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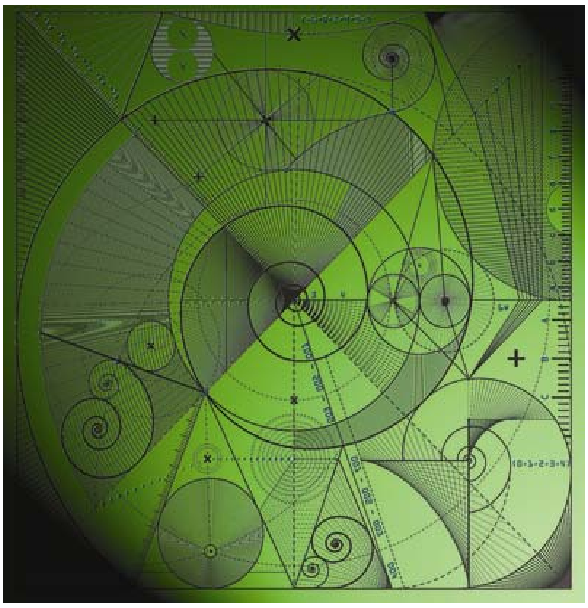

THE UNSTEADY FLOW OF A WEAKLY COMPRESSIBLE FLUID IN A THIN POROUS LAYER II: THREE-DIMENSIONAL THEORY

D. J. NEEDHAM*, S. LANGDON†, B. A. SAMSON‡, AND J. P. GILCHRIST‡

Abstract. We consider the problem of determining the pressure and velocity fields for a weakly compressible fluid flowing in a three-dimensional layer, composed of an inhomogeneous, anisotropic porous medium, with vertical side walls and variable upper and lower boundaries, in the presence of vertical wells injecting and/or extracting fluid. Numerical solution of this three-dimensional evolution problem may be expensive, particularly in the case that the depth scale of the layer h is small compared to the horizontal length scale l, a situation which occurs frequently in the application to oil reservoir recovery and which leads to significant stiffness in the numerical problem. Under the assumption that $\epsilon \propto h/l \ll 1$, we show that, to leading order in ϵ, the pressure field varies only in the horizontal directions away from the wells (the outer region). We construct asymptotic expansions in ϵ in both the inner (near the wells) and outer regions and use the asymptotic matching principle to derive expressions for all significant process quantities. The only computations required are for the solution of non-stiff linear, elliptic, two-dimensional boundary-value and eigenvalue problems. This approach, via the method of matched asymptotic expansions, takes advantage of the small aspect ratio of the layer, ϵ, at precisely the stage where full numerical computations become stiff, and also reveals the detailed structure of the dynamics of the flow, both in the neighborhood of wells and away from wells.

Key words. reservoir simulation, oil recovery, thin porous layer, matched asymptotics

AMS subject classifications. 35K15, 35K20, 76M45, 76S05, 86A99

1. Introduction.

The accurate and efficient simulation of fluid flow in oil and gas reservoirs is an essential tool in the management of hydrocarbon reserves. There has been a huge research effort in recent years to develop robust and accurate reservoir simulators based on numerical methods such as finite difference or finite element techniques; see for example recent reviews such as [5, 3] and the references therein. This kind of fully numerical approach has proved highly successful in modeling a wide variety of complicated physical processes in reservoirs, allowing the user to predict the effect of a change to well locations or production rates, for example. However, whilst reservoir simulators of this type will continue to play a crucial role in the industry, to use them takes considerable expertise and time, with long execution times often necessary for certain types of problems such as hydraulically fractured wells.

There is thus a place for cruder approximation techniques; solution schemes that may have some limitations in their accuracy, or in the range of situations that they can model, but which can allow a reservoir or production engineer to perform a rapid study of their reservoir in order to obtain a broad understanding of the dynamical processes and to make approximate costing forecasts.

Analytical approaches, although they may require some simplifying assumptions, can be extremely fast, and avoid the timestepping, stiffness and convergence issues seen with numerically based simulators. Also, given the speed and reliability of analytical results, there is a clear opportunity to exploit their use in history matching

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studies. Such schemes have thus received considerable attention in the literature over the years, both for well testing applications [11, 6, 17, 2, 10] and also for full field simulation problems, for porous media with homogeneous and anisotropic permeability [4, 20, 15], and for more complex problems involving inhomogeneous permeability and variable geometry [14, 16, 9]. For further details we refer to [13].

Here, we consider the problem of determining the pressure and velocity fields for a weakly compressible fluid flowing in a horizontal layer of porous medium with variable upper and lower boundaries. Vertical wells injecting and/or extracting fluid from the layer are considered as line sources and sinks respectively. Our approach to solving the problem rests on the assumption that the depth scale of the layer h is small compared to the length scale of the layer l, as is often the case in geophysical applications. We do, however, allow the porous layer to have both inhomogeneous and anisotropic permeability. As the ratio h/l decreases, efficient application of numerical schemes becomes harder, whilst the problem becomes more amenable to solution via matched asymptotic theory.

Here, we consider the case of fully three-dimensional flow, building on previous theory for a model two-dimensional problem [13]. We introduce the parameter $\epsilon \propto h/l$ (defined explicitly in (2.17) below), and construct asymptotic expansions for the solutions to the equations of motion of the fluid in increasing powers of ϵ, with $0 < \epsilon \ll 1$. In the vicinity of a well (the *inner* region) the pressure field is fully three-dimensional, but away from the wells (the *outer* region) the pressure field is only two-dimensional, to leading order in ϵ. This immediately leads to a reduction in complexity, but rather than solving the full equations of motion numerically in the inner and outer regions, we construct asymptotic expansions in both the inner and outer regions which are then matched, via the Van Dyke asymptotic matching principle [21], enabling us to derive explicit expressions for all significant process quantities.

We begin in §2 by stating the full equations of motion in the porous layer. Conservation of mass and momentum lead to a strongly parabolic linear initial-boundary value problem for the dynamic fluid pressure (from which the fluid velocity field can be deduced), with weighted Neumann boundary conditions, under the assumption that the walls are impenetrable to the fluid in the porous layer. This initial-boundary value problem has a unique solution, but its direct computation would be expensive, primarily due to stiffness when $0 < \epsilon \ll 1$. We thus consider the associated pseudo-steady state problem [PSSP], a linear strongly elliptic weighted Neumann problem, which also has a unique solution (up to the addition of a constant). The solution of [PSSP] is considered in §3. Subtracting the solution of the pseudo-steady state problem from the solution of the initial-boundary value problem leads to a linear, strongly parabolic, regular, homogeneous initial-boundary value problem with no singularities at the sources and sinks. The solution of this problem leads to a regular self-adjoint eigenvalue problem [EVP] whose solution is considered in §4.

Rather than solving [PSSP] and [EVP] directly, the solution to each problem is considered in the asymptotic limit $\epsilon \to 0$, via the method of matched asymptotic expansions. To solve [PSSP], we begin with the situation when the wells are well spaced (spacing $O(1)$ as $\epsilon \to 0$) and are away from the side walls of the layer, with generalisations for the cases of wells either close to a wall, or close together (within a distance of $O(\epsilon)$ as $\epsilon \to 0$), being considered in §3.1 and §3.2 respectively.

Whereas for the two-dimensional problem considered in [13] the asymptotic solutions to [PSSP] and [EVP] could be constructed analytically up to $O(\epsilon^2)$, for the three-dimensional problem considered here the outer problem (away from the wells) reduces to a linear, inhomogeneous, strongly elliptic two-dimensional boundary value problem [BVP], on the layer cross-sectional projection, that must in general be solved numerically. This can be achieved via standard finite or boundary element methods, and a detailed consideration of the numerical solution of [BVP] is described in [12]. In the inner regions, determination of the leading order terms reduces to the solution of a strongly elliptic problem whose solution can be written analytically in terms of the eigenvalues and corresponding eigenfunctions of a regular Sturm-Liouville eigenvalue problem, identical to that considered in [13]. The asymptotic solution of [EVP] in §4 reduces to a regular two-dimensional strongly elliptic problem, whose numerical solution can also be achieved via standard finite element methods in a very similar manner to the solution of [BVP], and this is also considered in [12]. Finally in §5 we draw some conclusions.

We remark further that full implementation details for an efficient numerical scheme for the computation of the dynamic fluid pressure and the fluid velocity field throughout the layer are provided in [12], where we also apply the theory to some simple model examples, demonstrating the exceptional computational efficiency of our approach via matched asymptotic expansions.

2. Equations of motion. As in Needham et. al [13] we again consider the flow of a weakly compressible fluid in the presence of sources and sinks, in a reservoir of porous medium with variable upper and lower boundary. The reservoir has permeability which is both inhomogeneous and anisotropic. Whilst in [13] we restricted attention to two-dimensional flow in a two-dimensional reservoir, we now extend the theory to fully three-dimensional flow in a three-dimensional reservoir. We adopt the same notation and the same physical model as in [13], and so omit a detailed description of the modelling here. Thus, following [13], the equations of motion of the fluid in the porous reservoir may be written as,

$$c_t\bar{\phi}\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right)\frac{\partial p}{\partial t}+\nabla.\mathbf{q}=\sum_{i=1}^{N}s_i\left(\frac{z}{h}\right)\frac{1}{l^2}\delta\left(\frac{x-x_i}{l}\right)\delta\left(\frac{y-y_i}{l}\right), \tag{2.1}$$

$$\mathbf{q}=-\underline{\underline{D}}\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right)(\nabla p+\rho_0 g\mathbf{k}), \tag{2.2}$$

for all $(x,y,z)\in M$, $t\in(0,\infty)$. Here (x,y,z) are rectangular cartesian coordinates with z pointing vertically upwards. The interior of the porous reservoir is denoted by $M\subset\mathbb{R}^3$ and its impermeable boundary by $\partial M\subset\mathbb{R}^3$, with $\bar{M}=M\cup\partial M$. The region $\bar{M}$ is taken as a finite section of a generalized cylinder which has its axis aligned with the z-axis and its cross section bounded by the simple closed piecewise smooth curve $\partial\Omega_l\subset\mathbb{R}^2$, which has interior $\Omega_l\subset\mathbb{R}^2$, with $\bar{\Omega}_l=\Omega_l\cup\partial\Omega_l$. Here $l>0$ is the horizontal length scale associated with $\bar{\Omega}_l$. The upper and lower boundary surfaces of the reservoir are described by

$$\left.\begin{array}{l} z=hz_+(x/l,y/l)\\ z=hz_-(x/l,y/l)\end{array}\right\}(x,y)\in\bar{\Omega}_l,$$

respectively, with $h(>0)$ being the vertical length scale associated with the reservoir, and $z_+,z_-:\bar{\Omega}_1\mapsto\mathbb{R}$ being such that

$$z_+,z_-\in C^1(\bar{\Omega}_1), \tag{2.3}$$

and

$$z_+(x,y)>z_-(x,y)\text{ for all }(x,y)\in\bar{\Omega}_1. \tag{2.4}$$

The normal fields on the upper and lower surfaces are then given by

$$\begin{aligned}\mathbf{n}_+(x,y) &= \left(-\tfrac{h}{l} z_{+x}(x,y), -\tfrac{h}{l} z_{+y}(x,y), 1\right)\\ \mathbf{n}_-(x,y) &= \left(\tfrac{h}{l} z_{-x}(x,y), \tfrac{h}{l} z_{-y}(x,y), -1\right)\end{aligned}$$

for all $(x,y) \in \bar{\Omega}_1$, with the normals directed out of $\bar{M}$. The situation is illustrated in Figure 2.1. The $N(\in \mathbb{N})$ vertical line sources/sinks embedded within $\bar{M}$, which

