36] A. S. Fokas and B. Pelloni. A transform method for linear evolution PDEs on a finite interval. *IMA J. Appl. Math.*, 70:564–587, 2005.

[37] A. S. Fokas and B. Pelloni. The Dirichlet-to-Neumann map for the elliptic sine-Gordon equation. *Nonlinearity*, 25:1011, 2012.

[38] A. S. Fokas and E. A. Spence. A new transform method I: domain-dependent fundamental solutions and integral representations. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 466(2120):2259–2281, 2010.

[39] A. S. Fokas and E. A. Spence. A new transform method II: the global relation and boundary-value problems in polar coordinates. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 466(2120):2283–2307, 2010.

[40] AS Fokas. Lax pairs: a novel type of separability. *Inverse Problems*, 25:1–44, 2009.

[41] A.S. Fokas and J. Lenells. The unified method: I. nonlinearizable problems on the half-line. *Journal of Physics A: Mathematical and Theoretical*, 45(19):195201, 2012.

[42] A.S. Fokas and J. Lenells. The unified method: II. NLS on the half-line with t-periodic boundary conditions. *Journal of Physics A: Mathematical and Theoretical*, 45(19):195202, 2012.

[43] A.S. Fokas and J. Lenells. The asymptotics of the Neumann value for the Nonlinear Schrödinger with t-periodic data. (preprint), 2013.

[44] C. S. Gardner, J.M. Greene, M. D. Kruskal, and R.M. Miura. Method for solving the Korteweg-deVries equation. *Physical Review Letters*, 19(19):1095–1097, 1967.

[45] E.S. Gutshabash and V.D. Lipovskii. Boundary value problem for the two-dimensional elliptic sine-gordon equation and its applications to the theory of the stationary josephson effect. *J. Math. Sciences*, 68, 1994.

[46] I.T. Habibulin. Backlund transformations and integrable initial-boundary value problems. In *Nonlinear and Turbulent Processes*, 1990.

[47] A. R. Its. The Riemann-Hilbert problem and integrable systems. *Notices Amer. Math. Soc.*, 50(11):1389–1400, 2003.

[48] I. M. Krichever. Methods of algebraic geometry in the theory of non-linear equations. *Russian Mathematical Surveys*, 32(6):185–213, 1977.

[49] R. E. Langer. The zeros of exponential sums and integrals. *Bull. AMS*, 37:213–239, 1931.

[50] P.D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Communications on pure and applied mathematics*, 21(5):467–490, 1968.

[51] D. Mantzavinos and A.S. Fokas. The unified method for the heat equation: I. non-separable boundary conditions and non-local constraints in one dimension. *European Journal of Applied Mathematics*, 24(06):857–886, 2013.

[52] Roman G Novikov. An inversion formula for the attenuated x-ray transformation. *Arkiv för matematik*, 40:145–167, 2002.

[53] B. Pelloni. The spectral representation of two-point boundary-value problems for third-order linear evolution partial differential equations. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 461:2965–2984, 2005.

[54] B. Pelloni. Spectral analysis of the elliptic sine-Gordon equation in the quarter plane. *Theoret. and Math. Phys.*, 160(1):1031–1041, 2009.

[55] B. Pelloni and D. A. Pinotsis. The elliptic sine-Gordon equation in a half plane. *Nonlinearity*, 23(1):77, 2010.

[56] B Pelloni and DA Smith. Spectral theory of some non-selfadjoint linear differential operators. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 469(2154), 2013.

[57] A Shabat and V Zakharov. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP*, 34:62–69, 1972.

[58] E.K. Sklyanin. Boundary conditions for integrable equations. *Funct. Anal. Appl.*, 21:86–87, 1987.

[59] Evgeni K Sklyanin. Boundary conditions for integrable quantum systems. *Journal of Physics A: Mathematical and General*, 21:2375, 1988.

[60] L. Y. Sung. *Initial-boundary value problems for linear dispersive evolution equations on the half-line*, pages 374–378. SIAM, Philadelphia, PA, 2000.

[61] S. Vetra-Carvalho. *The computation of spectral representations for evolution PDEs*. ABE Books, 2009.