

Department of Mathematics and Statistics

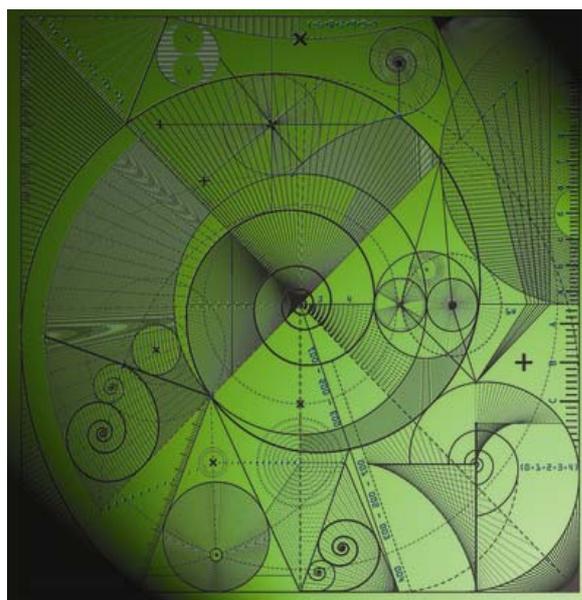
Preprint MPS-2012-15

9 July 2012

Acoustic trapped modes in a three- dimensional waveguide of slowly varying cross section

by

Simon N. Gaultier and Nicholas R. T. Biggs



Acoustic trapped modes in a three-dimensional waveguide of slowly varying cross section

Simon N. Gaulter and Nicholas R. T. Biggs

Department of Mathematics and Statistics, University of Reading,
Whiteknights Campus, Reading, RG6 6AX, UK

s.n.gaulter@pgr.reading.ac.uk

9th July 2012

Abstract

In this paper we develop an asymptotic scheme to approximate the trapped mode solutions to the time harmonic wave equation in a three-dimensional waveguide with a smooth but otherwise arbitrarily shaped cross section and a single, slowly varying ‘bulge’, symmetric in the longitudinal direction.

Extending the work in [4], we first employ a WKB-type ansatz to identify the possible *quasi-mode* solutions which propagate only in the thicker region, and hence find a finite *cut-on* region of oscillatory behaviour and asymptotic decay elsewhere. The WKB expansions are used to identify a *turning point* between the cut-on and *cut-off* regions. We note that the expansions are nonuniform in an interior layer centred on this point, and we use the method of matched asymptotic expansions to connect the cut-on and cut-off regions within this layer.

The behaviour of the expansions within the interior layer then motivates the construction of a uniformly valid asymptotic expansion. Finally, we use this expansion and the symmetry of the waveguide around the longitudinal centre, $x = 0$, to extract trapped mode wavenumbers, which are compared with those found using a numerical scheme and seen to be extremely accurate, even to relatively large values of the small parameter.

Keywords: trapped modes; quasi-modes; slowly varying waveguide; perturbation methods; WKB; turning point

1 Introduction

By *trapped modes*, we are referring to acoustic modes wherein we find a localised region of finite oscillatory energy, and which decay elsewhere. The existence of trapped modes in various types of topographically slowly varying waveguides has been demonstrated previously, in the cases of elastic plates and rods (e.g., [8, 10]), two-dimensional acoustic waveguides (e.g., [4, 11]) and in weakly curved quantum waveguides (e.g., [5]). Additionally, there has been previous research into the calculation of *quasi-modes*—perturbed modes of a slowly varying waveguide—in both two- and three-dimensional acoustic waveguides with a slow curvature (e.g., [2, 6]), and in two-dimensional elastic plates (e.g., [7]), amongst others.

In particular, in [4], an asymptotic scheme is developed to approximate the trapped mode solutions of the time-harmonic wave equation in a two-dimensional waveguide with a slowly varying symmetric bulge around the longitudinal centre $x = 0$. This slowness is incorporated into the scheme as a large longitudinal length scale, $L_x \sim 1/\epsilon$, where $\epsilon \ll 1$ is taken as the small parameter used to construct an asymptotic approximation to the solution. By expanding both the amplitude and phase in powers of ϵ , the cut-on and cut-off regions are determined by the relative magnitudes of the wavenumber, k , and the eigenvalues of the one-dimensional duct cross section at each x . A turning point between the two regions is identified, and the solution behaviour around this motivates the construction of a uniformly valid approximation, which is used to calculate the trapped mode eigensolutions.

In this paper, we extend this work to create an asymptotic approximation to the trapped mode solutions within a three-dimensional waveguide with a smooth and simply connected, but otherwise

arbitrarily shaped cross section with a slow, symmetric variation centred on the plane $x = 0$. In Section 2, the slowly varying nature of the waveguide boundary is introduced in terms of the small parameter, $0 < \epsilon \ll 1$, and the geometry is non-dimensionalised and mapped onto a stretched coordinate system, such that this parameter is incorporated into the differential operator.

In Section 3, we first determine the possible oscillatory or evanescent quasi-mode solutions (see, e.g., [2–4, 6]) that may be present within the waveguide by expanding the solution using a WKBJ-type ansatz of the form $\phi = A \exp(P)$, wherein the amplitude and phase terms are expanded as $A = A_0 + \epsilon A_1 + \dots$ and $P = \epsilon^{-1} P_{-1} + \epsilon P_1 + \dots$, respectively. A hierarchy of equations in the A_i and P_i terms is obtained, which we solve iteratively to obtain a leading order quasi-mode approximation to ϕ . It will be seen that the behaviour of the solution depends upon the relative sizes of the wavenumber, k , and the transverse eigenvalues of the waveguide cross section, which we denote $\lambda(x)$.

In Section 4, the existence of a turning point is discussed, around which the solution is either oscillatory or evanescent. Motivated by the quasi-mode solutions found in Section 3, we find this to be a point x_* , for which $\lambda(x_*) = k$. Taking established results in domain perturbation theory (e.g., [9, 12]), we are able to assert that a positive turning point will be found in a region for which the cross-sectional area is decreasing, such that we expect a solution to be oscillatory only in the bulge, up until the turning point, and decaying toward infinity.

An interior layer is then introduced around the turning point, wherein an interior solution is found and asymptotically matched with the oscillatory and evanescent solutions on either side of the region. We find that the interior solution takes the form of an Airy function in the x -direction, which motivates the uniformly valid expansion introduced in Section 6. Using this expansion, we are able to extract trapped mode wavenumbers by considering the symmetry of the waveguide about the $x = 0$ plane. As such, we find that to leading order, the wavenumbers of the trapped modes which are either symmetric or antisymmetric in the x -direction are those for which $\phi_x(0, y, z) = 0$ or $\phi(0, y, z) = 0$, respectively. We find there to be a sequence of trapped mode wavenumbers in each case, of the form $k_{n,p}$, $n, p \in \mathbb{N}$, for the Laplacian eigenvalues $\lambda_n(x)$ of the waveguide cross section at x .

Lastly, in Section 8, both the asymptotic scheme and a spectral collocation method are applied to calculate the trapped mode wavenumbers of two example problems, to demonstrate the accuracy of the former. The first test case considered is that of an azimuthally constant ‘cylindrical’ waveguide with a slow variation. Due to this azimuthal invariance, the problem in this particular geometry may be reduced to a sequence of two-dimensional problems, so that wavenumbers may be quickly extracted via the spectral method, and we see an excellent agreement between these and those obtained using the asymptotic scheme. Further benefits of the asymptotic method are shown in the second example problem, wherein no such reduction is possible and a three-dimensional spectral collocation or similar numerical method proves to be prohibitively expensive. In such a case, we see that the asymptotic scheme can still be used to efficiently approximate the trapped mode wavenumbers to a high degree of accuracy.

2 Formulation

We are interested in time harmonic solutions to the wave equation, and hence omit the common factor $\exp(-i\omega t)$ throughout. The problem is then equivalent to that of finding trapped mode solutions to the Helmholtz equation within a three-dimensional waveguide of arbitrary cross section and smooth boundary. Using standard cartesian coordinates with x oriented along the waveguide, we take the duct to be infinite in the x -direction and symmetric about the plane $x = 0$. We will take the cross section at x to be a simply connected domain $\bar{D}(x) \in \mathbb{R}^2$, with smooth boundary $\partial\bar{D}(x)$. We seek eigensolutions of

$$\begin{aligned} \bar{\phi}_{xx} + \bar{\phi}_{yy} + \bar{\phi}_{zz} + \bar{k}^2 \bar{\phi} &= 0 & \begin{cases} (y, z) \in \bar{D}(x), \\ -\infty < x < \infty, \end{cases} \\ \bar{\phi} &\rightarrow 0 & \text{as } x \rightarrow \pm\infty. \end{aligned} \tag{2.1}$$