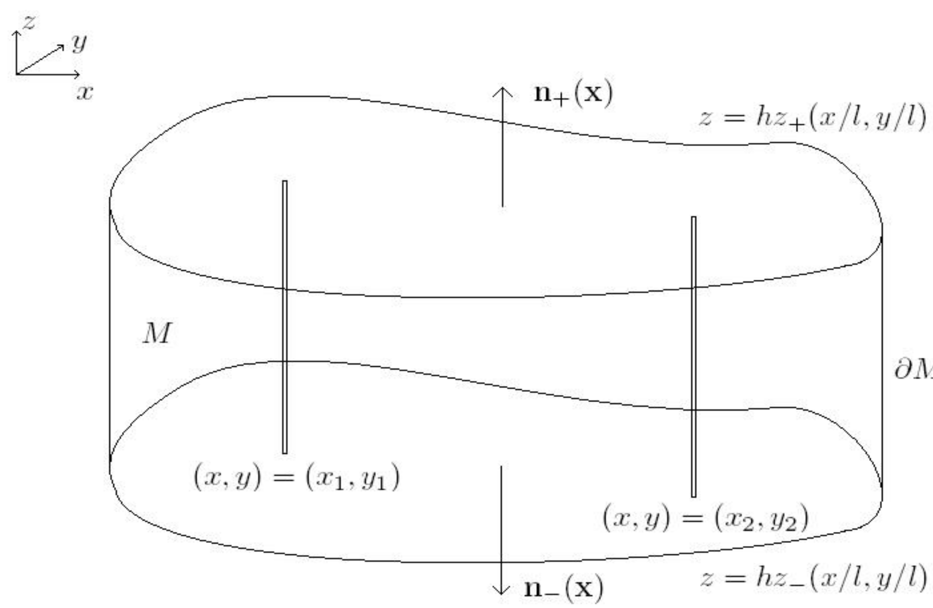

FIG. 2.1. *Porous layer $M \subset \mathbb{R}^3$, with impermeable boundary ∂M*

extend from the lower surface to the upper surface of $\bar{M}$, are located at

$$(x_i, y_i) \in \Omega_l, \quad i = 1, \dots, N.$$

The functions $s_i : \left[z_-\left(\frac{x_i}{l}, \frac{y_i}{l}\right), z_+\left(\frac{x_i}{l}, \frac{y_i}{l}\right)\right] \mapsto \mathbb{R}, \quad i = 1, \dots, N$, represent the line source/sink volumetric strengths, with

$$s_i \in C\left(\left[z_-\left(\frac{x_i}{l}, \frac{y_i}{l}\right), z_+\left(\frac{x_i}{l}, \frac{y_i}{l}\right)\right]\right), \quad i = 1, \dots, N.$$

The total volume flux from the i^{th} line source/sink is then

$$Q_i = \int_{hz_-(x_i/l, y_i/l)}^{hz_+(x_i/l, y_i/l)} s_i\left(\frac{\lambda}{h}\right) \mathrm{d}\lambda, \quad i = 1, \dots, N. \tag{2.5}$$

In (2.1) $\delta : \mathbb{R} \mapsto \mathbb{R}$ is the usual Dirac delta function. The operator $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, in terms of the cartesian coordinates (x,y,z), and $t \geq 0$ is time. The fluid velocity field and pressure field at each point in $\bar{M}$ are represented in (2.1) and (2.2) by

$$\begin{aligned}&\mathbf{q} = \mathbf{q}(\mathbf{r},t) = (u(\mathbf{r},t), v(\mathbf{r},t), w(\mathbf{r},t)),\\ &p = p(\mathbf{r},t),\end{aligned}$$

for each $(\mathbf{r},t) \in \bar{M} \times [0,\infty)$. The permeability tensor $\underline{D}(x/l, y/l, z/h)$ has the form

$$\underline{D}\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right) = \left(\begin{array}{ccc} D_0^H D_x(\frac{x}{l},\frac{y}{l},\frac{z}{h}) & 0 & 0 \\ 0 & D_0^H D_y(\frac{x}{l},\frac{y}{l},\frac{z}{h}) & 0 \\ 0 & 0 & D_0^L D_z(\frac{x}{l},\frac{y}{l},\frac{z}{h}) \end{array} \right),$$

with $D_x, D_y, D_z : \bar{M} \mapsto \mathbb{R}^+$ such that

$$D_x, D_y, D_z \in C^1(\bar{M}). \tag{2.6}$$

Here $D_0^H, D_0^L > 0$ are permeability scales in the horizontal and vertical directions respectively, with the functions D_x, D_y and D_z representing the variable permeabilities in the x, y and z directions respectively. We require the permeability in the reservoir to be bounded above zero, so that there is a constant $D_m > 0$ such that

$$D_x\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right), D_y\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right), D_z\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right) \geq D_m > 0, \tag{2.7}$$

for all $(x,y,z) \in \bar{M}$. Finally, $\rho_0 > 0$ is the reference density and p_0 is the reference pressure for the weakly compressible fluid (see [13]), g is the acceleration due to gravity, with $\mathbf{k}$ being the unit vector pointing vertically upwards, whilst $c_t = \Phi_0 \tilde{c}_t$, with $\tilde{c}_t$ being the isothermal expansion coefficient and $0 < \Phi_0 < 1$ being the mean porous matrix porosity. When the porous matrix porosity $\Phi : \bar{M} \mapsto (0,1)$ is not uniform, we may write

$$\Phi(x,y,z) = \Phi_0 \bar{\phi}\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right), \quad \text{for } (x,y,z) \in \bar{M},$$

where

$$\Phi_0 = \frac{1}{\mathrm{meas}(\bar{M})} \iiint_{\bar{M}} \Phi(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

and,

$$\iiint_{\bar{M}} \bar{\phi}\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \mathrm{meas}(\bar{M}),$$

with $\mathrm{meas}(\bar{M})$ being the measure (volume) of $\bar{M} \subset \mathbb{R}^3$. Throughout we will take $\bar{\phi} : \bar{M} \mapsto \mathbb{R}$ to have regularity

$$\bar{\phi} \in C^1(\bar{M}), \tag{2.8}$$

and to be bounded above zero on $\bar{M}$, so that there is a constant $\phi_m > 0$ such that

$$\bar{\phi}\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right) \geq \phi_m, \quad \text{for } (x,y,z) \in \bar{M}. \tag{2.9}$$

Note that when the porosity is uniform throughout the porous matrix then,

$$\bar{\phi}\left(\frac{x}{l},\frac{y}{l},\frac{z}{h}\right) = 1, \quad \text{for } (x,y,z) \in \bar{M}. \tag{2.10}$$

The boundary conditions to be applied on the impermeable boundary ∂M are

$$\begin{array}{ll} \mathbf{q}(\mathbf{r},t).\hat{\mathbf{n}}_l = 0 & \text{for all } (\mathbf{r},t) \in \partial M_H \times (0,\infty), \\ \mathbf{q}(\mathbf{r},t).\mathbf{n}_+ = 0 & \text{for all } (\mathbf{r},t) \in \partial M_+ \times (0,\infty), \\ \mathbf{q}(\mathbf{r},t).\mathbf{n}_- = 0 & \text{for all } (\mathbf{r},t) \in \partial M_- \times (0,\infty). \end{array} \tag{2.11}$$

Here $\partial M_H \subset \partial M$ is that part of ∂M representing the side walls of the boundary, whilst $\partial M_+, \partial M_- \subset \partial M$ represent the upper and lower surfaces of ∂M respectively, with $\partial M_+ \cup \partial M_- \cup \partial M_H = \partial M$. In addition, $\hat{\mathbf{n}}_l(x,y)$ for $(x,y) \in \partial\Omega_l$ represents the outward unit normal field to $\partial\Omega_l$. Finally we have the initial condition,

$$p(\mathbf{r},0) = p_0 f\left(\frac{x}{l}, \frac{y}{l}, \frac{z}{h}\right) - \rho_0 g z, \qquad \text{for all } (x,y,z) \in \bar{M},$$

with $f : \bar{M} \mapsto \mathbb{R}$ the prescribed initial pressure variation, with

$$f \in PC^1(\bar{M}) \cap C(\bar{M}),$$

where $PC^1(\bar{M})$ represents the class of piecewise continuously differentiable functions on $\bar{M}$. We now set

$$Q = \sum_{i=1}^{N} |Q_i| \quad (> 0).$$

The natural scales for the problem are then $x, y \sim l$ and $z \sim h$, from the geometry of the porous layer, whilst $s_i \sim Q/h$, via (2.5). The continuity equation (2.1) then requires $u \sim Q/(hl)$, $v \sim Q/(hl)$ and $w \sim Q/l^2$, whilst the momentum equation (2.2) requires $p \sim Q/(hD_0^H)$. We therefore introduce the dimensionless variables,

$$\begin{gathered} x = lx', \quad y = ly', \quad z = hz', \quad s_i = \tfrac{Q}{h} s_i', \\ u = \tfrac{Q}{hl} u', \quad v = \tfrac{Q}{hl} v', \quad w = \tfrac{Q}{l^2} w', \quad p = \left(\tfrac{Q}{hD_0^H}\right) p', \quad t = \tfrac{c_t l^2}{D_0^H} t'. \end{gathered} \tag{2.12}$$

On substitution from (2.12) into (2.1) and (2.2) (and dropping primes for convenience) we obtain the dimensionless equations of motion as

$$\bar{\phi}(x,y,z)\bar{p}_t + u_x + v_y + w_z = \sum_{i=1}^{N} s_i(z)\delta(x - x_i)\delta(y - y_i), \tag{2.13}$$

$$u = -D_x(x,y,z)\bar{p}_x, \tag{2.14}$$

$$v = -D_y(x,y,z)\bar{p}_y, \tag{2.15}$$

$$\epsilon^2 w = -D_z(x,y,z)\bar{p}_z, \tag{2.16}$$

for all $(x,y,z) \in M'$, $t \in (0,\infty)$, with M' being the domain occupied by the porous medium in dimensionless coordinates. Here,

$$p(x,y,z,t) = -\hat{\sigma} z + \bar{p}(x,y,z,t),$$

with $\bar{p}$ being the dynamic fluid pressure, and the dimensionless parameters ϵ and $\hat{\sigma}$ are given by

$$\epsilon = \sqrt{\frac{D_0^H}{D_0^L}}\frac{h}{l}, \qquad \hat{\sigma} = \frac{\rho_0 g h^2 D_0^H}{Q}. \tag{2.17}$$

The dimensionless domain is now

$$M' = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \Omega_1,\ z \in (z_-(x,y), z_+(x,y))\},$$

with closure $\bar{M}'$ and boundary $\partial M'$. The line source/sink locations are at $(x_i, y_i) \in \Omega_1$, $i = 1, \dots, N$. The volume flux conditions (2.5) become,

$$\alpha_i = \int_{z_-(x_i,y_i)}^{z_+(x_i,y_i)} s_i(\mu)\,\mathrm{d}\mu, \quad i = 1, \dots, N,$$

where

$$\alpha_i = \frac{Q_i}{Q}, \quad i = 1, \dots, N,$$

so that

$$|\alpha_i| = \frac{|Q_i|}{Q} \le 1, \text{ for } i = 1, \dots, N, \quad \text{ and } \sum_{i=1}^{N} |\alpha_i| = 1.$$

The boundary conditions (2.11) become, in dimensionless form,

$$(2.18) \qquad (u(\mathbf{r},t), v(\mathbf{r},t), w(\mathbf{r},t)).\hat{\mathbf{n}}_1 = 0, \quad \text{for all } (\mathbf{r},t) \in \partial M'_H \times (0,\infty),$$

$$(2.19) \quad w(\mathbf{r},t) - \{z_{+_x}(x,y)u(\mathbf{r},t) + z_{+_y}(x,y)v(\mathbf{r},t)\} = 0, \text{ for all } (\mathbf{r},t) \in \partial M'_+ \times (0,\infty),$$

$$(2.20) \quad w(\mathbf{r},t) - \{z_{-_x}(x,y)u(\mathbf{r},t) + z_{-_y}(x,y)v(\mathbf{r},t)\} = 0, \text{ for all } (\mathbf{r},t) \in \partial M'_- \times (0,\infty).$$

