

# Department of Mathematics

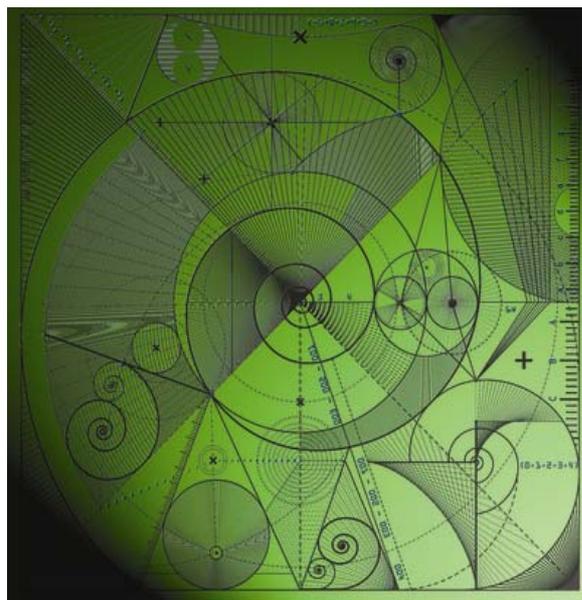
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## Models for Evolving Networks: with Applications in Telecommunication and Online Activities

by

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# Models for Evolving Networks: with Applications in Telecommunication and Online Activities

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## Abstract

Driven by a range of modern applications that includes telecommunications, e-business and on-line social interaction, recent ideas in complex networks can be extended to the case of time-varying connectivity. Here we propose a general framework for modelling and simulating such dynamic networks, and we explain how the long time behaviour may reveal important information about the mechanisms underlying the evolution.

## 1 Introduction

Many application areas give rise to connectivity patterns that change over time. As well as traditional contexts such as epidemiology [16, 28], examples are arising in modern applications such as mobile telecommunications, on-line trading, smart-metering and on-line social networking [2, 4, 6, 7, 11, 13, 16, 19, 24, 26]. Information such as ‘who called who’, ‘who tweeted who’, ‘who facebooked who’, and ‘people who bought his book also bought ...’ is naturally evolving over time and cannot be fully exploited through a static representation as a single time-average or snapshot. The motivation for our work is that these emerging, data-rich disciplines can generate large, highly-resolved network sequences that demand new models and computational tools.

Although we will draw on concepts from the well-studied ‘network growth’ context, where new vertices and accompanying edges are accumulated [3, 22], we are concerned here with a different time-dependent scenario where the population of vertices remains fixed from the outset, and the graph evolves through the appearance (birth) or the deletion (death) of edges. Specific examples that have recently received attention include

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- networks of mobile phone users with a link denoting current interaction [24, 26],
- transportation networks defined over a dynamic infrastructure [6],
- networks describing transient social interactions [25],
- correlated neural activity in response to a functional task [11].

Our aims here are (a) to set out and study some general options for describing such evolving networks in a stochastic setting and (b) to discuss practical challenges in interpreting and calibrating suitable models. In particular, we show how ideas from [11] can be extended to produce a wider class of models.

In section 2 we introduce a range of models under successively less restrictive assumptions. Section 3 focusses on a particularly promising model class and in section 4 we make some observations about long-time behaviour. In section 5 we give some illustrative simulations on synthetic data and then show computational results on a real evolving mobile phone network. Concluding remarks appear in section 6.

## 2 Stochastic models

For simplicity, we consider here undirected, unweighted graphs, defined on a set  $V$  of exactly  $n > 2$  vertices, with no self-loops. Extensions to directed graphs and self-loops follow naturally. Any such graph  $G$  may be represented by a symmetric  $(n \times n)$  adjacency matrix,  $A$ , with elements  $A_{ij} = A_{ji} = 1$  if the edge  $e = (i, j)$  is present, and zero otherwise.

Let  $S_n$  denote the set of all such graphs defined over these  $n$  vertices. We have  $|S_n| = 2^{\frac{n(n-1)}{2}} = M(n)$ , say. An evolving graph over discrete time steps is simply an ordered sequence,  $\{G_k\}$  for  $k = 0, 1, 2, \dots$ , within  $S_n$ . We think of the evolving graph as taking the particular state  $G_k$  at  $k$ th time step, that is, at time  $t_k$  from some monotonically increasing time sequence.

To introduce a stochastic element, suppose we have a set of conditional probabilities, defined for all possible networks,  $G_{k+1} \in S_n$ , given all of the networks earlier within the evolving sequence: say

$$P(G_{k+1}|G_k, G_{k-1}, \dots).$$

This set determines a probability distribution for the next element,  $G_{k+1}$ , in the sequence, given its history to date. It may be applied iteratively to generate successive further elements of the sequence.