We will consider individually both the sound soft and sound hard waveguide, i.e., that with either a Dirichlet or Neumann condition on the waveguide boundary, respectively, given by

$$\bar{\phi}(x, y, z) = 0 \quad \text{for } (y, z) \in \bar{D}(x), \tag{2.2a}$$

$$\nabla \bar{\phi}(x, y, z) \cdot \bar{\mathbf{n}}(x, y, z) = 0 \quad \text{for } (y, z) \in \partial\bar{D}(x), \tag{2.2b}$$

with $\bar{\mathbf{n}}(x, y, z)$ denoting the outward normal vector to $\bar{D}(x)$ at (x, y, z) . If we define a non-dimensionalisation constant L_{yz} , which characterises the length scale of the waveguide in the y - and z -directions, by

$$L_{yz} = \lim_{x \rightarrow \pm\infty} \left\{ \max_{\mathbf{p}_1, \mathbf{p}_2 \in \partial\bar{D}} \|\mathbf{p}_1 - \mathbf{p}_2\| \right\}$$

and similarly take L_x to characterise the length scale of the waveguide in the x -direction, we therefore take the waveguide to be slowly varying in the sense that $L_{yz}/L_x = \epsilon$, where $0 < \epsilon \ll 1$. We thus non-dimensionalise by introducing $\bar{\phi}(x, y, z) = \phi(\xi, \eta, \zeta)$, where $\xi = x/L_x$, $\eta = y/L_{yz}$ and $\zeta = z/L_{yz}$ on the stretched domain $D(\xi) = \bar{D}(x)/L_{yz}$. Transforming (2.1) into these stretched coordinates, we have the non-dimensional system

$$\begin{aligned} \epsilon^2 \phi_{\xi\xi} + \phi_{\eta\eta} + \phi_{\zeta\zeta} + k^2 \phi &= 0 & \begin{cases} (\eta, \zeta) \in D(\xi), \\ -\infty < \xi < \infty, \end{cases} \\ \phi \rightarrow 0 & \text{ as } \xi \rightarrow \pm\infty, \end{aligned} \quad (2.3)$$

where $k = \bar{k}L_{yz}$ is the non-dimensionalised wavenumber. The boundary conditions (2.2a) and (2.2b) are then equivalent to

$$\phi(\xi, \eta, \zeta) = 0 \quad \text{for } (\eta, \zeta) \in \partial D(\xi), \quad (2.5a)$$

$$\nabla_\epsilon \phi(\xi, \eta, \zeta) \cdot \mathbf{n}(\xi, \eta, \zeta) = 0 \quad \text{for } (\xi, \eta, \zeta) \in \partial D(\xi), \quad (2.5b)$$

respectively, where

$$\nabla_\epsilon = (\epsilon^2 \partial_\xi, \partial_\eta, \partial_\zeta)$$

is a skewed del operator and

$$\mathbf{n}(\xi, \eta, \zeta) = \text{diag}(\epsilon^{-1}L_{yz}, L_{yz}, L_{yz})\bar{\mathbf{n}}(x, y, z)$$

is the transformed normal vector.

3 WKBJ ansatz

We first seek quasi-mode solutions valid within the waveguide by expanding ϕ using a WKBJ-type ansatz (as in, e.g., [3]), of the form

$$\phi(\xi, \eta, \zeta) = A(\xi, \eta, \zeta) \exp\{P(\xi, \eta, \zeta)\} \quad (3.1)$$

where

$$\begin{aligned} A(\xi, \eta, \zeta) &= \sum_{j=0}^{\infty} \epsilon^j A_j(\xi, \eta, \zeta), \\ P(\xi, \eta, \zeta) &= \epsilon^{-1} P_{-1}(\xi, \eta, \zeta) + \sum_{j=1}^{\infty} \epsilon^j P_j(\xi, \eta, \zeta). \end{aligned}$$

We note that there is no P_0 term, since we observe that the $O(1)$ phase contribution can be subsumed into the leading order amplitude term. We also require the phase terms P_j to be either strictly real or strictly imaginary, since we are seeking modes which are either oscillatory or evanescent in different regions within the waveguide. By substituting the above ansatz into (2.3) and equating powers of ϵ , we extract a hierarchy of equations in A_j and P_j , from which we can determine the leading order terms of an asymptotic expansion for ϕ . At leading order, $O(\epsilon^{-2})$, we obtain

$$A_0 (P_{-1,\eta}^2 + P_{-1,\zeta}^2) = 0, \quad (3.2)$$

which, since we have specified that either $P_{-1} \in \mathbb{R}$ or $P_{-1} \in i\mathbb{R}$ and we require a non-zero leading order amplitude for a non-trivial solution, implies that $P_{-1}(\xi, \eta, \zeta) \equiv f(\xi)$, where f is a function to be determined. We find that the $O(\epsilon^{-1})$ terms are trivially satisfied and hence we turn to the $O(1)$ terms, at which point we consider separately the case of the sound soft boundary, as in (2.5a), and the sound hard boundary, as in (2.5b).

3.1 Sound soft boundary

Note that to satisfy (2.5a), it is sufficient to choose

$$A(\xi, \eta, \zeta) = 0 \quad \text{for } (\eta, \zeta) \in \partial D(\xi)$$

such that $A_j = 0$ on the boundary $\partial D(\xi)$ for all $j \in \mathbb{N}_0$. Henceforth we will denote $\nabla \equiv \nabla_{\eta\zeta} = (\partial_\eta, \partial_\zeta)$ to represent the two-dimensional del operator in η and ζ only, unless otherwise specified. Then, considering the $O(1)$ terms in the asymptotic expansion, we have that

$$(\nabla^2 + \lambda^2) A_0 = 0 \tag{3.3}$$

on the domain $D(\xi)$, with boundary condition $A_0 = 0$ on $\partial D(\xi)$, where we write $\lambda^2(\xi) = k^2 + f'^2(\xi)$ for notational convenience. We consider a Dirichlet eigenvalue problem over an arbitrarily shaped domain, $\Omega(\xi)$,

$$\begin{aligned} (\nabla^2 + \lambda^2) E &= 0 \quad \text{in } \Omega \\ E &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.4}$$

with eigensolutions $\lambda_n(\Omega)$, $E_n(\Omega)$ for all $\Omega \in \{\Omega(\xi) : \xi \in \mathbb{R}\}$ and $n \in \mathbb{N}$, where E_n are normalised such that

$$\iint_{\Omega} E_n^2 \, dS = 1.$$

We may then write the solutions of (3.3) as $A_{0;n} = \alpha_n(\xi) E_n$ with associated eigenvalues $\lambda_n(\xi)$. Now, briefly disregarding the subscript n notation for convenience, at $O(\epsilon)$ we have

$$(\nabla^2 + \lambda^2) A_1 + A_0 \nabla^2 P_1 + 2\nabla A_0 \cdot \nabla P_1 + A_0 f'' + 2A_{0,\xi} f' = 0, \tag{3.5}$$

which we multiply by A_0 and integrate over $D(\xi)$ to obtain the solvability condition

$$\iint_{D(\xi)} \left\{ A_0 (\nabla^2 + \lambda^2) A_1 + A_0^2 \nabla^2 P_1 + \nabla(A_0^2) \cdot \nabla P_1 + \frac{\partial}{\partial \xi} (A_0^2 f') \right\} dS = 0.$$

We apply equation (3.3) and use Green's identities, to yield

$$\oint_{\partial D(\xi)} \left\{ A_0 \frac{\partial A_1}{\partial n} - A_1 \frac{\partial A_0}{\partial n} + A_0^2 (\nabla P_1 \cdot \hat{\mathbf{n}}_{jk}) \right\} dC + \iint_{D(\xi)} \frac{\partial}{\partial \xi} (A_0^2 f') \, dS = 0,$$

where $\hat{\mathbf{n}}_{jk} \in \mathbb{R}^2$ denotes the outward unit normal to $\partial D(\xi)$. Then since the A_j terms vanish on ∂D , we retain only the condition

$$\iint_{D(\xi)} \frac{\partial}{\partial \xi} (A_0^2 f') \, dS = 0.$$

By the Reynolds transport theorem, and because the Dirichlet condition requires A_0 to vanish on the boundary, the derivative may be taken outside of the integral and the solvability condition reduces to

$$\frac{d}{d\xi} \left(\alpha^2(\xi) f'(\xi) \iint_{D(\xi)} E^2 \, dS \right) = \frac{d}{d\xi} (\alpha^2(\xi) f'(\xi)) = 0,$$

so that we have simply

$$\alpha(\xi) = \frac{\mathcal{A}}{|f'(\xi)|^{1/2}} \tag{3.6}$$

for some constant \mathcal{A} . Therefore, reapplying the subscript notation, we find the leading order quasi-mode solutions to be

$$\phi_n = \frac{\mathcal{A}_n}{|f'_n|^{1/2}} E_n \exp \{ \epsilon^{-1} f_n \} + O(\epsilon) \tag{3.7}$$

where

$$f_n(\xi) = \begin{cases} \pm i \int^\xi \sqrt{k^2 - \lambda_n^2(\bar{\xi})} \, d\bar{\xi} & \text{for } |k| \geq |\lambda_n(\xi)| \\ \pm \int^\xi \sqrt{\lambda_n^2(\bar{\xi}) - k^2} \, d\bar{\xi} & \text{for } |k| \leq |\lambda_n(\xi)|, \end{cases} \tag{3.8}$$

for constants \mathcal{A}_n .

3.2 Sound hard boundary

In the case of the sound hard waveguide, the boundary condition (2.5b) applied to the WKBJ ansatz tells us that, at leading order, we require

$$\{A_{0,\eta}\mathbf{j} + A_{0,\zeta}\mathbf{k}\} \cdot \mathbf{n} = 0 \quad (3.9)$$

on $\partial D(\xi)$. The $O(1)$ terms then provide the two-dimensional equation (3.3) as before, albeit now with the boundary condition upon A_0 given by

$$\nabla A_0(\xi, \eta, \zeta) \cdot \mathbf{n}_{jk}(\xi, \eta, \zeta) = 0 \quad \text{for } (\xi, \eta, \zeta) \in \partial D(\xi) \quad (3.10)$$

where $\nabla \equiv \nabla_{\eta\zeta}$, as specified above, and $\mathbf{n}_{jk} \in \mathbb{R}^2$ denotes the (η, ζ) -direction components of the outward normal, \mathbf{n} , to the boundary $\partial D(\xi)$.