Finally we have the initial condition,

$$(2.21) \qquad \bar{p}(\mathbf{r},0) = \bar{p}_0 f(\mathbf{r}), \quad \text{for all } \mathbf{r} \in \bar{M}',$$

and $\bar{p}_0 = p_0 h D_0^H / Q$. The full problem for consideration is now given by (2.13)–(2.16), (2.18)–(2.21), which we refer to as [IBVP]. To proceed it is convenient to introduce $d_i \subset \bar{M}'$ as follows,

$$d_i = \{(x,y,z) \in \bar{M}' : (x,y) = (x_i,y_i),\ z \in (z_-(x_i,y_i), z_+(x_i,y_i))\},$$

for each $i = 1, \dots, N$, and set

$$d = \bigcup_{i=1}^{N} d_i.$$

We require that a solution to [IBVP] has the following regularity (in accordance with the usual Dirac delta function formalism):

(i) $\bar{p} \in C((\bar{M}' \backslash \bar{d}) \times [0,\infty)) \cap C^1((\bar{M}' \backslash \bar{d}) \times (0,\infty)) \cap C^2((M' \backslash d) \times (0,\infty))$;

(ii) $\lim_{R_i \to 0}[R_i |\underline{\tilde{D}} \nabla p|]$ exists uniformly for $z \in [z_-(x_i,y_i), z_+(x_i,y_i)]$, $\theta \in [0, 2\pi)$, and for each $t \in (0,\infty)$; $i = 1, \dots, N$. Here (R_i, θ, z) are local cylindrical polar coordinates based at $(x,y,z) = (x_i, y_i, 0)$, with $R_i = ((x-x_i)^2 + (y-y_i)^2)^{1/2}$, for each $i = 1, \dots, N$, and $\underline{\tilde{D}}$ is as defined in (2.36);

(iii) $\lim_{R_i \to 0} R_i \left(\int_0^{2\pi} (\underline{\tilde{D}} \nabla p).\underline{\hat{R}}_i \,\mathrm{d}\theta \right) = s_i(z)$ uniformly for $z \in [z_-(x_i,y_i), z_+(x_i,y_i)]$ and for each $t \in (0,\infty)$; $i = 1, \dots, N$. Here $\underline{\hat{R}}_i$ is the radial unit vector in the cylindrical polar coordinates (R_i, θ, z).

It is now convenient to introduce the associated pseudo-steady state problem to [IBVP], namely,

$$(2.22) \quad \hat{u}_x + \hat{v}_y + \hat{w}_z = \sum_{i=1}^{N} s_i(z)\delta(x-x_i)\delta(y-y_i) - \hat{\alpha}_T \bar{\phi}(x,y,z), \quad (x,y,z) \in M',$$

$$
\left.\begin{aligned}
\hat{u} &= -D_x(x,y,z)\hat{p}_x \\
\hat{v} &= -D_y(x,y,z)\hat{p}_y \\
\epsilon^2\hat{w} &= -D_z(x,y,z)\hat{p}_z
\end{aligned}\right\}(x,y,z)\in M', \tag{2.23}
$$

$$
(\hat{u}(\mathbf{r}),\hat{v}(\mathbf{r}),\hat{w}(\mathbf{r})).\hat{\mathbf{n}}_1 = 0, \quad \text{for all } \mathbf{r}\in\partial M'_H, \tag{2.24}
$$

$$
\hat{w}(\mathbf{r}) - \{z_{+x}(x,y)\hat{u}(\mathbf{r}) + z_{+y}(x,y)\hat{v}(\mathbf{r})\} = 0, \quad \text{for all } \mathbf{r}\in\partial M'_+, \tag{2.25}
$$

$$
\hat{w}(\mathbf{r}) - \{z_{-x}(x,y)\hat{u}(\mathbf{r}) + z_{-y}(x,y)\hat{v}(\mathbf{r})\} = 0, \quad \text{for all } \mathbf{r}\in\partial M'_-, \tag{2.26}
$$

which we will refer to as [PSSP]. Corresponding to (i)–(iii) a solution to [PSSP] has the following regularity:

(Pi) $\hat{p}\in C^1(\bar{M}'\backslash\bar{d})\cap C^2(M'\backslash d)$;

(Pii) $\lim_{R_i\to 0}[R_i|\tilde{\underline{D}}\nabla\hat{p}|]$ exists uniformly for $z\in[z_-(x_i,y_i),z_+(x_i,y_i)]$, $\theta\in[0,2\pi)$; $i=1,\ldots,N$;

(Piii) $\lim_{R_i\to 0} R_i\left(\int_0^{2\pi}(\tilde{\underline{D}}\nabla\hat{p}).\hat{\underline{R}}_i\,\mathrm{d}\theta\right) = s_i(z)$ uniformly for $z\in[z_-(x_i,y_i),z_+(x_i,y_i)]$; $i=1,\ldots,N$.

The constant $\hat{\alpha}_T$ is given by

$$
\hat{\alpha}_T = \frac{1}{\mathrm{meas}(\bar{M}')}\sum_{i=1}^{N}\alpha_i.
$$

with $\mathrm{meas}(\bar{M}')$ being the measure (volume) of $\bar{M}'\subset\mathbb{R}^3$. Now, following standard theory for linear, strongly elliptic weighted Neumann problems (see, for example [8] or [19, chapters 8,9]), we have:

THEOREM 2.1. *For each $\epsilon>0$, [PSSP] has a unique (up to the addition of a constant in $\hat{p}$) solution $\hat{u},\hat{v},\hat{w},\hat{p}:\bar{M}'\mapsto\mathbb{R}$.*

We remark that when $\hat{\alpha}_T=0$, then the solution to [PSSP] is, in fact, a steady state of [IBVP].

We now return to [IBVP] and introduce $\tilde{u},\tilde{v},\tilde{w},\tilde{p}:\bar{M}'\times[0,\infty)\mapsto\mathbb{R}$ as

$$
\tilde{u} = u-\hat{u}, \quad \tilde{v} = v-\hat{v}, \quad \tilde{w} = w-\hat{w}, \quad \tilde{p} = \bar{p}-\hat{p}-\hat{\alpha}_T t, \tag{2.27}
$$

where $\hat{u},\hat{v},\hat{w},\hat{p}:\bar{M}'\mapsto\mathbb{R}$ represents that unique solution to [PSSP] which has

$$
\iiint_{\bar{M}'}\hat{p}(x,y,z)\bar{\phi}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{\bar{M}'}\bar{p}_0 f(x,y,z)\bar{\phi}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{2.28}
$$

so that, via (2.21), (2.27),

$$
\iiint_{\bar{M}'}\tilde{p}(x,y,z,0)\bar{\phi}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0. \tag{2.29}
$$

It is now straightforward to establish that $u,v,w,\bar{p}:\bar{M}'\mapsto\mathbb{R}$ is a solution to [IBVP] if and only if $\tilde{u},\tilde{v},\tilde{w},\tilde{p}:\bar{M}'\mapsto\mathbb{R}$ is a solution to the following linear, strongly parabolic, regular, initial-boundary value problem, namely

$$
\bar{\phi}(x,y,z)\tilde{p}_t - \left\{(D_x(x,y,z)\tilde{p}_x)_x + (D_y(x,y,z)\tilde{p}_y)_y + \left(\epsilon^{-2}D_z(x,y,z)\tilde{p}_z\right)_z\right\} = 0, \tag{2.30}
$$
$$
\text{for } (x,y,z,t)\in M'\times(0,\infty),
$$

$$
[\tilde{\underline{D}}(\mathbf{r})\nabla\tilde{p}(\mathbf{r},t)].\hat{\mathbf{n}}_1 = 0, \quad \text{for all } (\mathbf{r},t)\in\partial M'_H\times(0,\infty), \tag{2.31}
$$

$$
[\tilde{\underline{D}}(\mathbf{r})\nabla\tilde{p}(\mathbf{r},t)].(-\epsilon^2 z_{+x}(x,y), -\epsilon^2 z_{+y}(x,y), 1) = 0, \tag{2.32}
$$
$$
\text{for all } (\mathbf{r},t)\in\partial M'_+\times(0,\infty),
$$

$$[\underline{\tilde{D}}(\mathbf{r})\nabla\tilde{p}(\mathbf{r},t)].(-\epsilon^2 z_{-x}(x,y), -\epsilon^2 z_{-y}(x,y), 1) = 0, \quad \text{for all } (\mathbf{r},t) \in \partial M'_- \times (0,\infty), \tag{2.33}$$

$$\tilde{p}(\mathbf{r},0) = \bar{p}_0 f(\mathbf{r}) - \hat{p}(\mathbf{r}) = \tilde{p}_0(\mathbf{r}), \quad \text{for all } \mathbf{r} \in \bar{M}', \tag{2.34}$$

with regularity

$$\tilde{p} \in C((\bar{M}' \times [0,\infty))\backslash(\bar{d} \times \{0\})) \cap C^1(\bar{M}' \times (0,\infty)) \cap C^2(M' \times (0,\infty)), \tag{2.35}$$

after which

$$\left.\begin{aligned} \tilde{u} &= -D_x(x,y,z)\tilde{p}_x, \\ \tilde{v} &= -D_y(x,y,z)\tilde{p}_y, \\ \epsilon^2\tilde{w} &= -D_z(x,y,z)\tilde{p}_z, \end{aligned}\right\} (\mathbf{r},t) \in M' \times (0,\infty).$$

Here

$$\underline{\tilde{D}}(\mathbf{r}) = \begin{pmatrix} -D_x(x,y,z) & 0 & 0 \\ 0 & -D_y(x,y,z) & 0 \\ 0 & 0 & -D_z(x,y,z) \end{pmatrix}, \tag{2.36}$$

for all $\mathbf{r} \in \bar{M}'$. The strongly parabolic problem (2.30)–(2.35) has a unique solution in $\bar{M}' \times [0,\infty)$ (see for example [8, Chapter 3]), and we now construct this solution. To this end we first consider the following self-adjoint eigenvalue problem in $\bar{M}'$,

$$(D_x(x,y,z)\phi_x)_x + (D_y(x,y,z)\phi_y)_y + \left(\epsilon^{-2}D_z(x,y,z)\phi_z\right)_z + \lambda\bar{\phi}(x,y,z)\phi = 0, \quad \text{for } (x,y,z) \in M',$$

$$[\underline{\tilde{D}}(\mathbf{r})\nabla\phi(\mathbf{r})].\hat{\mathbf{n}}_1 = 0, \quad \text{for all } \mathbf{r} \in \partial M'_H,$$

$$[\underline{\tilde{D}}(\mathbf{r})\nabla\phi(\mathbf{r})].(-\epsilon^2 z_{+x}(x,y), -\epsilon^2 z_{+y}(x,y), 1) = 0, \quad \text{for all } \mathbf{r} \in \partial M'_+,$$

$$[\underline{\tilde{D}}(\mathbf{r})\nabla\phi(\mathbf{r})].(-\epsilon^2 z_{-x}(x,y), -\epsilon^2 z_{-y}(x,y), 1) = 0, \quad \text{for all } \mathbf{r} \in \partial M'_-.$$