Let us also suppose that the evolving graph respects the natural Markov property, so that the most recent graph in the sequence is sufficient to define the statistics of its future evolution: then we must have well defined probabilities

$$P(G_{k+1}|G_k), \tag{1}$$

for all ordered pairs  $(G_{k+1}, G_k)$  in  $S_n \times S_n$ . Hence a total of  $2^{n(n-1)}$  probabilities is required in this general setting. This number seems prohibitively large if we wish to study general properties or calibrate models to data.

We may reduce this complexity by imposing symmetry. If there is nothing distinguished about any of the vertices that effects the evolution, then we may argue that the probabilities should not change if the vertices are permuted. (Note this is not the case for range dependent graphs introduced in subsection 3.3, where the vertices are embedded within some underlying ordering). In this *permutable* case the specification is reduced by a factor of  $n!$ . For  $n$  large, Stirling's approximation gives  $n! = O(2^{n \log_2 n})$ , so this cannot provide much relief!

A more useful simplification is to assume *edge independence*. Here, at each time step the probability for the appearance or disappearance of each edge,  $e$ , is independent of those for all other edges, yet all such probabilities are conditional on properties of  $G_k$ . So for each edge  $e$  and each  $G_k$  we must specify the probability

$$P(e \in G_{k+1} | G_k). \quad (2)$$

Then we can reassemble the full graph transition probabilities in (1) from those for the independent edge transitions in (2), to give

$$P(G_{k+1} | G_k) = \prod_{e \in G_{k+1}} P(e \in G_{k+1} | G_k) \prod_{e \notin G_{k+1}} (1 - P(e \in G_{k+1} | G_k)). \quad (3)$$

This radically reduces the need to specify graph to graph transition probabilities, since at worst we now require  $n(n-1)2^{-1+n(n-1)/2}$  probabilities. In the next section, we consider how these may be defined as suitable functions of  $e$  and  $G_k$ .

### 3 Independent edge birth and death dynamics

We will introduce a class of independent edge models by specifying edge birth and death dynamics. Let us assume we have well defined terms

$$\alpha(e) = P(e \in G_{k+1} | e \notin G_k), \quad \omega(e) = P(e \notin G_{k+1} | e \in G_k), \quad (4)$$

that denote the edge birth probability and edge death probability, respectively. For simplicity, this notation suppresses their possible dependence on  $G_k$ . Then to define a model we simply give  $\alpha(e)$  and  $\omega(e)$  as functions of properties of  $G_k$ , when  $e$  is not present, or present, in  $G_k$  respectively. We will propose three useful cases.

### 3.1 Births and deaths dependent upon degree

Suppose the edge  $e$  that connects vertices  $v_i$  and  $v_j$  is not in  $G_k$ . Let  $d_i$  and  $d_j$  denote the degree of vertices  $v_i$  and  $v_j$  within  $G_k$ , respectively. Then let us define

$$\alpha(e) = F_\alpha(d_i, d_j),$$

where  $F_\alpha$  is any continuous mapping from pairs of integers onto the interval  $[0,1]$ . In the undirected edge case that we consider in this work, symmetry demands  $F_\alpha(z_1, z_2) = F_\alpha(z_2, z_1)$  for all nonnegative integers  $z_1$  and  $z_2$ . For example,  $F_\alpha$  might be monotonically increasing in both arguments, meaning that edges are more likely to appear between vertices of higher degree. Such a case is given by

$$F_\alpha(d_i, d_j) = \frac{d_i d_j + a}{d_i d_j + a + b}$$

for positive reals  $a$  and  $b$ . This mirrors the concepts of *preferential attachment* and *assortativity* found in static models [3, 21].

Similarly suppose  $e$  is in  $G_k$ , and connects vertices  $v_i$  and  $v_j$ . Then we may define

$$\omega(e) = F_\omega(d_i, d_j),$$

where  $F_\omega$  is any continuous mapping from pairs of integers onto the interval  $[0,1]$  (again such that  $F_\omega(z_1, z_2) = F_\omega(z_2, z_1)$ ). For example  $F_\omega$  may be monotonically decreasing in both arguments, meaning that edges are less likely to disappear between vertices of higher degree.

This type of degree-dependent activity where certain individuals act as *hubs* or *authorities*, [17], or, in Gladwell's terminology *connectors*, *mavens* and *salespeople*, [8], may be appropriate for modelling social and business networking communities, such as LinkedIn, and for on-line marketplaces, such as ebay, where 'powersellers' may attract more business.