Similarly to the sound soft waveguide, we consider a general eigenvalue problem over a domain $\Omega(\xi)$,

$$\begin{aligned} (\nabla^2 + \tilde{\lambda}^2) \tilde{E} &= 0 \quad \text{in } \Omega, \\ \nabla \tilde{E} \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (3.11)$$

where \mathbf{n} is the outward normal to the domain, and write its eigenvalues and normalised eigenfunctions as $\tilde{\lambda}_n$, \tilde{E}_n , respectively, for $n \in \mathbb{N}$. Now given that the leading order amplitude in the sound hard duct satisfies (3.3) and (3.10), it admits the eigensolutions $A_0 \equiv \tilde{A}_{0;n} = \tilde{\alpha}_n(\xi)\tilde{E}_n$, with eigenvalues $\tilde{\lambda}_n$, for $n \in \mathbb{N}$. We again briefly omit the subscript notation and hence the solvability condition from the $O(\epsilon)$ terms can be written as

$$\iint_{D(\xi)} \left\{ \tilde{A}_0 (\nabla^2 + \tilde{\lambda}^2) A_1 + \tilde{A}_0^2 \nabla^2 P_1 + \nabla(\tilde{A}_0^2) \cdot \nabla P_1 + \frac{\partial}{\partial \xi} (\tilde{A}_0^2 \tilde{f}') \right\} dS = 0, \quad (3.12)$$

which, through an application of Green's identities and (3.3), is equivalent to

$$\oint_{\partial D(\xi)} \left\{ \tilde{A}_0 \nabla A_1 \cdot \hat{\mathbf{n}}_{jk} - A_1 \nabla \tilde{A}_0 \cdot \hat{\mathbf{n}}_{jk} + \tilde{A}_0^2 \nabla P_1 \cdot \hat{\mathbf{n}}_{jk} \right\} dC + \iint_{D(\xi)} \frac{\partial}{\partial \xi} (\tilde{A}_0^2 \tilde{f}') dS = 0, \quad (3.13)$$

where $\hat{\mathbf{n}}_{jk}$ denotes the outward unit normal to $\partial D(\xi)$. The $O(\epsilon)$ terms of the Neumann boundary condition require that

$$\left\{ \tilde{A}_0 \tilde{f}' \mathbf{i} + (A_{1,\eta} + \tilde{A}_0 P_{1,\eta}) \mathbf{j} + (A_{1,\zeta} + \tilde{A}_0 P_{1,\zeta}) \mathbf{k} \right\} \cdot \mathbf{n} = 0$$

which can be written as

$$\tilde{A}_0 \tilde{f}' n_i + \left\{ \nabla A_1 + \tilde{A}_0 \nabla P_1 \right\} \cdot \mathbf{n}_{jk} = 0, \quad (3.14)$$

on $\partial D(\xi)$, where \mathbf{n}_{jk} is as before, and similarly n_i denotes the component of the normal vector acting in the ξ -direction. Thus, applying both this and condition (3.10) to the solvability condition yields

$$- \oint_{\partial D(\xi)} \tilde{A}_0^2 \tilde{f}' \frac{n_i}{|\mathbf{n}_{jk}|} dC + \iint_{D(\xi)} \frac{\partial}{\partial \xi} (\tilde{A}_0^2 \tilde{f}') dS = 0. \quad (3.15)$$

We denote by \mathbf{v} the rate of change of the waveguide boundary with respect to ξ and see from Appendix A that (3.15) is equivalent to

$$\oint_{\partial D(\xi)} \tilde{A}_0^2 \tilde{f}' \mathbf{v} \cdot \hat{\mathbf{n}}_{jk} dC + \iint_{D(\xi)} \frac{\partial}{\partial \xi} (\tilde{A}_0^2 \tilde{f}') dS = 0.$$

We may thus apply the Reynolds transport theorem to obtain, as in the Dirichlet case,

$$\frac{d}{d\xi} \left(\tilde{f}' \tilde{\alpha}^2(\xi) \iint_{D(\xi)} E^2 dS \right) = 0,$$

so that, as before,

$$\tilde{\alpha}(\xi) = \frac{\tilde{\mathcal{A}}}{|\tilde{f}'(\xi)|^{1/2}},$$

for constant $\tilde{\mathcal{A}}$. Hence, reintroducing the subscript notation, we have the sound hard quasi-mode solutions

$$\tilde{\phi}_n = \frac{\tilde{\mathcal{A}}_n}{|\tilde{f}'_n|^{1/2}} \tilde{E}_n \exp \left\{ \epsilon^{-1} \tilde{f}_n \right\} + O(\epsilon) \quad (3.16)$$

where

$$\tilde{f}_n(\xi) = \begin{cases} \pm i \int^{\xi} \sqrt{k^2 - \tilde{\lambda}_n^2(\bar{\xi})} d\bar{\xi} & \text{for } |k| \geq |\tilde{\lambda}_n(\xi)| \\ \pm \int^{\xi} \sqrt{\tilde{\lambda}_n^2(\bar{\xi}) - k^2} d\bar{\xi} & \text{for } |k| \leq |\tilde{\lambda}_n(\xi)|, \end{cases} \quad (3.17)$$

for constants $\tilde{\mathcal{A}}_n$.

These solutions thus imply that the presence of oscillations in the ξ -direction depend upon the relative magnitudes of the wavenumber, k , and the eigenvalues, $\lambda_n(\xi)$, of the Laplacian over the domain $D(\xi)$. In the case presented here, in which we have a waveguide symmetric about the plane $\xi = 0$, we require, for a trapped mode solution, a region of oscillatory energy up to a particular point, $0 \leq |\xi| \leq |\xi_*|$, and decay for $|\xi| \geq |\xi_*|$. In fact, we may restrict our attention to just the $\xi \geq 0$ half of the waveguide, since it is symmetric, and we see from (3.17) that we hence require a turning point $\xi_* > 0$ satisfying $\lambda_n(\xi_*) = k$ and $\lambda'_n(\xi_*) > 0$ for such a solution to exist.

4 Turning point

We briefly discuss a result in domain perturbation theory for a necessary condition for a suitable turning point to exist in the Dirichlet waveguide. Given an arbitrary, piecewise smooth domain $\Omega(x) \subset \mathbb{R}^2$ and a domain $\Omega^\delta(x) \subset \mathbb{R}^2$, with a small (i.e., $O(\delta)$ for $0 < \delta \ll 1$), but not necessarily localised, perturbation from $\Omega(x)$, it follows from the work of [9], [12] that the perturbed Dirichlet eigenvalues can be expressed as a uniform asymptotic expansion in δ . In particular, if we denote by λ_j a particular Dirichlet eigenvalue on $\Omega(x)$, and by λ_j^δ the associated perturbed eigenvalue on $\Omega^\delta(x)$, then we can expand

$$\lambda_j^\delta = \lambda_j + \delta \bar{\lambda}_{j,1} + \delta^2 \bar{\lambda}_{j,2} + \dots$$

We then find that for the leading order variation, $\text{sgn}(\bar{\lambda}_{j,1}) = -\text{sgn}(f)$, wherein f is the normal distance between the boundaries $\partial\Omega$ and $\partial\Omega^\delta$. We may say, then, that a decrease in the size of the domain coincides with an increase in the magnitude of the eigenvalues of the Dirichlet Laplacian on it. In terms of our sound soft waveguide, a necessary condition for the existence of a turning point $\xi_* > 0$, with the properties described above, is for the size of the waveguide cross section to be decreasing at ξ_* .

In the example waveguide geometry that we will return to in Section 8.1, for instance, we consider a rotationally symmetric waveguide with a circular cross section of radius $h(\xi)$. It is straightforward to solve the eigenproblem over the cross section analytically and show that the eigenvalues are given by $\lambda_{mn}(\xi) = \beta_{mn}/h(\xi)$, where β_{mn} denotes the n th root of the Bessel J_m function. Hence we see directly in this case that the magnitude of the eigenvalues are inversely proportional to the radius of the cross section, as expected.

Unfortunately, no such result holds in general for the Neumann boundary and thus we have no simple sufficient condition on the waveguide geometry for a trapped mode solution to exist within the sound hard duct. However, we may be able to make an assertion about the behaviour of the eigenvalues of the Neumann Laplacian over its cross section, particularly for certain ‘simple’ geometries. Returning to the example of the waveguide of circular cross section, as mentioned above, we find that the Neumann eigenvalues over the circle of radius $h(\xi)$ are given similarly by $\lambda_{mn}(\xi) = \beta'_{mn}/h(\xi)$, where β'_{mn} denotes the n th root of the derivative of the Bessel J_m function.

5 Asymptotic matching

Given the quasi-mode solutions (3.7), with (3.8), and the existence of a turning point, $\xi_* > 0$, we now discuss the behaviour of the solution near this turning point, and show that the asymptotic expansions exhibit nonuniformity within a small layer around it. To do so, we consider the related reflection-transmission problem formed by the half-waveguide $D(\xi)$ in $\xi \in [0, \infty)$ and the constant duct $D(\xi) \equiv D(0)$ for $\xi \in (-\infty, 0]$.