We will denote this eigenvalue problem by [EVP], with $\lambda \in \mathbb{C}$ being the eigenvalue parameter. It follows from (2.6) and (2.7) that this is a regular, self-adjoint eigenvalue problem. It then follows from established theory (see for example [19]) that the eigenvalues of [EVP] are all real and given by $\lambda = \lambda_j(\epsilon)$, $j = 0,1,2,\ldots$, where

$$0 = \lambda_0(\epsilon) < \lambda_1(\epsilon) \le \lambda_2(\epsilon) \le \ldots \tag{2.37}$$

with $\lambda_j(\epsilon) \to +\infty$ as $j \to \infty$, and an eigenvalue is repeated in the ordering (2.37) according to its geometric multiplicity (the dimension of the eigenspace corresponding to that eigenvalue). To each occurence $\lambda_j(\epsilon)$ in the ordering (2.37) there corresponds an eigenfunction $\phi_j : \bar{M}' \mapsto \mathbb{R}$, $j = 0,1,2,\ldots$, with

$$\phi_0(x,y,z;\epsilon) = (\text{meas}(\bar{M}'))^{-1/2}, \quad \text{for all } (x,y,z) \in \bar{M}',$$

and

$$\langle\phi_i,\phi_j\rangle = \iiint_{\bar{M}'} \bar{\phi}(x,y,z)\phi_i(x,y,z;\epsilon)\phi_j(x,y,z;\epsilon)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \delta_{ij},$$

for $i,j = 0,1,2,\ldots$, and δ_{ij} being the Kronecker delta symbol. Without repeating details (see [13]), it is now straightforward to obtain the solution to (2.30)–(2.35) as

$$\tilde{p}(\mathbf{r},t) = \sum_{n=1}^{\infty} a_n(\epsilon)\mathrm{e}^{-\lambda_n(\epsilon)t}\phi_n(\mathbf{r};\epsilon), \quad \text{for all } (\mathbf{r},t) \in \bar{M}' \times [0,\infty), \tag{2.38}$$

with $a_0(\epsilon) = 0$, via (2.29), and

$$a_j(\epsilon) = \iiint_{\bar{M}'} \tilde{p}_0(u,v,w)\bar{\phi}(u,v,w)\phi_j(u,v,w;\epsilon)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \tag{2.39}$$

for $j = 1, 2, \ldots$. We observe immediately from (2.38), with (2.37), that

$$\tilde{p}(\mathbf{r},t) \to 0 \quad \text{as } t \to \infty,$$

uniformly for $\mathbf{r} \in \bar{M}'$, and that, in addition,

$$\tilde{p}_x(\mathbf{r},t), \tilde{p}_y(\mathbf{r},t), \tilde{p}_z(\mathbf{r},t) \to 0 \quad \text{as } t \to \infty,$$

uniformly for $\mathbf{r} \in \bar{M}'$. In fact, we have established:

THEOREM 2.2. *For each $\epsilon > 0$, [IBVP] has a unique solution $u, v, w, \bar{p} : \bar{M}' \times [0,\infty) \mapsto \mathbb{R}$ given by*

$$\begin{aligned}
\bar{p}(\mathbf{r},t) &= \hat{\alpha}_T t + \hat{p}(\mathbf{r}) + \tilde{p}(\mathbf{r},t),\\
u(\mathbf{r},t) &= \hat{u}(\mathbf{r}) - D_x(\mathbf{r})\tilde{p}_x(\mathbf{r},t),\\
v(\mathbf{r},t) &= \hat{v}(\mathbf{r}) - D_y(\mathbf{r})\tilde{p}_y(\mathbf{r},t),\\
w(\mathbf{r},t) &= \hat{w}(\mathbf{r}) - \epsilon^{-2}D_z(\mathbf{r})\tilde{p}_z(\mathbf{r},t),
\end{aligned}$$

for all $(\mathbf{r},t) \in \bar{M}' \times [0,\infty)$. Here $\tilde{p} : \bar{M}' \times [0,\infty) \mapsto \mathbb{R}$ is given by (2.38), (2.39), and $\hat{u}, \hat{v}, \hat{w}, \hat{p} : \bar{M}' \mapsto \mathbb{R}$ is that solution to [PSSP] which satisfies the constraint (2.28). Moreover

$$\begin{aligned}
\bar{p}(\mathbf{r},t) &= \hat{\alpha}_T t + \hat{p}(\mathbf{r}) + O(\mathrm{e}^{-\lambda_1(\epsilon)t}),\\
u(\mathbf{r},t) &= \hat{u}(\mathbf{r}) + O(\mathrm{e}^{-\lambda_1(\epsilon)t}),\\
v(\mathbf{r},t) &= \hat{v}(\mathbf{r}) + O(\mathrm{e}^{-\lambda_1(\epsilon)t}),\\
w(\mathbf{r},t) &= \hat{w}(\mathbf{r}) + O(\mathrm{e}^{-\lambda_1(\epsilon)t}),
\end{aligned}$$

as $t \to \infty$, uniformly for $\mathbf{r} \in \bar{M}'$.

To complete the solution to the problem [IBVP] we must determine $\lambda_n(\epsilon)$ (> 0) and its corresponding eigenfunction $\phi_n : \bar{M}' \mapsto \mathbb{R}$ for each $n = 1, 2, \ldots$, together with the pseudo-steady state $\hat{p}, \hat{u}, \hat{v}, \hat{w} : \bar{M}' \mapsto \mathbb{R}$ which satisfies the constraint (2.28). In the next two sections we thus focus attention on the study of [PSSP] and [EVP] in turn.

In particular, for a thin porous layer, the parameter ϵ, which measures the aspect ratio of the layer, is small, provided that

$$\frac{h}{l} \ll \left(\frac{D_0^L}{D_0^H}\right)^{1/2},$$

which we will take to be the case. Thus $0 < \epsilon \ll 1$, and in the next two sections we will consider the structure of the solutions to [PSSP] and [EVP] in the asymptotic limit $\epsilon \to 0$, via the method of matched asymptotic expansions.

3. Asymptotic solution to the pseudo-steady state problem [PSSP] as $\epsilon \to 0$. In this section we develop the uniform asymptotic structure to the solution of the pseudo-steady state problem [PSSP] (given by (2.22)–(2.26)) in the limit $\epsilon \to 0$, via the method of matched asymptotic expansions. We recall that existence and uniqueness, for each $\epsilon > 0$, follows from Theorem 2.1, and, following Theorem 2.2, we require that solution to [PSSP] which satisfies the constraint

$$\iiint_{\bar{M}'} \hat{p}(x,y,z)\bar{\phi}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = I_0, \tag{3.1}$$

where the constant I_0 is given by

$$I_0 = \bar{p}_0 \iiint_{\bar{M}'} f(x,y,z)\bar{\phi}(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Due to the initial scalings in the nondimensionalization (2.12), we anticipate that $\hat{p},\hat{u},\hat{v},\hat{w} : \bar{M}' \mapsto \mathbb{R}$ are such that

$$\hat{p},\hat{u},\hat{v},\hat{w} = O(1) \tag{3.2}$$

as $\epsilon \to 0$, uniformly for,

$$\mathbf{r} \in \bar{M}' \backslash \bigcup_{i=1}^{N} \delta_i^\epsilon = \bar{N}'_\epsilon,$$

where δ_i^ϵ is an $\mathrm{O}(\epsilon)$ neighbourhood of $\bar{d}_i$, for each $i = 1,\ldots,N$. Thus, following (3.2), we introduce the outer region ($\bar{N}'_\epsilon$) asymptotic expansions

$$\begin{aligned}\hat{p}(\mathbf{r};\epsilon) &= \hat{p}_0(\mathbf{r}) + \epsilon\hat{p}_1(\mathbf{r}) + O(\epsilon^2),\\ \hat{u}(\mathbf{r};\epsilon) &= \hat{u}_0(\mathbf{r}) + \epsilon\hat{u}_1(\mathbf{r}) + O(\epsilon^2),\\ \hat{v}(\mathbf{r};\epsilon) &= \hat{v}_0(\mathbf{r}) + \epsilon\hat{v}_1(\mathbf{r}) + O(\epsilon^2),\\ \hat{w}(\mathbf{r};\epsilon) &= \hat{w}_0(\mathbf{r}) + \epsilon\hat{w}_1(\mathbf{r}) + O(\epsilon^2),\end{aligned} \tag{3.3}$$

as $\epsilon \to 0$, uniformly for $\mathbf{r} \in \bar{N}'_\epsilon$. We now substitute from (3.3) into [PSSP], and condition (3.1). At leading order we obtain the following problem for $\hat{p}_0,\hat{u}_0,\hat{v}_0,\hat{w}_0 : \bar{M}' \mapsto \mathbb{R}$, namely,

$$\hat{u}_{0x} + \hat{v}_{0y} + \hat{w}_{0z} = \sum_{i=1}^{N} s_i(z)\delta(x-x_i)\delta(y-y_i) - \hat{\alpha}_T\bar{\phi}(x,y,z), \quad (x,y,z) \in M', \tag{3.4}$$

$$\hat{u}_0 = -D_x(x,y,z)\hat{p}_{0x}, \quad (x,y,z) \in M', \tag{3.5}$$

$$\hat{v}_0 = -D_y(x,y,z)\hat{p}_{0y}, \quad (x,y,z) \in M', \tag{3.6}$$

$$0 = -D_z(x,y,z)\hat{p}_{0z}, \quad (x,y,z) \in M', \tag{3.7}$$

$$(\hat{u}_0(\mathbf{r}),\hat{v}_0(\mathbf{r}),\hat{w}_0(\mathbf{r})).\hat{\mathbf{n}}_1 = 0, \quad \text{for all } \mathbf{r} \in \partial M'_H, \tag{3.8}$$

$$\hat{w}_0(\mathbf{r}) - \{z_{+x}(x,y)\hat{u}_0(\mathbf{r}) + z_{+y}(x,y)\hat{v}_0(\mathbf{r})\} = 0, \quad \text{for all } \mathbf{r} \in \partial M'_+, \tag{3.9}$$

$$\hat{w}_0(\mathbf{r}) - \{z_{-x}(x,y)\hat{u}_0(\mathbf{r}) + z_{-y}(x,y)\hat{v}_0(\mathbf{r})\} = 0, \quad \text{for all } \mathbf{r} \in \partial M'_-, \tag{3.10}$$

$$\iiint_{\bar{M}'} \hat{p}_0(\mathbf{r})\bar{\phi}(\mathbf{r})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = I_0. \tag{3.11}$$

We now construct the solution to (3.4)–(3.11). As a consequence of (2.7), we obtain from (3.7),

$$\hat{p}_0(x,y,z) = A(x,y), \quad (x,y,z) \in \bar{M}',$$

with $A : \bar{\Omega}_1 \mapsto \mathbb{R}$ to be determined. Equations (3.5) and (3.6) then give

$$\begin{aligned}\hat{u}_0(x,y,z) &= -D_x(x,y,z)A_x(x,y),\\ \hat{v}_0(x,y,z) &= -D_y(x,y,z)A_y(x,y),\end{aligned} \quad (x,y,z) \in \bar{M}', \tag{3.12}$$

with boundary condition (3.8) requiring

$$D_x(x,y,z)A_x(x,y)n_x(x,y) + D_y(x,y,z)A_y(x,y)n_y(x,y) = 0, \quad \mathbf{r} \in \partial M'_H, \tag{3.13}$$

where we have written

$$\hat{\mathbf{n}}_1(\mathbf{r}) = (n_x(x,y), n_y(x,y), 0), \quad \mathbf{r} \in \partial M'_H.$$

We next substitute from (3.12) into (3.4) which becomes

$$\begin{aligned} \hat{w_0}_z = \sum_{i=1}^{N} s_i(z)\delta(x-x_i)\delta(y-y_i) - \hat{\alpha}_T\bar{\phi}(x,y,z) \\ +[D_x(x,y,z)A_x(x,y)]_x + [D_y(x,y,z)A_y(x,y)]_y, \quad (x,y,z) \in M'. \end{aligned} \tag{3.14}$$