5.1 Region of nonuniformity

We consider for clarity just the sound-soft waveguide, but note that the analysis shown also holds for the sound-hard duct. Given that the bulge maximum is at $\xi = 0$ and the waveguide tapers with increasing ξ , we can write the general quasi-mode solutions in the cut-on and cut-off regions as

$$\phi_o = \frac{IE \exp \left\{ i\epsilon^{-1} \int^\xi \sqrt{k^2 - \lambda^2(\bar{\xi})} d\bar{\xi} \right\} + RE \exp \left\{ -i\epsilon^{-1} \int^\xi \sqrt{k^2 - \lambda^2(\bar{\xi})} d\bar{\xi} \right\}}{(k^2 - \lambda^2(\xi))^{1/4}} + O(\epsilon) \quad \text{for } k \geq \lambda(\xi), \quad (5.1a)$$

$$\phi_d = \frac{TE \exp \left\{ -\epsilon^{-1} \int^\xi \sqrt{\lambda^2(\bar{\xi}) - k^2} d\bar{\xi} \right\}}{(\lambda^2(\xi) - k^2)^{1/4}} + O(\epsilon) \quad \text{for } k \leq \lambda(\xi), \quad (5.1b)$$

respectively, where the oscillatory solution is comprised of an incident and reflected component. We expect there to be a layer of nonuniformity between these two regions within which these expansions break down around a turning point ξ_* , where $\lambda(\xi_*) = k$, and consider the behaviour of the quasi-mode solutions as $\xi \rightarrow \xi_*$. Taking the Taylor series of $\lambda^2(\xi)$ near $\xi = \xi_*$, we have that

$$\lambda^2 = \lambda^2(\xi_*) + 2(\xi - \xi_*)\lambda'(\xi_*)\lambda(\xi_*) + O((\xi - \xi_*)^2).$$

Writing $\Delta = 2\lambda'(\xi_*)\lambda(\xi_*)$, so that $|k^2 - \lambda^2(\xi)| \sim |\Delta(\xi - \xi_*)|$, we can consider (5.1b), for example, to see that

$$\int^\xi \sqrt{\lambda^2(\bar{\xi}) - k^2} d\bar{\xi} \sim \Delta^{1/2} \int^{\xi - \xi_*} (\xi_* - \bar{\xi})^{1/2} d\bar{\xi} = \frac{2}{3} \Delta^{1/2} (\xi_* - \xi)^{3/2},$$

from which it is evident that the quasi-mode expansions break down in the region where $(\xi - \xi_*) = O(\epsilon^{2/3})$.

5.2 Interior layer

Having found the region within which the WKBJ expansion breaks down, we place an interior layer of width $O(\epsilon^{2/3})$ in this region by considering a solution in terms of the stretched variable $\chi = \epsilon^{-2/3}(\xi - \xi_*)$, which is $O(1)$ in the region of interest. We suppose that $\phi = X(\chi)E$, where $E_{\eta\eta} + E_{\zeta\zeta} + \lambda^2 E = 0$, and hence reformulate the original equation

$$\epsilon^2 \phi_{\xi\xi} + \phi_{\eta\eta} + \phi_{\zeta\zeta} + k^2 \phi = 0$$

such that it becomes

$$\epsilon^2 \left\{ \epsilon^{-4/3} X'' E + 2\epsilon^{-2/3} X' E_\xi + X E_{\xi\xi} \right\} + X E_{\eta\eta} + X E_{\zeta\zeta} + k^2 X E = 0.$$

Then by taking the Taylor expansions of E and its derivatives near ξ_* , we have

$$X (E_{\eta\eta} + E_{\zeta\zeta} + \lambda^2(\xi) A_0) \Big|_{\xi=\xi_*} + \epsilon^{2/3} \left\{ X'' E \Big|_{\xi=\xi_*} + X \chi \frac{\partial}{\partial \xi} (E_{\eta\eta} + E_{\zeta\zeta}) \Big|_{\xi=\xi_*} + X \chi \lambda^2(\xi_*) E_\xi \Big|_{\xi=\xi_*} \right\} + O(\epsilon^{4/3}) = 0.$$

The $O(1)$ terms vanish and we are left with

$$\begin{aligned} & \epsilon^{2/3} \left\{ X'' E|_{\xi=\xi_*} - X \chi \frac{\partial}{\partial \xi} (\lambda^2(\xi) E)|_{\xi=\xi_*} + X \chi \lambda^2(\xi_*) E_\xi|_{\xi=\xi_*} \right\} + O(\epsilon^{4/3}) = 0, \\ \implies & \epsilon^{2/3} \left\{ X'' E|_{\xi=\xi_*} - 2\lambda'(\xi_*) \lambda(\xi_*) X \chi E|_{\xi=\xi_*} \right\} + O(\epsilon^{4/3}) = 0, \\ \implies & \epsilon^{2/3} \left\{ X'' E|_{\xi=\xi_*} - \Delta X \chi E|_{\xi=\xi_*} \right\} + O(\epsilon^{4/3}) = 0. \end{aligned}$$

Hence, at leading order, we require

$$X'' - \Delta \chi X = 0, \quad (5.2)$$

whose solution is given by

$$X(\chi) = F \text{Ai}(\Delta^{1/3} \chi) + G \text{Bi}(\Delta^{1/3} \chi), \quad (5.3)$$

where Ai and Bi are the Airy functions of the first and second kind, and F and G are constants. Given (5.2), we see that $\Delta = 2\lambda'(\xi_*)\lambda(\xi_*) > 0$, where $\lambda(\xi)$ is an eigenvalue for the domain $D(\xi)$, corresponds to a cut-on region $\xi \leq \xi_*$, in the thicker part of the waveguide.

5.3 Matching coefficients

Given the tapering geometry of the waveguide, we seek solutions which are oscillatory for $\xi \leq \xi_*$ and decay in $\xi \geq \xi_*$. Following this assumption, we rewrite the expansion (5.1a)–(5.1b) as

$$\phi_o = \frac{IE \exp \left\{ i\epsilon^{-1} \int_{\xi}^{\xi_*} \sqrt{k^2 - \lambda^2(\bar{\xi})} d\bar{\xi} \right\} + RE \exp \left\{ -i\epsilon^{-1} \int_{\xi}^{\xi_*} \sqrt{k^2 - \lambda^2(\bar{\xi})} d\bar{\xi} \right\}}{(k^2 - \lambda^2(\xi))^{1/4}} + O(\epsilon) \quad \text{for } \xi \leq \xi_*, \quad (5.4a)$$

$$\phi_d = \frac{TE \exp \left\{ -\epsilon^{-1} \int_{\xi_*}^{\xi} \sqrt{\lambda^2(\bar{\xi}) - k^2} d\bar{\xi} \right\}}{(\lambda^2(\xi) - k^2)^{1/4}} + O(\epsilon) \quad \text{for } \xi \geq \xi_*. \quad (5.4b)$$

Then in the oscillatory region, we have that, as $\xi \rightarrow \xi_*^-$,

$$\phi_o \sim \frac{IE \exp \left\{ -i\frac{2}{3} \Delta^{1/2} (-\chi)^{3/2} \right\} + RE \exp \left\{ i\frac{2}{3} \Delta^{1/2} (-\chi)^{3/2} \right\}}{\epsilon^{1/6} \Delta^{1/4} (-\chi)^{1/4}},$$

whereas the Airy function in (5.3) has the asymptotic expansion (see [1]) as $\chi \rightarrow -\infty$

$$FE \text{Ai}(\Delta^{1/3} \chi) \sim \frac{FE \exp \left\{ -i\frac{2}{3} \Delta^{1/2} (-\chi)^{3/2} - i\frac{\pi}{4} \right\} + FE \exp \left\{ i\frac{2}{3} \Delta^{1/2} (-\chi)^{3/2} + i\frac{\pi}{4} \right\}}{2\sqrt{\pi} \Delta^{1/12} (-\chi)^{1/4}}$$

and we hence require the constants F , R and I to satisfy

$$\begin{aligned} F &= \frac{2\sqrt{\pi} e^{i\pi/4}}{\epsilon^{1/6} \Delta^{1/6}} I, \\ R &= e^{-i\pi/2} I \end{aligned} \quad (5.5)$$

in order to match these outer and inner expansions. Similarly, if we look at expansion (5.4b) as $\xi \rightarrow \xi_*^+$, we see that

$$\phi_d \sim \frac{TE \exp \left\{ -\frac{2}{3} \Delta^{1/2} \chi^{3/2} \right\}}{\epsilon^{1/6} \Delta^{1/4} \chi^{1/4}},$$

which matches with the interior expansion for $\chi \rightarrow \infty$

$$FE \text{Ai}(\Delta^{1/3} \chi) \sim \frac{FE \exp \left\{ -\frac{2}{3} \Delta^{1/2} \chi^{3/2} \right\}}{2\sqrt{\pi} \Delta^{1/12} \chi^{1/4}}$$

provided that

$$T = \frac{\epsilon^{1/6} \Delta^{1/6}}{2\sqrt{\pi}} F = e^{-i\pi/4} I. \quad (5.6)$$

We therefore have the exterior expansions

$$\phi_{o,ex} = \frac{2IE e^{-i\pi/4} \cos\left(\epsilon^{-1} \int_{\xi}^{\xi_*} \sqrt{k^2 - \lambda^2(\bar{\xi})} d\bar{\xi} - \frac{\pi}{4}\right)}{(k^2 - \lambda^2(\xi))^{1/4}} + O(\epsilon) \quad \text{for } \xi \leq \xi_*, \quad (5.7a)$$

$$\phi_{d,ex} = \frac{IE \exp\left\{-\epsilon^{-1} \int_{\xi}^{\xi_*} \sqrt{\lambda^2(\bar{\xi}) - k^2} d\bar{\xi} - i\frac{\pi}{4}\right\}}{(\lambda^2(\xi) - k^2)^{1/4}} + O(\epsilon) \quad \text{for } \xi \geq \xi_*, \quad (5.7b)$$

and the interior expansion

$$\phi_{in} = \frac{2IE\sqrt{\pi}e^{i\pi/4} \text{Ai}(\epsilon^{-2/3}\Delta^{1/3}(\xi - \xi_*))}{\epsilon^{1/6}\Delta^{1/6}} \quad \text{for } |\xi - \xi_*| = O(\epsilon^{2/3}), \quad (5.8)$$

giving us a one-term composite expansion, valid in the region of oscillatory behaviour $0 \leq \xi \leq \xi_* + O(\epsilon^{2/3})$,

$$\begin{aligned} \phi_{o,comp} = & \frac{2IE e^{-i\pi/4} \cos\left(\epsilon^{-1} \int_{\xi}^{\xi_*} \sqrt{k^2 - \lambda^2(\bar{\xi})} d\bar{\xi} - \frac{\pi}{4}\right)}{(k^2 - \lambda^2(\xi))^{1/4}} - \frac{2IE e^{-i\pi/4} \cos\left(\epsilon^{-1} \frac{2}{3} \Delta^{1/2} (\xi_* - \xi)^{3/2} - \frac{\pi}{4}\right)}{\Delta^{1/4} (\xi_* - \xi)^{1/4}} \\ & + \frac{2IE\sqrt{\pi}e^{i\pi/4} \text{Ai}(\epsilon^{-2/3}\Delta^{1/3}(\xi - \xi_*))}{\epsilon^{1/6}\Delta^{1/6}} + O(\epsilon), \quad (5.9) \end{aligned}$$

and a composite expansion valid in $\xi_* - O(\epsilon^{2/3}) \leq \xi < \infty$,

$$\begin{aligned} \phi_{d,comp} = & \frac{IE \exp\left\{-\epsilon^{-1} \int_{\xi}^{\xi_*} \sqrt{\lambda^2(\bar{\xi}) - k^2} d\bar{\xi} - i\frac{\pi}{4}\right\}}{(\lambda^2(\xi) - k^2)^{1/4}} - \frac{IE \exp\left\{-\epsilon^{-1} \frac{2}{3} \Delta^{1/2} (\xi - \xi_*)^{3/2} - i\frac{\pi}{4}\right\}}{\Delta^{1/4} (\xi - \xi_*)^{1/4}} \\ & + \frac{2IE\sqrt{\pi}e^{i\pi/4} \text{Ai}(\epsilon^{-2/3}\Delta^{1/3}(\xi - \xi_*))}{\epsilon^{1/6}\Delta^{1/6}} + O(\epsilon). \quad (5.10) \end{aligned}$$

6 Uniform expansion

Having observed the appearance of an Airy function of the first kind in the interior layer connecting the oscillatory and evanescent quasi-mode solutions, and indeed motivated by its use in constructing a uniformly valid expansion for the two-dimensional waveguide in [4], we now seek a uniformly valid expression for ϕ by expanding in terms of Airy functions. We introduce the new ansatz

$$\phi(\xi, \eta, \zeta) = B(\xi, \eta, \zeta) \text{Ai}\left(\epsilon^{-2/3}g(\xi)\right) + C(\xi, \eta, \zeta) \text{Ai}'\left(\epsilon^{-2/3}g(\xi)\right), \quad (6.1)$$

where

$$B(\xi, \eta, \zeta) = \sum_{j=0}^{\infty} \epsilon^{2j/3} B_j(\xi, \eta, \zeta), \quad C(\xi, \eta, \zeta) = \sum_{j=1}^{\infty} \epsilon^{2j/3} C_j(\xi, \eta, \zeta), \quad g(\xi) = \sum_{j=0}^{\infty} \epsilon^{2j/3} g_j(\xi),$$

and we restrict $g(\xi)$ to be real. The Ai' function is included to form an orthogonal basis, but we note that only the Ai function appears to leading order in the interior layer expansion (5.3), and so omit an $O(1)$ Ai' coefficient. We also note here that the presence of an interior layer of width $O(\epsilon^{2/3})$ suggests that our ansatz should be expressed in powers of $\epsilon^{2/3}$. Inserting this ansatz into the governing equation (2.3) yields a new hierarchy of equations in powers of ϵ , each in coefficients of the linearly independent functions Ai and Ai' . As with the quasi-mode expansions, we will consider first the sound soft waveguide, with a Dirichlet condition on its surface, followed by the sound hard duct with a Neumann condition.

6.1 Sound soft boundary

In the sound soft case, the boundary condition (2.5a) under this ansatz resolves simply to the requirement that

$$B_j(\xi, \eta, \zeta) = C_j(\xi, \eta, \zeta) = 0 \quad \text{for } (\eta, \zeta) \in \partial D(\xi) \text{ and } j \in \mathbb{N}. \quad (6.2)$$

So looking first at the $O(1)$ coefficients of Ai, we have

$$(\nabla^2 + \lambda^2) B_0 = 0 \quad (6.3)$$

where we now define $\lambda^2(\xi) = k^2 + g_0'^2(\xi)g_0(\xi)$. We denote the eigensolutions of (6.3) over $D(\xi)$, with boundary condition $B_0(\xi, \eta, \zeta) = 0$ for $(\eta, \zeta) \in \partial D(\xi)$, by $B_{0;n} = \beta_n(\xi)E_n$, with associated eigenvalues $\lambda_n(\xi)$, where E_n and λ_n are the eigensolutions of (3.4) for $n \in \mathbb{N}$.

For the sake of notational convenience, we take it as understood for the time being that we are dealing with an arbitrary n th eigensolution and omit the n -subscript notation. Since the eigenvalue $\lambda(\xi)$ is now known from (6.3), we have an expression for $g_0(\xi)$, taking care to note the separation between cases where $\lambda(\xi) \leq k$. As in the notation above, if we assume there to be a turning point $\xi_* > 0$ such that $\lambda(\xi_*) = k$ and $\lambda'(\xi_*) > 0$, then we may write

$$g_0 \equiv g_0(\xi) = \begin{cases} - \left(\frac{3}{2} \int_{\xi}^{\xi_*} \sqrt{k^2 - \lambda^2(\bar{\xi})} d\bar{\xi} \right)^{2/3} & \text{for } \xi \leq \xi_*, \\ \left(\frac{3}{2} \int_{\xi_*}^{\xi} \sqrt{\lambda^2(\bar{\xi}) - k^2} d\bar{\xi} \right)^{2/3} & \text{for } \xi \geq \xi_*. \end{cases} \quad (6.4)$$

We then find an expression identical to (6.3) in C_1 from the $O(\epsilon^{2/3})$ coefficients of Ai', such that $C_1 = \gamma(\xi)E$ with eigenvalue λ . Now the $O(\epsilon^{2/3})$ coefficients of Ai are

$$(\nabla^2 + \lambda^2) B_1 + (g_0'^2 g_1 + 2g_0 g_0' g_1') B_0 = 0,$$

which we multiply through by B_0 and integrate over $D(\xi)$ to yield the solvability condition

$$\begin{aligned} & \iint_{D(\xi)} B_0 (\nabla^2 + \lambda^2) B_1 + (g_0'^2 g_1 + 2g_0 g_0' g_1') B_0^2 dS = 0 \\ \implies & \iint_{D(\xi)} B_0 \nabla^2 B_1 - B_1 \nabla^2 B_0 dS + (g_0'^2 g_1 + 2g_0 g_0' g_1') \beta^2 \iint_{D(\xi)} E^2 dS = 0. \end{aligned} \quad (6.5)$$

The Dirichlet boundary condition causes the first integral to disappear upon an application of Green's Theorem, which along with the orthonormality of E leaves just

$$\begin{aligned} & g_0' g_1 + 2g_0 g_1' = 0 \\ \implies & g_1 = \mathcal{G} g_0^{-1/2} \end{aligned} \quad (6.6)$$

for some constant \mathcal{G} . Turning to the $O(\epsilon^{4/3})$ coefficients of Ai', we have that

$$(\nabla^2 + \lambda^2) C_2 + B_0 g_0'' + 2B_0 g_0' = 0, \quad (6.7)$$

which we once again integrate with B_0 and, given that C_2 vanishes on the boundary, find

$$\begin{aligned} & \iint_{D(\xi)} \{ B_0 (\nabla^2 + \lambda^2) C_2 + B_0^2 g_0'' + 2B_0 B_{0,\xi} g_0' \} dS = 0, \\ \implies & \iint_{D(\xi)} \frac{\partial}{\partial \xi} (B_0^2 g_0') dS = 0. \end{aligned}$$

Since B_0 vanishes on the boundary $\partial D(\xi)$, we may take the derivative outside of the integral, such that

$$\begin{aligned} & \frac{d}{d\xi} \left(\beta^2 g_0' \iint_{D(\xi)} E^2 dS \right) = 0, \\ \implies & \beta = \mathcal{B} |g_0'|^{-1/2} \end{aligned} \quad (6.8)$$

for constant \mathcal{B} . Hence, to leading order, we have the uniformly valid asymptotic expansions

$$\phi_n = \mathcal{B}_n |g'_{0;n}|^{-1/2} E_n \text{Ai} \left(\epsilon^{-2/3} g_{0;n} \right) + O(\epsilon^{2/3}), \quad (6.9)$$

for constants \mathcal{B}_n , where in a slight amendment to the notation in (6.4), to highlight the dependence of ξ_* upon k ,

$$g_{0;n} \equiv g_{0;n}(\xi; k) = \begin{cases} - \left(\frac{3}{2} \int_{\xi}^{\xi_*(k)} \sqrt{k^2 - \lambda_n^2(\bar{\xi})} d\bar{\xi} \right)^{2/3} & \text{for } \xi \leq \xi_*(k), \\ \left(\frac{3}{2} \int_{\xi_*(k)}^{\xi} \sqrt{\lambda_n^2(\bar{\xi}) - k^2} d\bar{\xi} \right)^{2/3} & \text{for } \xi \geq \xi_*(k). \end{cases} \quad (6.10)$$