A direct integration of (3.14), together with an application of the boundary condition (3.10), gives

$$\begin{aligned} \hat{w}_0(x,y,z) = \sum_{i=1}^{N} F_i(z)\delta(x-x_i)\delta(y-y_i) - \hat{\alpha}_T \int_{z_-(x,y)}^{z} \bar{\phi}(x,y,\lambda)\,\mathrm{d}\lambda \\ + \int_{z_-(x,y)}^{z} \{[D_x(x,y,\lambda)A_x(x,y)]_x + [D_y(x,y,\lambda)A_y(x,y)]_y\}\,\mathrm{d}\lambda \\ -z_{-x}(x,y)D_x(x,y,z_-(x,y))A_x(x,y) - z_{-y}(x,y)D_y(x,y,z_-(x,y))A_y(x,y), \\ \text{for } (x,y,z) \in \bar{M}', \end{aligned}$$

where

$$F_i(z) = \int_{z_-(x_i,y_i)}^{z} s_i(\lambda)\,\mathrm{d}\lambda, \quad z \in [z_-(x_i,y_i), z_+(x_i,y_i)], \tag{3.15}$$

for each $i = 1, \ldots, N$. It remains to apply the boundary condition (3.9). The application of (3.9), using (3.12) and (3.15), finally requires that

$$\begin{aligned} \int_{z_-(x,y)}^{z_+(x,y)} \{[D_x(x,y,\lambda)A_x(x,y)]_x + [D_y(x,y,\lambda)A_y(x,y)]_y\}\,\mathrm{d}\lambda \\ + \{z_{+x}(x,y)D_x(x,y,z_+(x,y)) - z_{-x}(x,y)D_x(x,y,z_-(x,y))\}\,A_x(x,y) \\ + \Big\{z_{+y}(x,y)D_y(x,y,z_+(x,y)) - z_{-y}(x,y)D_y(x,y,z_-(x,y))\Big\}\,A_y(x,y) \\ -\hat{\alpha}_T\hat{\phi}(x,y) + \sum_{i=1}^{N} \alpha_i\delta(x-x_i)\delta(y-y_i) = 0, \quad (x,y) \in \Omega_1, \end{aligned} \tag{3.16}$$

with $\hat{\phi} : \bar{\Omega}_1 \mapsto \mathbb{R}$ given by

$$\hat{\phi}(x,y) = \int_{z_-(x,y)}^{z_+(x,y)} \bar{\phi}(x,y,\lambda)\,\mathrm{d}\lambda,$$

for $(x,y) \in \bar{\Omega}_1$, and representing the depth integrated porosity of the porous layer. We observe that

$$\begin{aligned} \hat{\phi} \in C^1(\bar{\Omega}_1), \\ \hat{\phi}(x,y) \geq \hat{\phi}_m(>0) \quad \text{for } (x,y) \in \bar{\Omega}_1, \end{aligned}$$

for some positive constant $\hat{\phi}_m$, via (2.8), (2.9), (2.3) and (2.4). We also note that in the case of uniform porosity, we have,

$$\hat{\phi}(x,y) = h(x,y) \quad \text{for } (x,y) \in \bar{\Omega}_1$$

via (2.10), where $h : \bar{\Omega}_1 \mapsto \mathbb{R}$ is defined by

$$h(x,y) = z_+(x,y) - z_-(x,y), \quad \text{for all } (x,y) \in \bar{\Omega}_1,$$

and so $h \in C^1(\bar{\Omega}_1)$. After a little manipulation, (3.16) simplifies, to give

$$[\bar{D}_x(x,y)A_x(x,y)]_x + [\bar{D}_y(x,y)A_y(x,y)]_y = -\sum_{i=1}^{N} \alpha_i \delta(x-x_i)\delta(y-y_i) + \hat{\alpha}_T \hat{\phi}(x,y), \quad \text{for } (x,y) \in \Omega_1,$$

where $\bar{D}_x, \bar{D}_y : \bar{\Omega}_1 \mapsto \mathbb{R}$ are defined by

$$\bar{D}_x(x,y) = \int_{z_-(x,y)}^{z_+(x,y)} D_x(x,y,\lambda)\,\mathrm{d}\lambda, \quad \bar{D}_y(x,y) = \int_{z_-(x,y)}^{z_+(x,y)} D_y(x,y,\lambda)\,\mathrm{d}\lambda,$$

for $(x,y) \in \bar{\Omega}_1$, with

$$\begin{gathered} \bar{D}_x, \bar{D}_y \in C^1(\bar{\Omega}_1), \\ \bar{D}_x(x,y), \bar{D}_y(x,y) \geq \bar{D}_0 > 0 \quad \text{for all } (x,y) \in \bar{\Omega}_1, \end{gathered}$$

via (2.6) and (2.7), for some positive constant $\bar{D}_0$. Note that $\bar{D}_x(x,y)$ and $\bar{D}_y(x,y)$ at any $(x,y) \in \bar{\Omega}_1$ represent the depth integrated permeabilities of the porous layer in the x and y directions respectively. In addition, an integration of boundary condition (3.13) results in the boundary condition

$$\bar{D}_x(x,y)A_x(x,y)n_x(x,y) + \bar{D}_y(x,y)A_y(x,y)n_y(x,y) = 0, \quad (x,y) \in \partial\Omega_1.$$

Thus, $A : \bar{\Omega} \mapsto \mathbb{R}$ is determined as the solution to the linear, inhomogeneous, strongly elliptic boundary value problem,

$$\begin{gathered} \hat{\nabla}.(\underline{\hat{D}}(\hat{\mathbf{r}})\hat{\nabla} A) = -\sum_{i=1}^{N} \alpha_i \delta(\hat{\mathbf{r}} - \hat{\mathbf{r}}_i) + \hat{\alpha}_T \hat{\phi}(\hat{\mathbf{r}}), \quad \hat{\mathbf{r}} \in \Omega, \\ (\underline{\hat{D}}(\hat{\mathbf{r}})\hat{\nabla} A).\hat{\mathbf{n}}(\hat{\mathbf{r}}) = 0, \quad \hat{\mathbf{r}} \in \partial\Omega, \\ \iint_{\bar{\Omega}} \hat{\phi}(\hat{\mathbf{r}}) A(\hat{\mathbf{r}})\,\mathrm{d}x\,\mathrm{d}y = I_0, \end{gathered}$$

where we have dropped the subscript on Ω_1, $\partial\Omega_1$ for convenience, $\hat{\mathbf{r}} = (x,y)$, $\hat{\mathbf{r}}_i = (x_i, y_i)$, $i = 1, \ldots, N$, and $\hat{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is the two-dimensional gradient operator on $\bar{\Omega}$. Here,

$$\underline{\hat{D}}(\hat{\mathbf{r}}) = \begin{pmatrix} \bar{D}_x(\hat{\mathbf{r}}) & 0 \\ 0 & \bar{D}_y(\hat{\mathbf{r}}) \end{pmatrix}, \quad \hat{\mathbf{r}} \in \bar{\Omega}, \tag{3.17}$$

and

$$\hat{\mathbf{n}}(\hat{\mathbf{r}}) = (n_x(\hat{\mathbf{r}}), n_y(\hat{\mathbf{r}})), \quad \hat{\mathbf{r}} \in \partial\Omega.$$

We will refer to this problem as [BVP]. A solution to [BVP] will have the following regularity requirements:

(Bi) $A \in C^1(\bar{\Omega} \backslash \cup_{i=1}^N \hat{\mathbf{r}}_i) \cap C^2(\Omega \backslash \cup_{i=1}^N \hat{\mathbf{r}}_i)$;

(Bii) $\lim_{R_i \to 0}[R_i |\underline{\hat{D}} \nabla A|]$ exists uniformly for $\theta \in [0, 2\pi)$; $i = 1, \ldots, N$;

(Biii) $\lim_{R_i \to 0} R_i \left(\int_0^{2\pi} (\underline{\hat{D}} \nabla A).\underline{\hat{R}}_i \, \mathrm{d}\theta \right) = -\alpha_i$; $i = 1, \ldots, N$.

Here $\hat{\mathbf{r}}_i = (x_i, y_i) \in \Omega$, $i = 1, \ldots, N$, and (R_i, θ) and $\underline{\hat{R}}_i$ are as defined in §2 (and can now be regarded as plane polar coordinates on Ω based at $(x, y) = (x_i, y_i)$).

REMARK 3.1. *It follows from classical theory for strongly elliptic boundary value problems (see for example [8]) that [BVP] has a unique solution.*

In particular, with $A : \bar{\Omega} \mapsto \mathbb{R}$ being the solution to [BVP], we have

$$A(x,y) = \frac{-\alpha_i}{4\pi(\bar{D}_x^i \bar{D}_y^i)^{\frac{1}{2}}} \log \left[\frac{(x-x_i)^2}{\bar{D}_x^i} + \frac{(y-y_i)^2}{\bar{D}_y^i} \right] + A_0^i + O\left(([x-x_i]^2 + [y-y_i]^2)^{\frac{1}{2}} \right), \tag{3.18}$$

as $(x, y) \to (x_i, y_i)$, with $A_0^i \in \mathbb{R}$ being a globally determined constant, and $i = 1, \ldots, N$. It follows from (3.18) that, for each $i = 1, \ldots, N$,

$$\nabla A(x,y) \sim \frac{-\alpha_i}{2\pi(\bar{D}_x^i \bar{D}_y^i)^{\frac{1}{2}}} \left(\frac{(x-x_i)^2}{\bar{D}_x^i} + \frac{(y-y_i)^2}{\bar{D}_y^i} \right)^{-1} \left(\frac{(x-x_i)}{\bar{D}_x^i}\mathbf{i} + \frac{(y-y_i)}{\bar{D}_y^i}\mathbf{j} \right), \tag{3.19}$$

as $(x, y) \to (x_i, y_i)$, with $\mathbf{i}$ and $\mathbf{j}$ being unit vectors in the x and y directions respectively, and,

$$\bar{D}_x^i = \bar{D}_x(\hat{\mathbf{r}}_i), \qquad \bar{D}_y^i = \bar{D}_y(\hat{\mathbf{r}}_i),$$

for $i = 1, \ldots, N$.

In general, except for particularly simple boundaries $\partial\Omega$, permeabilities $\bar{D}_x(\hat{\mathbf{r}})$, $\bar{D}_y(\hat{\mathbf{r}})$, and line source/sink locations $\hat{\mathbf{r}}_i \in \partial\Omega$, $i = 1, \ldots, N$, [BVP] will need to be solved numerically. However, [BVP] is a two-dimensional, non-stiff, regular, strongly elliptic problem, and numerical solution via finite or boundary element methods can be achieved rapidly and accurately. A detailed consideration of the numerical solution of [BVP] is provided in [12]. We observe that once the solution $A : \bar{\Omega} \mapsto \mathbb{R}$ to [BVP] has been determined, the solution to the leading order problem is given as,

$$\begin{aligned} \hat{p}_0(x,y,z) &= A(x,y), \quad (x,y,z) \in \bar{M}', \\ \hat{u}_0(x,y,z) &= -D_x(x,y,z)A_x(x,y), \quad (x,y,z) \in \bar{M}', \\ \hat{v}_0(x,y,z) &= -D_y(x,y,z)A_y(x,y), \quad (x,y,z) \in \bar{M}', \\ \hat{w}_0(x,y,z) &= \int_{z_-(x,y)}^{z} \{[D_x(x,y,\lambda)A_x(x,y)]_x + [D_y(x,y,\lambda)A_y(x,y)]_y\} \, \mathrm{d}\lambda \\ &\quad -\hat{\alpha}_T \int_{z_-(x,y)}^{z} \bar{\phi}(x,y,\lambda) \, \mathrm{d}\lambda, \quad (x,y,z) \in \bar{M}'. \end{aligned} \tag{3.20}$$

It is worth noting here, via (3.18), (3.19) and (3.20), that

$$\begin{aligned} \hat{p}_0(x,y,z) &= O(\log R_i), \\ \hat{u}_0(x,y,z) &= O(R_i^{-1}), \\ \hat{v}_0(x,y,z) &= O(R_i^{-1}), \\ \hat{w}_0(x,y,z) &= O(R_i^{-2}), \end{aligned} \tag{3.21}$$

as $R_i \to 0$, uniformly for $\theta \in [0, 2\pi)$ and $z \in [z_-(x_i, y_i), z_+(x_i, y_i)]$, for each $i = 1, \ldots, N$.