6.2 Sound hard boundary

In the case of the sound hard boundary, we return to (6.1) under boundary condition (2.5b). Expanding this boundary condition in powers of ϵ and collecting coefficients of Ai and Ai' , we once again extract a hierarchy of equations that must be satisfied. Firstly, taking the $O(1)$ coefficients of the Ai terms, we have the condition

$$\nabla B_0 \cdot \mathbf{n}_{jk} = 0$$

on $\partial D(\xi)$. Thus at leading order, we have the PDE (6.3) with a Neumann boundary condition on $\partial D(\xi)$. Analogously to the sound soft case, we will denote the eigensolutions of such as $\tilde{B}_{0;n} = \tilde{\beta}_n(\xi) \tilde{E}_n$, with eigenvalues $\tilde{\lambda}_n(\xi)$, as in (3.11). Once again, we also extract a similar PDE for C_1 from the $O(\epsilon^{2/3})$ coefficients of Ai' with a Neumann boundary condition on ∂D , and thus we find the eigensolutions of C_1 to be $\tilde{C}_{1;n} = \tilde{\gamma}_n(\xi) \tilde{E}_n$, with eigenvalues $\tilde{\lambda}_n$. We shall briefly remove the subscript notation for clarity. Then the Neumann solvability condition is found to be identical to (6.6), and so we turn to the $O(\epsilon^{4/3})$ coefficients of Ai' , wherein we find the next solvability condition

$$\begin{aligned} & \iint_{D(\xi)} \tilde{B}_0 \left(\nabla^2 + \tilde{\lambda}^2 \right) C_2 + \frac{\partial}{\partial \xi} \left(\tilde{B}_0^2 \tilde{g}'_0 \right) dS = 0 \\ \implies & \oint_{\partial D(\xi)} \tilde{B}_0 \nabla C_2 \cdot \hat{\mathbf{n}}_{jk} - C_2 \nabla \tilde{B}_0 \cdot \hat{\mathbf{n}}_{jk} dC + \iint_{D(\xi)} \frac{\partial}{\partial \xi} \left(\tilde{B}_0^2 \tilde{g}'_0 \right) dS = 0. \end{aligned}$$

Returning to the hierarchy of equations extracted from the boundary condition, we find that the $O(\epsilon^{4/3})$ coefficients of Ai' yield the condition

$$\nabla C_2 \cdot \mathbf{n}_{jk} = \tilde{B}_0 \tilde{g}'_0 n_i \quad (6.11)$$

on $\partial D(\xi)$, such that the solvability condition becomes

$$\iint_{D(\xi)} \frac{\partial}{\partial \xi} \left(\tilde{B}_0^2 \tilde{g}'_0 \right) dS - \oint_{\partial D(\xi)} \tilde{B}_0^2 \tilde{g}'_0 \frac{n_i}{|\mathbf{n}_{jk}|} dC = 0.$$

Hence using the property given in Appendix A, the Reynolds transport theorem and the orthonormality of \tilde{E} , we have that

$$\frac{d}{d\xi} \left(\tilde{\beta}(\xi) \tilde{g}'_0(\xi) \right) = 0$$

and so may fix $\beta(\xi)$ up to constant multiple, as

$$\tilde{\beta}(\xi) = \tilde{\mathcal{B}} |\tilde{g}'_0|^{-1/2},$$

as in (6.8) in the sound soft case. Thus, reintroducing the subscript notation, we have the uniformly valid solution for the sound hard duct

$$\tilde{\phi}_n = \tilde{\mathcal{B}}_n |\tilde{g}'_{0;n}|^{-1/2} \tilde{E}_n \text{Ai} \left(\epsilon^{-2/3} \tilde{g}_{0;n} \right) + O(\epsilon^{2/3}) \quad (6.12)$$

with

$$\tilde{g}_{0;n} \equiv \tilde{g}_{0;n}(\xi; k) = \begin{cases} - \left(\frac{3}{2} \int_{\xi}^{\xi_*(k)} \sqrt{k^2 - \tilde{\lambda}_n^2(\bar{\xi})} d\bar{\xi} \right)^{2/3} & \text{for } \xi \leq \xi_*(k), \\ \left(\frac{3}{2} \int_{\xi_*(k)}^{\xi} \sqrt{\tilde{\lambda}_n^2(\bar{\xi}) - k^2} d\bar{\xi} \right)^{2/3} & \text{for } \xi \geq \xi_*(k). \end{cases} \quad (6.13)$$

7 Trapped modes

Given the leading order uniform expansions (6.9) and (6.12), valid for $\xi \in [0, \infty)$, we notice that by considering a reflection of this semi-infinite duct through the (η, ζ) -plane, we may extract leading order trapped mode solutions by considering those solutions which are either antisymmetric or symmetric about $\xi = 0$. We hence choose appropriate wavenumbers, k , for which either $\phi(0, \eta, \zeta) = 0$ or $\phi_\xi(0, \eta, \zeta) = 0$, respectively. Taking just the Dirichlet expansion, for example, this is equivalent to requiring either

$$\text{Antisymmetric: } \text{Ai} \left(\epsilon^{-2/3} g_{0;n}(0; k) \right) = 0, \quad (7.1a)$$

$$\text{Symmetric: } \text{Ai}' \left(\epsilon^{-2/3} g_{0;n}(0; k) \right) = 0, \quad (7.1b)$$

such that the trapped mode wavenumbers k are solutions of either

$$\text{Antisymmetric: } \epsilon^{-2/3} g_{0;n}(0; k_{np}^a) = z_p \quad (7.2a)$$

$$\text{Symmetric: } \epsilon^{-2/3} g_{0;n}(0; k_{np}^s) = z'_p \quad (7.2b)$$

where we take z_p and z'_p to be the zeros of Ai and Ai' respectively, ordered such that $0 > z_1 > z_2 \dots$, and similarly for z'_p .

8 Numerical verification

We note that the full numerical solution of (2.1) is, in an arbitrarily shaped waveguide, far from straightforward and in general, computationally expensive. In the case of a waveguide with circular cross section, however, we may use the azimuthal periodicity of the solution to reduce the problem to a two-dimensional one, to which we may apply a spectral collocation method (see [13, 14]) to demonstrate the accuracy of the asymptotic scheme.

8.1 Waveguide with circular cross section

We consider first the case of the rotationally symmetric, cylindrical waveguide with a slow, uniform bulge around $x = 0$. It is natural to describe this geometry in cylindrical coordinates and we are thus seeking trapped mode solutions of

$$\epsilon^2 r^2 \phi_{\xi\xi} + r^2 \phi_{rr} + r \phi_r + \phi_{\theta\theta} + k^2 r^2 \phi = 0 \quad \begin{cases} 0 \leq r < h(\xi, \theta), \\ 0 \leq \theta \leq 2\pi, \\ -\infty < \xi < \infty, \end{cases} \quad (8.1)$$

$$\phi \rightarrow 0 \quad \text{as } \xi \rightarrow \pm\infty,$$

where we will consider the example waveguide profile $h(\xi, \theta) \equiv h(\xi) = 1 + (h_1 - 1) \text{sech } \xi$, such that we are considering a duct with unit radius as ξ approaches $\pm\infty$ and bulges to a maximum radius of h_1 at $\xi = 0$. We will present the numerics for the sound soft duct, so that we also require $\phi(\xi, h(\xi), \theta) = 0$, although note that it is a straightforward adjustment to instead consider the sound hard case, wherein $\phi_r(\xi, h(\xi), \theta) = 0$. In this case, the waveguide cross section at any ξ is merely a circle of radius $h(\xi)$ and, as such, we are able to derive the transverse eigenvalues $\lambda(\xi)$ and uniform solution analytically. Following the methods of Section 6, we obtain the uniformly valid, leading order eigensolutions

$$\phi_{mn}(\xi, r, \theta) \sim \frac{\mathcal{A}_{mn} |g_{0;mn}(\xi)|^{1/4} \cos(m\theta) J_m(r \lambda_{mn}(\xi)) \text{Ai}(\epsilon^{-2/3} g_{0;mn}(\xi))}{h(\xi) |k^2 - \lambda_{mn}^2(\xi)|^{1/4}} \quad (8.2)$$

with associated eigenvalues $\lambda_{mn}(\xi) = \beta_{mn}/h(\xi)$, where β_{mn} denotes the n th root of the Bessel J_m function. We note here that, in fact, the Dirichlet Laplacian on the circular cross section has, for all except the zeroth azimuthal mode, double eigenvalues, with either a cosine or sine eigenfunction. In this unforced problem we choose, without loss of generality, the cosine solution to preserve the zeroth azimuthal mode. Given the waveguide topography under consideration, it is clear that, for

all positive ξ , $h'(\xi) \leq 0$, and hence $\lambda'_{mn}(\xi) > 0$. Thus, as discussed in Section 4, we may reasonably anticipate the existence of one or more turning points per wavenumber, and hence associated trapped mode solutions.

We notice that the separability and behaviour of (8.2) in the θ -direction permits the rewriting of (8.1) as

$$\epsilon^2 r^2 \phi_{\xi\xi} + r^2 \phi_{rr} + r \phi_r + (k^2 r^2 - m^2) \phi = 0 \quad \begin{cases} 0 \leq r < h(\xi), \\ -\infty < \xi < \infty, \end{cases}$$

$$\phi \rightarrow 0 \quad \text{as } \xi \rightarrow \pm\infty, \quad (8.3)$$

such that the θ -derivative has been removed and hence the three-dimensional problem has been reduced to m two-dimensional problems. In general this modification will not be possible and, to apply a full three-dimensional spectral collocation method with M , N and P interpolation points in the spatial directions will require the construction and calculation of the eigenvalues of an $(M \times N \times P)$ -by- $(M \times N \times P)$ differentiation matrix—not a computationally cheap exercise.