We now proceed to $O(\epsilon)$. The problem for $\hat{p}_1, \hat{u}_1, \hat{v}_1, \hat{w}_1 : \bar{M}' \mapsto \mathbb{R}$ is similar to the leading order problem and is not repeated here. We obtain

$$
\begin{aligned}
\hat{p}_1(x,y,z) &= B(x,y), \quad (x,y,z) \in \bar{M}', \\
\hat{u}_1(x,y,z) &= -D_x(x,y,z)B_x(x,y), \quad (x,y,z) \in \bar{M}', \\
\hat{v}_1(x,y,z) &= -D_y(x,y,z)B_y(x,y), \quad (x,y,z) \in \bar{M}', \\
\hat{w}_1(x,y,z) &= \textstyle\int_{z_-(x,y)}^{z} \{(D_x(x,y,\lambda)B_x(x,y))_x + (D_y(x,y,\lambda)B_y(x,y))_y\}\, \mathrm{d}\lambda \\
&\quad -z_{-x}(x,y)D_x(x,y,z_-(x,y))B_x(x,y) - z_{-y}(x,y)D_y(x,y,z_-(x,y))B_y(x,y), \\
&\qquad \text{for } (x,y,z) \in \bar{M}',
\end{aligned}
\tag{3.22}
$$

where $B : \bar{\Omega} \mapsto \mathbb{R}$ is the solution to the strongly elliptic boundary value problem,

$$
\begin{gathered}
\hat{\nabla}.(\underline{\hat{D}}(\hat{\mathbf{r}})\hat{\nabla}B) = 0, \quad \hat{\mathbf{r}} \in \Omega, \\
(\underline{\hat{D}}(\hat{\mathbf{r}})\hat{\nabla}B).\hat{\mathbf{n}}(\hat{\mathbf{r}}) = 0, \quad \hat{\mathbf{r}} \in \partial\Omega, \\
\int\!\!\int_{\bar{\Omega}} \hat{\bar{\phi}}(\hat{\mathbf{r}})B(\hat{\mathbf{r}})\, \mathrm{d}x\, \mathrm{d}y = 0.
\end{gathered}
$$

The unique solution $B \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ is given by

$$
B(x,y) = 0, \quad (x,y) \in \bar{\Omega},
$$

and so

$$
\hat{p}_1(x,y,z) = \hat{u}_1(x,y,z) = \hat{v}_1(x,y,z) = \hat{w}_1(x,y,z) = 0, \quad (x,y,z) \in \bar{M}',
$$

via (3.22). The outer region asymptotic expansions are now complete to $O(\epsilon^2)$, and we have

$$
\begin{aligned}
\hat{p}(\mathbf{r};\epsilon) &= A(x,y) + O(\epsilon^2), \\
\hat{u}(\mathbf{r};\epsilon) &= -D_x(x,y,z)A_x(x,y) + O(\epsilon^2), \\
\hat{v}(\mathbf{r};\epsilon) &= -D_y(x,y,z)A_y(x,y) + O(\epsilon^2), \\
\hat{w}(\mathbf{r};\epsilon) &= \textstyle\int_{z_-(x,y)}^{z} \{[D_x(x,y,\lambda)A_x(x,y)]_x + [D_y(x,y,\lambda)A_y(x,y)]_y\}\, \mathrm{d}\lambda \\
&\quad -\hat{\alpha}_T \textstyle\int_{z_-(x,y)}^{z} \bar{\phi}(x,y,\lambda)\, \mathrm{d}\lambda + O(\epsilon^2),
\end{aligned}
\tag{3.23}
$$

as $\epsilon \to 0$, uniformly for $\mathbf{r} \in \bar{N}'_\epsilon$. Here $A : \bar{\Omega} \mapsto \mathbb{R}$ is the solution to [BVP]. We now observe from (3.23) with (3.18) and (3.19) that all of the regularity requirements in (Pi), together with the limit conditions (Pii) and (Piii) are not satisfied as $(x,y) \to (x_i,y_i)$ for each $z \in [z_-(x_i,y_i), z_+(x_i,y_i)]$, with $i = 1,\ldots,N$ (although depth integrated forms are satisfied). We conclude (as was anticipated earlier) that the outer region asymptotic expansions (3.3) become non-uniform when $\mathbf{r} \in \delta_i^\epsilon$ as $\epsilon \to 0$ $(i = 1,\ldots,N)$. To obtain a uniform asymptotic representation of the solution to [PSSP] when $\mathbf{r} \in \delta_i^\epsilon$ as $\epsilon \to 0$, we must therefore introduce an inner region at each line source/sink location $(x,y) = (x_i,y_i)$, $i = 1,\ldots,N$. We now consider the inner region in the neighbourhood of $(x,y) = (x_i,y_i)$ in detail. In the inner region,

$$
(x,y) = (x_i,y_i) + O(\epsilon), \quad z = O(1),
$$

as $\epsilon \to 0$, with, from (3.18), (3.19), (3.21) and (3.23),

$$
\hat{p} = \frac{-\alpha_i}{2\pi(\bar{D}_x^i\bar{D}_y^i)^{1/2}} \log\epsilon + O(1), \quad \hat{u} = O(\epsilon^{-1}), \quad \hat{v} = O(\epsilon^{-1}), \quad \hat{w} = O(\epsilon^{-2}),
$$

as $\epsilon \to 0$. Thus, in the inner region we write,

$$(x,y) = (x_i, y_i) + \epsilon(X,Y), \tag{3.24}$$

with $(X,Y) \in \mathbb{R}^2$ such that $X, Y = O(1)$ as $\epsilon \to 0$, together with

$$\hat{p} = \frac{-\alpha_i}{2\pi(\bar{D}_x^i \bar{D}_y^i)^{1/2}} \log\epsilon + P, \quad \hat{u} = \epsilon^{-1}U, \quad \hat{v} = \epsilon^{-1}V, \quad \hat{w} = \epsilon^{-2}W, \tag{3.25}$$

where $P, U, V, W : \mathbb{R}^2 \times [z_-(x_i,y_i), z_+(x_i,y_i)] \mapsto \mathbb{R}$ are such that $P, U, V, W = O(1)$ as $\epsilon \to 0$. We now introduce inner region asymptotic expansions as

$$\begin{aligned} P(X,Y,z;\epsilon) &= P_0(X,Y,z) + O(\epsilon), \\ U(X,Y,z;\epsilon) &= U_0(X,Y,z) + O(\epsilon), \\ V(X,Y,z;\epsilon) &= V_0(X,Y,z) + O(\epsilon), \\ W(X,Y,z;\epsilon) &= W_0(X,Y,z) + O(\epsilon), \end{aligned} \tag{3.26}$$

as $\epsilon \to 0$, $(X,Y,z) \in \mathbb{R}^2 \times [z_-(x_i,y_i), z_+(x_i,y_i)]$. We substitute from (3.24)–(3.26) into the full problem [PSSP], to obtain the leading order problem as

$$U_{0X} + V_{0Y} + W_{0z} = s_i(z)\delta(X)\delta(Y), \tag{3.27}$$

$$\begin{aligned} U_0 &= -\tilde{D}_x(z) P_{0X}, \\ V_0 &= -\tilde{D}_y(z) P_{0Y}, \qquad (X,Y,z) \in D, \\ W_0 &= -\tilde{D}_z(z) P_{0z}, \end{aligned} \tag{3.28}$$

$$W_0(X,Y,z_+^i) = 0, \quad W_0(X,Y,z_-^i) = 0, \quad (X,Y) \in \mathbb{R}^2. \tag{3.29}$$

Here $z_\pm^i = z_\pm(x_i,y_i)$, $D = \mathbb{R}^2 \times (z_-^i, z_+^i)$ and $\tilde{D}_\alpha(z) = D_\alpha(x_i,y_i,z)$, for $z \in [z_-^i, z_+^i]$ and $\alpha = x, y$ or z. We remark that the spatial domain for this leading order problem is now the unbounded region in (X,Y,z) space contained between the coordinate planes $z = z_-^i$ and $z = z_+^i$. The problem (3.27)–(3.29) is completed by applying the asymptotic matching principle of Van Dyke [21]. It is straightforward to establish that matching of $\hat{p}$ is sufficient, after which matching of $\hat{u}$, $\hat{v}$ and $\hat{w}$ follows automatically. We must apply Van Dyke's matching principle to the outer region asymptotic expansion for $\hat{p}$ taken to $O(\epsilon)$, (3.23), with the inner region asymptotic expansion for $\hat{p}$ taken to $O(1)$, (3.25) and (3.26). The appropriate matching condition is

$$P_0(X,Y,z) = -\frac{\alpha_i}{4\pi(\bar{D}_x^i \bar{D}_y^i)^{1/2}} \log\left(\frac{X^2}{\bar{D}_x^i} + \frac{Y^2}{\bar{D}_y^i}\right) + A_0^i + O\left(\left(\frac{X^2}{\bar{D}_x^i} + \frac{Y^2}{\bar{D}_y^i}\right)^{-1/2}\right)$$
$$\text{as } (X^2+Y^2) \to \infty \text{ uniformly for } z \in [z_-^i, z_+^i].$$

Finally, the regularity conditions (Pi)–(Piii) require:

(Ii) $P_0 \in C^1(\bar{D}\backslash\bar{I}) \cap C^2(D\backslash I)$, where $I = \{(0,0)\} \times (z_-^i, z_+^i)$;

(Iii) $\lim_{\tilde{R}_i \to 0}[\tilde{R}_i |\underline{\hat{D}}\nabla P_0|]$ exists uniformly for $z \in [z_-^i, z_+^i]$, $\theta \in [0, 2\pi)$.

(Iiii) $\lim_{\tilde{R}_i \to 0} \tilde{R}_i \left(\int_0^{2\pi} (\underline{\hat{D}}\nabla P_0).\underline{\check{R}}_i \,\mathrm{d}\theta\right) = -s_i(z)$, uniformly for $z \in [z_-^i, z_+^i]$.

Here $(\tilde{R}_i, \theta, z)$ are local cylindrical polar coordinates based at $(X,Y,z) = (0,0,0)$, with $X = \tilde{R}_i\cos\theta$, $Y = \tilde{R}_i \sin\theta$ and $\tilde{R}_i = (X^2+Y^2)^{1/2}$, $\underline{\check{R}}_i$ is the radial unit vector in the cylindrical polar coordinates $(\tilde{R}_i, \theta, z)$, and

$$\underline{\hat{D}}(z) = \begin{pmatrix} \tilde{D}_x(z) & 0 & 0 \\ 0 & \tilde{D}_y(z) & 0 \\ 0 & 0 & \tilde{D}_z(z) \end{pmatrix},$$

for all $z \in [z_-^i, z_+^i]$.

We can now eliminate U_0, V_0 and W_0 via (3.28), and obtain the following strongly elliptic problem for P_0, namely,

$$\nabla.(\underline{\hat{D}}(z)\nabla P_0) = -s_i(z)\delta(X)\delta(Y), \quad (X,Y,z) \in D, \tag{3.30}$$

$$P_{0z}(X,Y,z_+^i) = 0, \quad (X,Y) \in \mathbb{R}^2, \tag{3.31}$$

$$P_{0z}(X,Y,z_-^i) = 0, \quad (X,Y) \in \mathbb{R}^2, \tag{3.32}$$

$$P_0(X,Y,z) = -\frac{\alpha_i}{4\pi(\bar{D}_x^i\bar{D}_y^i)^{1/2}} \log\left(\frac{X^2}{\bar{D}_x^i} + \frac{Y^2}{\bar{D}_y^i}\right) + A_0^i + O\left[\left(\frac{X^2}{\bar{D}_x^i} + \frac{Y^2}{\bar{D}_y^i}\right)^{-1/2}\right]$$

$$\text{as } (X^2+Y^2) \to \infty \text{ uniformly for } z \in [z_-^i, z_+^i], \tag{3.33}$$

together with regularity conditions (Ii)–(Iiii). To make analytical progress we will take

$$\tilde{D}_x(z) = \tilde{D}_y(z) \stackrel{\text{def}}{=} \tilde{D}_h(z), \quad z \in [z_-^i, z_+^i] \tag{3.34}$$

in what follows, so that permeability in the horizontal directions is equal, but still dependent upon $z \in [z_-^i, z_+^i]$. A consequence of (3.34) is then that

$$\bar{D}_x^i = \bar{D}_y^i \stackrel{\text{def}}{=} \bar{D}_h^i.$$

It follows from standard theory that the strongly elliptic boundary value problem (3.30)–(3.33) with regularity conditions (Ii)–(Iiii) has a unique solution. Moreover, the solution can be written as

$$P_0(\tilde{R}_i, z) = F_i(\tilde{R}_i, z),$$

where $F_i : (0,\infty) \times [z_-^i, z_+^i] \mapsto \mathbb{R}$ is given by

$$F_i(\tilde{R}_i, z) = \left(A_0^i + \frac{\alpha_i}{4\pi\bar{D}_h^i}\log\bar{D}_h^i - \frac{\alpha_i}{2\pi\bar{D}_h^i}\log\tilde{R}_i\right) + \sum_{j=1}^{\infty} B_j K_0(\bar{\lambda}_j^{1/2}\tilde{R}_i)\psi_j(z),$$

$$(\tilde{R}_i, z) \in (0,\infty) \times [z_-^i, z_+^i]. \tag{3.35}$$

Here $K_0(\cdot)$ is the usual modified Bessel function of order zero (see [1, chapter 9]), whilst $\bar{\lambda}_r \in \mathbb{R}$ and $\psi_r : [z_-^i, z_+^i] \mapsto \mathbb{R}$ for $r = 0,1,2,\ldots$ are the eigenvalues and corresponding eigenfunctions of the following regular Sturm-Liouville eigenvalue problem,

$$(\tilde{D}_z(z)\psi_z)_z + \bar{\lambda}\tilde{D}_h(z)\psi = 0, \quad z \in (z_-^i, z_+^i),$$
$$\psi_z(z_-^i) = \psi_z(z_+^i) = 0,$$

which we refer to as [SL]. The eigenvalues of [SL] have

$$0 = \bar{\lambda}_0 < \bar{\lambda}_1 < \bar{\lambda}_2 < \ldots,$$

(see e.g. [7, chapters 7,8]) with $\bar{\lambda}_r \to \infty$ as $r \to \infty$, and the corresponding eigenfunctions are normalised so that

$$\langle\psi_j, \psi_k\rangle = \int_{z_-^i}^{z_+^i} \tilde{D}_h(s)\psi_j(s)\psi_k(s)\,\mathrm{d}s = \delta_{jk},$$

for $j, k = 0, 1, 2, \ldots$. The constants B_r, $r = 1, 2, \ldots$ are given by

$$B_r = \frac{1}{2\pi} \int_{z_-^i}^{z_+^i} s_i(s)\psi_r(s)\,\mathrm{d}s, \quad r = 1, 2, \ldots. \tag{3.36}$$

The functions U_0, V_0 and W_0 are now obtained directly from (3.28) via (3.35) and (3.36). The only remaining question is how to actually compute the eigenvalues and corresponding eigenfunctions of [SL]. This is straightforward and is addressed in [13]. The solution to the leading order problem is now complete. The asymptotic expansion for $\hat{p}$ in the inner region is thus,

$$\hat{p}(\tilde{R}_i, z; \epsilon) = \frac{-\alpha_i}{2\pi \bar{D}_h^i} \log \epsilon + F_i(\tilde{R}_i, z) + O(\epsilon)$$

as $\epsilon \to 0$, with $(\tilde{R}_i, z) \in (0, \infty) \times [z_-^i, z_+^i]$, and $F_i(\tilde{R}_i, z)$ given by (3.35). To obtain an approximation to $\hat{p}$ close to the i^{th} line source/sink, we obtain the structure of $F_i(\tilde{R}_i, z)$, with $\tilde{R}_i$ small, as

$$F_i(\tilde{R}_i, z) = \frac{-s_i(z)}{2\pi \tilde{D}_h^i(z)} \log \tilde{R}_i + \left\{ A_0^i + \frac{\alpha_i}{4\pi \bar{D}_h^i} \log \bar{D}_h^i - \sum_{j=1}^{\infty} B_j \left[\gamma + \log\left(\frac{1}{2} \bar{\lambda}_j^{1/2} \right) \right] \psi_j(z) \right\} + O(\tilde{R}_i^2 \log \tilde{R}_i)$$

as $\tilde{R}_i \to 0$ uniformly for $z \in [z_-^i, z_+^i]$, via (3.35) and (3.36). Here $\gamma = 0.57721\ldots$ is Euler's constant.

The asymptotic structure of [PSSP] as $\epsilon \to 0$ is now complete. However, two minor extensions are worthy of further consideration at this stage.

3.1. A line source/sink close to the boundary. In the above, the locations of the line source/sinks $(x_i, y_i) \in \Omega_1$ are such that the horizontal distance from $(x_i, y_i) \in \Omega_1$ to the boundary $\partial\Omega_1$ remains finite as $\epsilon \to 0$. In this extension we consider the situation when the k^{th} line source/sink location $(x_k, y_k) \in \Omega_1$ is close to the boundary $\partial\Omega_1$, and in particular lies within a distance $O(\epsilon)$ of $\partial\Omega_1$ as $\epsilon \to 0$. To formalise this we let $(\bar{x}_k, \bar{y}_k) \in \partial\Omega_1$ be the closest point on the boundary to $(x_k, y_k) \in \Omega_1$. The vector $(\bar{x}_k - x_k, \bar{y}_k - y_k)$ will then cut the boundary $\partial\Omega_1$ orthogonally, and we write

$$(\bar{x}_k - x_k, \bar{y}_k - y_k) = O(\epsilon)$$

as $\epsilon \to 0$. The structure of the outer region to [PSSP] remains unchanged. However, the inner region to [PSSP] at $(x, y) = (x_k, y_k)$ now encompasses part of the boundary $\partial\Omega_1$ in an ϵ-neighbourhood of $(\bar{x}_k, \bar{y}_k) \in \partial\Omega_1$, and so the leading order problem in this inner region, when $i = k$, is modified. We set

$$(\bar{x}_k - x_k, \bar{y}_k - y_k) = \epsilon(l_1, l_2),$$

with constants $l_1, l_2 = O(1)$ as $\epsilon \to 0$, and introduce inner coordinates based at (x_k, y_k), so that, in the inner region,

$$(x, y) = (x_k, y_k) + \epsilon(X, Y).$$

The boundary in the inner region now becomes the straight line in the (X, Y) plane passing through the point $(X, Y) = (l_1, l_2)$ and in the direction of the vector $(-l_2, l_1)$.

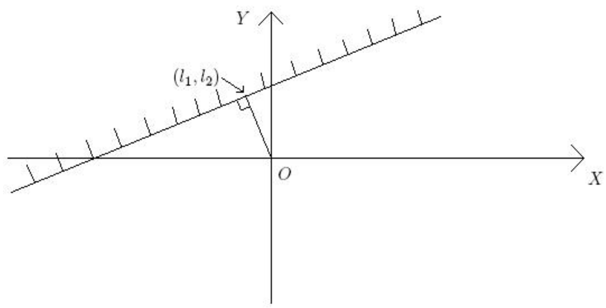

FIG. 3.1. *Domain $\mathcal{H}$ for modified inner region problem*

The domain for this modified inner region problem is shown in Figure 3.1, and is referred to as $\mathcal{H}$. Without repeating details, the inner expansion for $\hat{p}$ is now

$$\hat{p}(X, Y, z; \epsilon) = \frac{-\alpha_k}{2\pi \bar{D}_h^k} \log \epsilon + P_0(X, Y, z) + O(\epsilon)$$

as $\epsilon \to 0$ with $(X, Y, z) \in \mathcal{H} \times [z_-^k, z_+^k]$. The solution to the leading order problem is then

$$P_0(X, Y, z) = F_k(\tilde{R}_k, z) + F_k(\tilde{R}'_k, z) - A_0^k + \frac{\alpha_k}{4\pi \bar{D}_h^k} \log\left(\frac{4(l_1^2 + l_2^2)}{\bar{D}_h^k}\right), \quad (X, Y, z) \in \mathcal{H} \times [z_-^k, z_+^k],$$

with $F_k(\cdot, \cdot)$ as defined in (3.35) and

$$\tilde{R}_k = (X^2 + Y^2)^{1/2}, \quad \tilde{R}'_k = ((X - 2l_1)^2 + (Y - 2l_2)^2)^{1/2}.$$

3.2. Two closely located line source/sinks. In this extension, we consider the situation when the k^{th} and $(k+1)^{\text{th}}$ line source/sinks are within $O(\epsilon)$ separation of each other in Ω_1. We write, with $(x_k, y_k), (x_{k+1}, y_{k+1}) \in \Omega_1$,

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) + \epsilon(l_1, l_2),$$

with the constants $l_1, l_2 = O(1)$ as $\epsilon \to 0$. The structure of the outer region to [PSSP] remains unchanged. However, the inner region to [PSSP] at $(x, y) = (x_k, y_k)$ contains both of the line source/sinks at both (x_k, y_k) and (x_{k+1}, y_{k+1}). In terms of the inner coordinates (X, Y), with $(x, y) = (x_k, y_k) + \epsilon(X, Y)$, these are located at $(X, Y) = (0, 0)$ and $(X, Y) = (l_1, l_2)$. Without repeating details, the inner expansion for $\hat{p}$ is now

$$\hat{p}(X, Y, z; \epsilon) = \frac{-\alpha_k}{2\pi \bar{D}_h^k} \log \epsilon + P_0(X, Y, z) + O(\epsilon)$$

as $\epsilon \to 0$ with $(X, Y, z) \in \mathbb{R}^2 \times [z_-^k, z_+^k]$. The solution to the leading order problem is then

$$P_0(X, Y, z) = F_k(\tilde{R}_k, z) + F'_k(\tilde{R}'_k, z) - A_0^k + \frac{\alpha_{k+1}}{4\pi \bar{D}_h^k} \log\left(\frac{(l_1^2 + l_2^2)}{\bar{D}_h^k}\right), \quad (X, Y, z) \in \mathbb{R}^2 \times [z_-^k, z_+^k],$$

with $F_k(\cdot,\cdot)$ as defined in (3.35) and $F'_k(\cdot,\cdot)$ obtained from $F_k(\cdot,\cdot)$ by replacing α_k with α_{k+1} and $s_k(\cdot)$ with $s_{k+1}(\cdot)$, whilst

$$\tilde{R}_k = (X^2 + Y^2)^{1/2}, \quad \tilde{R}'_k = ((X - l_1)^2 + (Y - l_2)^2)^{1/2}.$$

The asymptotic solution to [PSSP] as $\epsilon \to 0$ uniformly for $(x, y, z) \in \bar{M}'$ is now complete. We next turn our attention to the eigenvalue problem [EVP].