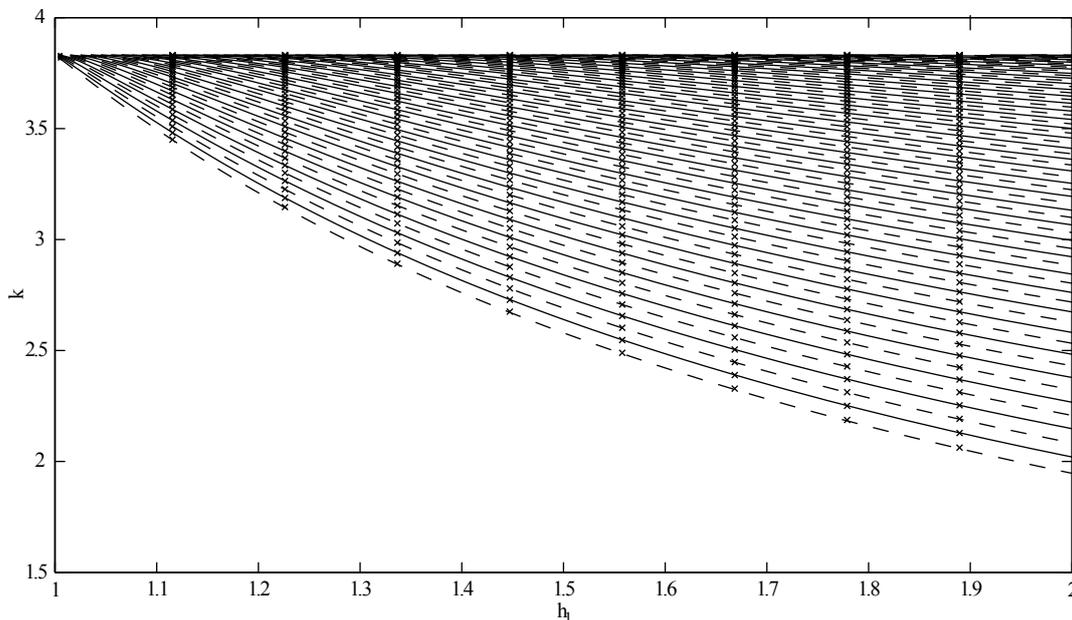


Figure 1: Trapped mode wavenumbers for the first azimuthal and radial modes in the sound soft waveguide with profile $h(\xi) = 1 + (h_1 - 1) \operatorname{sech} \xi$ and a slow parameter $\epsilon = 0.1$. Solid and dashed lines represent antisymmetric and symmetric trapped modes, respectively, and crosses denote spectral approximations.

In order to apply the spectral method, we map the domain $\Omega = \{(\xi, r) : -\infty < \xi < \infty, 0 \leq r \leq h(\xi)\}$ onto a semi-infinite rectangular domain $\Omega' = \{(\xi, \rho) : -\infty < \xi < \infty, -1 \leq \rho \leq 1\}$ via the transformation $\rho = 2r/h(\xi) - 1$, such that it follows from (8.3) that the function $\psi(\xi, \rho) = \phi(\xi, r)$ satisfies

$$\epsilon^2 (\rho + 1)^2 h^2 (\psi_{\xi\xi} + 2\psi_{\rho\xi}\rho_{\xi} + \psi_{\rho\rho}\rho_{\xi\xi} + \psi_{\rho\rho}\rho_{\xi}^2) + 4(\rho + 1)^2 \psi_{\rho\rho} + 4(\rho + 1) \psi_{\rho} + ((\rho + 1)^2 h^2 k^2 - 4m^2) \psi = 0$$

for $\rho \in [-1, 1)$, $\xi \in (-\infty, \infty)$, where

$$\rho_{\xi} = -\frac{(\rho + 1) h'}{h}, \quad \rho_{\xi\xi} = \frac{(\rho + 1) (2h'^2 - hh'')}{h^2}$$

and with boundary conditions $\psi \rightarrow 0$ as $\xi \rightarrow \pm\infty$ and $\psi(\xi, 1) = 0$. We may simplify the problem further by noting that we are seeking either symmetric or antisymmetric modes, and may consider only the half plane $\xi \in [0, \infty)$, with the respective boundary conditions $\psi_{\xi}(0, \rho) = 0$ or $\psi(0, \rho) = 0$

and $\psi \rightarrow 0$ as $\xi \rightarrow \infty$. We then apply a Chebyshev interpolant in the ρ -direction and a Laguerre interpolant in the ξ -direction to discretise (8.4) and approximate the solution.

Taking $\epsilon = 0.1$ as our small parameter, Figure 1 depicts the comparisons between the asymptotic scheme and spectral approximation for the first azimuthal and radial modes, i.e., for $m = n = 1$, using a spectral grid of 60-by-15 points in the ξ -, ρ -directions, respectively. This is sufficient to give the ‘tenth’ longitudinal wavenumber, k_{115}^a , to 8d.p. of accuracy. It can be seen that the leading order asymptotic approximation is in excellent agreement with the numerical scheme. Indeed, turning to Table 1, we see that if we fix $h_1 = 1.5$ as a sample bulge radius, the first wavenumber carries the greatest error at $\approx 0.136\%$, and in fact the average error over the first ten calculated wavenumbers is just $\approx 0.027\%$.

$k_{mnp}^{a;s}$	Asymptotic	Spectral	Rel. error
k_{111}^s	2.579452576	2.582965631	0.1360%
k_{111}^a	2.639524568	2.638546418	0.0371%
k_{112}^s	2.691204857	2.692021554	0.0303%
k_{112}^a	2.743867464	2.743506731	0.0132%
k_{113}^s	2.792622563	2.793105341	0.0173%
k_{113}^a	2.841074058	2.840910293	0.0058%
k_{114}^s	2.886647825	2.887005486	0.0124%
k_{114}^a	2.931540143	2.931467017	0.0025%
k_{115}^s	2.974076448	2.974364182	0.0097%
k_{115}^a	3.015787193	3.015760316	0.0009%

Table 1: Relative error between the asymptotic and numerical calculations of the wavenumbers of the first ten symmetric and antisymmetric longitudinal modes for the duct profile $h(\xi) = 1 + (h_1 - 1) \operatorname{sech} \xi$, with $h_1 = 1.5$ and small parameter $\epsilon = 0.1$.

It is also interesting to note that, whilst we have exploited the ‘small’ parameter ϵ in order to construct an asymptotic scheme, we now find that the accuracy of the approximation remains fairly high with much larger ϵ . Table 2 shows the relative error for the first symmetric and antisymmetric trapped mode wavenumbers for the $m = n = 1$ modes with varying ϵ , wherein we see that even with a value of $\epsilon = 1$, the relative error for the first symmetric trapped mode is only $\approx 1.156\%$.

ϵ	Symmetric			Antisymmetric		
	Asymptotic	Spectral	Rel. error	Asymptotic	Spectral	Rel. error
0.001	2.5547226	2.554759281	0.0014%	2.555346426	2.555336403	0.0004%
0.2008	2.604205704	2.611047438	0.2620%	2.720389611	2.718692029	0.0624%
0.4006	2.652040945	2.665238773	0.4952%	2.867699109	2.865803923	0.0661%
0.6004	2.69830863	2.717850757	0.7190%	2.999704018	2.999052109	0.0217%
0.8002	2.743081633	2.769085764	0.9391%	3.118231954	3.119676788	0.0463%
1	2.786426339	2.818998892	1.1555%	3.224699631	3.228506471	0.1179%

Table 2: Relative error between the asymptotic and spectral calculations of the first symmetric and antisymmetric mode wavenumbers for varying ‘small’ parameter ϵ . The duct profile is again $h(\xi) = 1 + (h_1 - 1) \operatorname{sech} \xi$, with $h_1 = 1.5$.

8.2 Elliptical waveguide

The method presented here can be applied to any waveguide geometry featuring a smooth cross section and slow, axially symmetric perturbation, and is of the most interest in any case where it is extremely difficult or computationally expensive to numerically solve the full three-dimensional problem. We demonstrate this by considering a waveguide which is elliptical at large $|\xi|$ and has a localised, one-sided bulge around $\xi = 0$. Once more in cylindrical coordinates, we consider the

problem described by (8.1), with the waveguide profile given instead by

$$h(\xi, \theta) = \frac{2 + 2(h_1 - 1) \operatorname{sech} \xi \exp \left\{ -16 \sin \left(\frac{\theta}{2} - \frac{\pi}{4} \right)^2 \right\}}{\sqrt{\cos^2(\theta) + 4 \sin^2(\theta)}}, \quad (8.4)$$

and a bulge height $h_1 = 1.5$, as shown in Figures 2(a) and 2(b). We map this geometry onto a

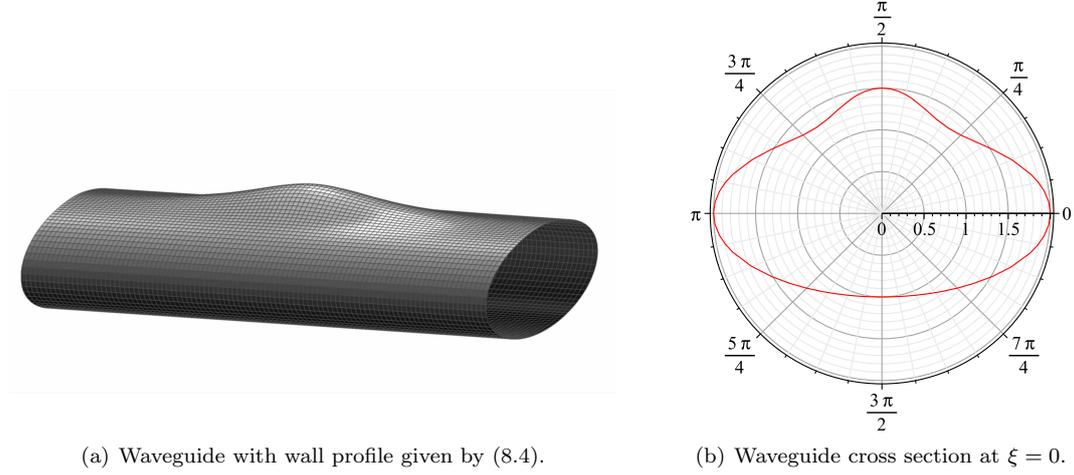


Figure 2: Elliptical waveguide with an axially symmetric bulge around $\xi = 0$.

circular cross section cylinder of unit radius in order to apply a spectral collocation method, using the transformation $\rho = r/h(\xi, \theta)$. Thus (8.1) becomes

$$\begin{aligned} \epsilon^2 \rho^2 h^2 (\psi_{\xi\xi} + 2\psi_{\rho\xi}\rho_{\xi} + \psi_{\rho\rho}\rho_{\xi\xi} + \psi_{\rho\rho}\rho_{\xi}^2) + \rho^2 \psi_{\rho\rho} + \rho\psi_{\rho} + \\ \psi_{\rho\rho}\rho_{\theta}^2 + 2\psi_{\theta\rho}\rho_{\theta} + \psi_{\rho\rho}\rho_{\theta\theta} + \psi_{\theta\theta} + \rho^2 h^2 k^2 \psi = 0, \end{aligned}$$

where, similarly to the previous case,

$$\begin{aligned} \rho_{\xi} &= -\frac{(\rho + 1) h_{\xi}}{h}, & \rho_{\xi\xi} &= \frac{(\rho + 1) (2h_{\xi}^2 - hh_{\xi\xi})}{h^2}, \\ \rho_{\theta} &= -\frac{(\rho + 1) h_{\theta}}{h}, & \rho_{\theta\theta} &= \frac{(\rho + 1) (2h_{\theta}^2 - hh_{\theta\theta})}{h^2} \end{aligned}$$

and such that $\rho \in [0, 1)$, $\theta \in [0, 2\pi]$ and $\xi \in (-\infty, \infty)$. It is of note here that, following the methods of [13], we map ρ onto the interval $[0, 1)$, as opposed to the ‘full’ Chebyshev interval $[-1, 1)$, and later discard the negative Chebyshev interpolation points. This ensures a much more even spacing of Chebyshev points in the ρ -direction and thus far greater accuracy for a marginal computational cost. We note, however, that even with such a modification, a full spectral collocation method with M , N and P interpolation points in the ξ -, r - and θ -directions, respectively, involves calculating the eigenvalues of an $(M \times N \times P)$ -by- $(M \times N \times P)$ matrix. For the purposes of this verification, we apply this method with an interpolation grid of $(18 \times 18 \times 20)$ points, such that the problem is discretised into a 6480-by-6480 matrix.

Comparing the results of this numerical scheme with our leading order asymptotic results, we see from Table 3 that for the first eight wavenumbers, they are once again in excellent agreement. Furthermore, once the wavenumbers have been found, the full leading order solution is immediately obtained from substitution into (6.9) - (6.10). Figure 3 shows the cross section of the first four leading order antisymmetric trapped mode solutions (k_{np} for $n = 1 \dots 4$) inside this particular waveguide geometry, taken slightly away from the bulge centre, at $\xi = 0.2$.

$k_{np}^{(a;s)}$	Asymptotic	Spectral	Rel. error
k_{11}^s	1.785610713	1.781332958	0.2401%
k_{11}^a	1.803862094	1.798011732	0.3254%
k_{12}^s	1.81886783	1.814169297	0.2590%
k_{12}^a	1.833185541	1.828951766	0.2315%
k_{13}^s	1.845360806	1.842142284	0.1747%
k_{13}^a	1.856266363	1.853653319	0.1410%
k_{14}^s	1.865307241	1.863482336	0.0979%
k_{14}^a	1.872942432	1.87174977	0.0637%

Table 3: Relative error between the asymptotic and spectral collocation calculations of the first eight wavenumbers for the duct described in profile by (8.4), with small parameter $\epsilon = 0.1$ and $h_1 = 1.5$.

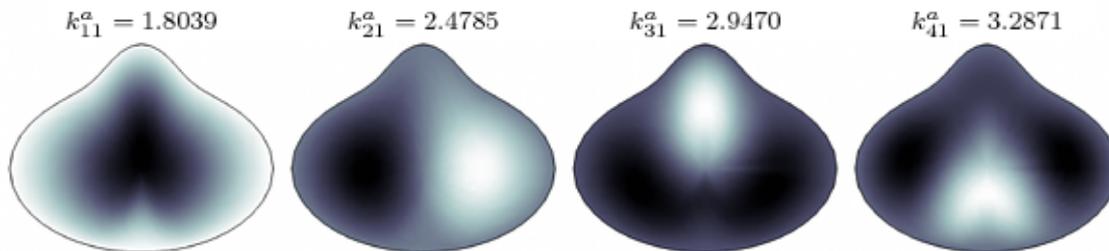


Figure 3: Cross sections of the first four (in the azimuthal / radial directions) antisymmetric trapped mode solutions at $\xi = 0.2$, in the sound-soft waveguide given in profile by (8.4), with a bulge height of $h_1 = 1.5$.

9 Conclusions

In this paper, we have presented an extension of the asymptotic scheme derived in [4] for calculating the trapped mode solutions inside a two-dimensional sound soft or sound hard waveguide, and shown that it is applicable and numerically accurate in the case of a three-dimensional duct. In doing so, we have replaced a computationally expensive and cumbersome numerical solution with a method involving only the solution of a nonlinear algebraic expression.

We also note that, despite the construction of the scheme hinging on the slowly varying nature of the waveguide geometry (i.e., a variation over a length scale $1/\epsilon$ for small ϵ), the leading order approximations obtained are still accurate in some cases to a relative error of $\approx 1\%$ when the ‘slow’ parameter is as large as $O(1)$.

Furthermore, we have introduced no restrictions on the overall size or shape of the perturbation, other than the smoothness of the cross section and symmetry around $x = 0$, such that the method presented here may be applied to a variety of waveguide geometries in which we have an axially symmetric perturbation. We propose to extend this work to encompass other waveguide geometries with relaxed conditions, wherein the cross section need not be smooth, and to consider the case of trapped modes in an elastic plate.

Acknowledgements

The authors wish to thank Professor Michael Levitin and Professor Michael Baines for their helpful discussions and interest in this work. We are also grateful to the EPSRC, who financially support this research.

A Appendix

Given a smooth surface $\mathbf{S} = (x, y, z)$, parameterised by

$$x = x, \quad y = \eta(x, \gamma), \quad z = \zeta(x, \gamma),$$

it is clear that the rate of change of the cross section with x is given by $\mathbf{v} = (\eta_x, \zeta_x)$. We also have that the surface normal is given by the cross product

$$\mathbf{n} = \frac{\partial \mathbf{S}}{\partial x} \times \frac{\partial \mathbf{S}}{\partial \gamma} = (\eta_x \zeta_\gamma - \eta_\gamma \zeta_x) \mathbf{i} - \zeta_\gamma \mathbf{j} + \eta_\gamma \mathbf{k},$$

and hence

$$\mathbf{v} \cdot \mathbf{n}_{jk} = \zeta_x \eta_\gamma - \eta_x \zeta_\gamma = -n_i.$$

References

- [1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55. Dover publications, 1964.
- [2] A. T. I. Adamou, D. Gridin, and R. V. Craster. Acoustic quasi-modes in slowly varying cylindrical tubes. *The Quarterly Journal of Mechanics and Applied Mathematics*, 58(3):419–438, 2005.
- [3] C.M. Bender and S.A. Orszag. *Advanced mathematical methods for scientists and engineers: Asymptotic methods and perturbation theory*, volume 1. Springer Verlag, 1978.
- [4] Nicholas R.T. Biggs. Wave trapping in a two-dimensional sound-soft or sound-hard acoustic waveguide of slowly-varying width. *Wave Motion*, 49(1):24 – 33, 2012.
- [5] P. Duclos and P. Exner. Curvature-induced bound states in quantum waveguides in two and three dimensions. *Reviews in Mathematical Physics*, 7(1):73–102, 1995.
- [6] Dmitri Gridin and Richard V Craster. Quasi-modes of a weakly curved waveguide. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 459(2039):2909–2931, 2003.
- [7] Dmitri Gridin and Richard V. Craster. Lamb quasi-modes in curved plates. *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, 460(2046):1831–1847, 2004.
- [8] Dmitri Gridin, Richard V Craster, and Alexander T.I Adamou. Trapped modes in curved elastic plates. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 461(2056):1181–1197, 2005.
- [9] Tosio Kato. *Perturbation theory for linear operators*. Springer, Berlin, 1995.
- [10] Julia Postnova and Richard V. Craster. Trapped modes in topographically varying elastic waveguides. *Wave Motion*, 44(3):205 – 221, 2007.
- [11] Julia Postnova and Richard V. Craster. Trapped modes in elastic plates, ocean and quantum waveguides. *Wave Motion*, 45(4):565 – 579, 2008.
- [12] F. Rellich. *Perturbation theory of eigenvalue problems*. Notes on mathematics and its applications. Gordon and Breach, 1969.
- [13] Lloyd Trefethen. *Spectral methods in MATLAB*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2000.
- [14] J. A. C. Weideman and S. C. Reddy. A MATLAB differentiation matrix suite. *ACM Transactions of Mathematical Software*, 26:465 – 519, 2000.