4. Asymptotic solution to the eigenvalue problem [EVP] as $\epsilon \to 0$. In this section we develop the asymptotic solution to the eigenvalue problem [EVP] as $\epsilon \to 0$. As in [13], we first employ the theory developed by Ramm [18] to establish that the set of eigenvalues to [EVP], (2.37), with $\epsilon > 0$, splits into two disjoint subsets as $\epsilon \to 0^+$, which we denote by

$$S_+ = \left\{\lambda_1^+(\epsilon), \lambda_2^+(\epsilon), \ldots\right\}, \qquad S_- = \left\{\lambda_0^-(\epsilon), \lambda_1^-(\epsilon), \ldots\right\},$$

with

$$0 = \lambda_0^-(\epsilon) < \lambda_1^-(\epsilon) < \ldots, \qquad 0 < \lambda_1^+(\epsilon) < \lambda_2^+(\epsilon) < \ldots,$$

and, in particular,

$$\lambda_n^-(\epsilon) = O(n^2), \quad \lambda_n^+(\epsilon) = O(n^2\epsilon^{-2}) \tag{4.1}$$

as $\epsilon \to 0^+$, uniformly for $n = 1, 2, \ldots$. We will focus attention on the eigenvalues and corresponding eigenfunctions in the set S_-, so that in [EVP], we have,

$$\lambda(\epsilon) = O(1) \text{ as } \epsilon \to 0,$$

via (4.1). Thus we expand $\phi : \bar{M}' \mapsto \mathbb{R}$ in the form,

$$\phi(x, y, z; \epsilon) = \tilde{\psi}(x, y, z) + \epsilon^2\hat{\psi}(x, y, z) + o(\epsilon^2) \text{ as } \epsilon \to 0 \tag{4.2}$$

uniformly for $(x, y, z) \in \bar{M}'$, whilst we expand

$$\lambda(\epsilon) = \tilde{\lambda} + \epsilon^2\hat{\lambda} + o(\epsilon^2) \text{ as } \epsilon \to 0. \tag{4.3}$$

On substitution from (4.2) and (4.3) into [EVP], we obtain the leading order problem as,

$$\Big(D_z(x, y, z)\tilde{\psi}_z\Big)_z = 0, \quad (x, y, z) \in M', \tag{4.4}$$

$$\Big(\underline{\tilde{D}}(\mathbf{r})\nabla\tilde{\psi}(\mathbf{r})\Big).\hat{\mathbf{n}}_1 = 0, \quad \text{for all } \mathbf{r} \in \partial M'_H, \tag{4.5}$$

$$\tilde{\psi}_z(x, y, z_+(x, y)) = 0, \quad \text{for all } (x, y) \in \Omega_1, \tag{4.6}$$

$$\tilde{\psi}_z(x, y, z_-(x, y)) = 0, \quad \text{for all } (x, y) \in \Omega_1. \tag{4.7}$$

A direct integration of (4.4) gives

$$\tilde{\psi}_z(x, y, z) = \frac{\tilde{B}(x, y)}{D_z(x, y, z)}, \quad (x, y, z) \in \bar{M}'.$$

Application of boundary conditions (4.6) and (4.7) then require $\tilde{B}(x, y) = 0$ for all $(x, y) \in \bar{\Omega}_1$, and so

$$\tilde{\psi}(x, y, z) = \tilde{A}(x, y), \quad (x, y, z) \in \bar{M}', \tag{4.8}$$

with $\tilde{A} : \bar{\Omega}_1 \mapsto \mathbb{R}$ such that $\tilde{A} \in C^1(\bar{\Omega}_1) \cap C^2(\Omega_1)$. Boundary condition (4.5) then requires, after an integration,

$$\left(\underline{\hat{D}}(\hat{\mathbf{r}})\hat{\nabla}\tilde{A}\right).\hat{\mathbf{n}}_1(\hat{\mathbf{r}}) = 0, \quad \hat{\mathbf{r}} \in \partial\Omega_1,$$

where $\hat{\mathbf{r}} = (x, y) \in \partial\Omega_1$ and $\underline{\hat{D}}(\hat{\mathbf{r}})$ is defined in (3.17), whilst $\hat{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. At $O(\epsilon^2)$ we obtain an inhomogeneous version of (4.4)–(4.7). As in [13], the solvability requirement on this inhomogeneous boundary value problem provides a strongly elliptic partial differential equation which must be satisfied by $\tilde{A}(\hat{\mathbf{r}})$, $\hat{\mathbf{r}} \in \Omega_1$, namely

$$\hat{\nabla}.\left(\underline{\hat{D}}(\hat{\mathbf{r}})\hat{\nabla}\tilde{A}\right) + \tilde{\lambda}\hat{\phi}(\hat{\mathbf{r}})\tilde{A} = 0, \quad \hat{\mathbf{r}} \in \Omega_1.$$

Thus $\tilde{A} : \bar{\Omega} \mapsto \mathbb{R}$ and $\tilde{\lambda} \in \mathbb{R}$ satisfy the regular self-adjoint eigenvalue problem,

$$\hat{\nabla}.\left(\underline{\hat{D}}(\hat{\mathbf{r}})\hat{\nabla}\tilde{A}\right) + \tilde{\lambda}\hat{\phi}(\hat{\mathbf{r}})\tilde{A} = 0, \quad \hat{\mathbf{r}} \in \Omega,$$
$$\left(\underline{\hat{D}}(\hat{\mathbf{r}})\hat{\nabla}\tilde{A}\right).\hat{\mathbf{n}}(\hat{\mathbf{r}}) = 0, \quad \hat{\mathbf{r}} \in \partial\Omega,$$

where the subscripts on Ω_1, $\partial\Omega_1$, $\hat{\mathbf{n}}_1(\hat{\mathbf{r}})$ have been dropped for convenience, and we recall that $\hat{\phi} : \bar{\Omega} \mapsto \mathbb{R}$ is given by $\hat{\phi}(x, y) = \int_{z_-(x,y)}^{z_+(x,y)} \bar{\phi}(x, y, \lambda)\,\mathrm{d}\lambda$, for all $(x, y) \in \bar{\Omega}$. We refer to this eigenvalue problem as [EVP]$'$. Now, established theory (see for example [19]) determines that the set of eigenvalues of [EVP]$'$ is given by $\tilde{\lambda} = \tilde{\lambda}_r \in \mathbb{R}$, $r = 0, 1, 2, \ldots$, with,

$$0 = \tilde{\lambda}_0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \ldots \tag{4.9}$$

and $\tilde{\lambda}_r = O(r^2)$ as $r \to \infty$. Corresponding to each eigenvalue $\tilde{\lambda}_r$ in the ordering (4.9) there is a unique normalized eigenfunction $\tilde{A}_r : \bar{\Omega} \mapsto \mathbb{R}$ such that

$$\int\int_{\bar{\Omega}} \hat{\phi}(x, y)\tilde{A}_i(x, y)\tilde{A}_j(x, y)\,\mathrm{d}x\,\mathrm{d}y = \delta_{ij},$$

for $i, j = 0, 1, 2, \ldots$. We observe that

$$\tilde{A}_0(\hat{\mathbf{r}}) = \left\{\int\int_{\bar{\Omega}} \hat{\phi}(u, v)\,\mathrm{d}u\,\mathrm{d}v\right\}^{-1/2} = (\mathrm{meas}(\bar{M}'))^{-1/2}, \quad \hat{\mathbf{r}} \in \bar{\Omega}.$$

Thus, we have established for [EVP], via (4.2), (4.3), (4.8), that,

$$\lambda_r^-(\epsilon) = \tilde{\lambda}_r[1 + O(\epsilon^2)], \text{ as } \epsilon \to 0,$$

for $r = 0, 1, 2, \ldots$, with corresponding normalised eigenfunction

$$\phi_r^-(x, y, z; \epsilon) = \tilde{A}_r(x, y) + O(\epsilon^2), \text{ as } \epsilon \to 0,$$

uniformly for $(x, y, z) \in \bar{M}'$.

We can now use the above theory to obtain the following expression for $\tilde{p} : \bar{M}' \times [0, \infty) \mapsto \mathbb{R}$, via (2.38) and (2.39),

$$\tilde{p}(x, y, z, t) = \sum_{r=1}^{\infty} c_r \mathrm{e}^{-\tilde{\lambda}_r t}\tilde{A}_r(x, y) + O(\epsilon^2\mathrm{e}^{-\tilde{\lambda}_1 t}, \mathrm{e}^{-t/\epsilon^2}) \tag{4.10}$$

as $\epsilon \to 0$, uniformly for $(x, y, z, t) \in \bar{M}' \times [\delta, \infty)$, for any fixed $\delta > 0$. Here the coefficients c_r, $r = 1, 2, \ldots$, are given by

$$c_r = \int\int_{\bar{\Omega}} \langle \tilde{p}_0 \rangle (u, v) \tilde{A}_r(u, v) \, \mathrm{d}u \, \mathrm{d}v,$$

where $\langle \tilde{p}_0 \rangle : \bar{\Omega} \mapsto \mathbb{R}$ is given by

$$\langle \tilde{p}_0 \rangle (x, y) = \int_{z_-(x,y)}^{z_+(x,y)} \tilde{p}_0(x, y, s) \bar{\phi}(x, y, s) \, \mathrm{d}s, \quad (x, y) \in \bar{\Omega},$$

with $\tilde{p}_0 : \bar{M}' \mapsto \mathbb{R}$ as given in (2.34). We observe from (4.10) that,

$$\tilde{p}(x, y, z, t) \sim (c_1 \tilde{A}_1(x, y) + O(\epsilon^2)) \mathrm{e}^{-\tilde{\lambda}_1 t}$$

as $t \to \infty$, uniformly for $(x, y, z) \in \bar{M}'$. Therefore, the transient part of the solution to [IBVP] decays exponentially as $t \to \infty$ with rate $\tilde{\lambda}_1$. In dimensionless variables, the time scale for transient relaxation is thus $t_s \sim (\tilde{\lambda}_1)^{-1}$,which, in dimensional variables is, via (2.12),

$$t_s^d \sim \frac{c_t l^2}{D_0^H} (\tilde{\lambda}_1)^{-1}.$$

It remains to consider the numerical solution of both [BVP] and [EVP]$'$, which can be achieved efficiently as both are two-dimensional regular, strongly elliptic problems on $\bar{\Omega}$. For details, and numerical experiments demonstrating the exceptional efficiency of our approach, we refer to [12].

5. Conclusions. We have considered the unsteady flow of a weakly compressible fluid in a horizontal layer of an inhomogeneous and anisotropic porous medium with variable upper and lower boundaries, in the presence of vertical line sources and sinks. We have derived a strongly parabolic linear initial-boundary value problem for the dynamic fluid pressure, and shown that this problem has a unique solution. We have then constructed the solution to this problem when the layer aspect ratio $0 < \epsilon \ll 1$, via the method of matched asymptotic expansions. First, we have derived a matched asymptotic solution to the pseudo-steady state problem. The solution in the outer region is given in terms of the solution of a linear, inhomogeneous, strongly elliptic two-dimensional boundary value problem, whose numerical solution is straightforward (see [12]). In the inner regions the solution is given in terms of the eigenvalues and eigenvectors of a regular Sturm-Liouville eigenvalue problem [SL], whose numerical solution is again straightforward (again, see [12]).

By subtracting the solution of the pseudo-steady state problem from the solution of the initial-boundary value problem we have then constructed a strongly parabolic homogeneous evolution problem with no singularities at the line sources and sinks, whose solution can be written in terms of the eigenvalues and eigenfunctions of a regular self-adjoint eigenvalue problem, and represents the transient behaviour. Asymptotic solution of this problem when $0 < \epsilon \ll 1$ reduces to solution of a regular two-dimensional strongly elliptic problem, whose numerical solution is once again straightforward [12].

It has further been shown that the solution of the initial-boundary value problem approaches the solution to the pseudo-steady state problem, with a linear, homogeneous compression or expansion term, through terms exponentially small with respect to time t as $t \to \infty$. Generalisations to cases where a line source or sink is near a boundary wall, or where line sources and sinks are not well spaced, have also been considered, in §3.1 and §3.2 respectively.

For a full description of the entire computational procedure required to obtain numerical approximations to the pressure and flow fields throughout the layer, and examples demonstrating the application of the theory to some simple situations, we refer to [12]. We finally remark that since the initial-boundary value problem is solved for a general C^1 initial condition, the effect of time dependent transient effects due to temporal changes in the well discharge rates can easily be accounted for.

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