### 3.2 Births and deaths dependent upon local clustering

Localised clustering, the ability for connections to be transitive, is a basic ingredient of small world networks [31]. In applications involving networks of social ties, it may be natural to assume that edges evolve so as to triangulate second neighbours and strengthen clique-formation [1].

Suppose some possible edge  $e$  connects vertices  $v_i$  and  $v_j$ , but is not in  $G_k$ . Let  $r_{ij}$  denote the number of adjacent vertices that  $v_i$  and  $v_j$  have in common. Then  $r_{ij}$  is the  $i, j$ th element of  $A_k^2$  (the square of the adjacency matrix for  $G_k$ ). Then let us define

$$\alpha(e) = F_\alpha(d_i, d_j, r_{ij}),$$

where  $F_\alpha$  is any continuous mapping from triples of integers onto the interval  $[0,1]$ . Note that  $r_{ij} \leq \min\{d_i, d_j\}$ . As before we require  $F_\alpha(z_1, z_2, z_3) = F_\alpha(z_2, z_1, z_3)$  for all nonnegative integers  $z_1$  and  $z_2$ . For example  $F_\alpha$  may be monotonically decreasing in both  $z_1 - z_3$  and  $z_2 - z_3$ , but increasing in  $z_3$ , so that that edges are likely to appear between vertices that have many adjacent vertices in common. Such a case is given by

$$F_\alpha(d_i, d_j, r_{ij}) = \frac{1 + r_{ij}}{\sqrt{1 + d_i d_j}}.$$

Similarly, suppose  $e$  is in  $G_k$ , and connects vertices  $v_i$  and  $v_j$ . Then we may define

$$\omega(e) = F_\omega(d_i, d_j, r_{ij}),$$

where  $F_\omega$  is any continuous mapping from from triples of integers onto the interval  $[0,1]$ , with  $F_\omega(z_1, z_2, z_3) = F_\omega(z_2, z_1, z_3)$ . For example  $F_\omega$  may be monotonically decreasing in  $z_3$  meaning that edges are less likely to disappear between vertices of with many common adjacencies. Such a case is given by

$$F_\omega(d_i, d_j, r_{ij}) = \frac{1 + \sqrt{d_i d_j}}{1 + r_{ij}}.$$

### 3.3 Births and deaths dependent upon edge range

In some circumstances, it is reasonable to assume that connections between vertices are determined in part by their relative locations in some physical or abstract space [18, 23, 29]. This concept of location in space may be more than geographical; there is evidence for a more general ‘social distance’ metric that, in principle, could be inferred from the network data [30]. Specifically, for range dependent graphs [9, 10, 12, 14, 15] the vertices are considered to have an underlying (generally unknown) ordering on the integer lattice, and the *range* of any possible edge  $e$  connecting vertices  $v_i$  and  $v_j$ , is defined by the distance  $m(e) = |i - j|$ . To generate an instance of the graph, each edge is then created independently, with a probability given by some predetermined function of its range.

These ideas were extended in [11] to the dynamic network setting. In that case, we define a range-dependent birth rate and death rate

$$\begin{aligned} \alpha(e) &= F_\alpha(m(e)), \\ \omega(e) &= F_\omega(m(e)), \end{aligned}$$

where  $F_\alpha$  and  $F_\omega$  are continuous mappings from the integers onto the interval  $[0,1]$ .

Typically we may choose  $F_\alpha$  to be monotonically decreasing so that longer range edges are less likely to arise. The case where

$$\frac{F_\alpha(m)}{F_\omega(m)} = \theta^{m^2},$$

for constant  $0 \leq \theta \leq 1$  was studied in [11], and shown to be attractive from a model calibration perspective.

This case of range-dependent edge evolution is special because not only are the births or deaths each for edge independent from each other, but each edge depends only on its own immediate history (and of course its own range). In the newly proposed degree dependent and cluster dependent models of subsections 3.1 and 3.2, the births and deaths depend on  $G_k$  in a more sophisticated way, and the evolution of each edge cannot be determined without determining the evolution of some or all other possible edges. For the range-dependent case though, whether or not  $e$  is in  $G_{k+1}$  depends only on whether it is in  $G_k$ , and the corresponding forms assumed for  $\alpha(e)$  and  $\omega(e)$ . As pointed out in [11], if  $p_k(e) = P(e \in G_k)$ , then

$$p_{k+1}(e) = \alpha(e)(1 - p_k(e)) + (1 - \omega(e))p_k(e).$$

So as  $k$  becomes large such an evolving network burns out its initial starting point and we have

$$p_k(e) \rightarrow p_\infty(e) = \frac{\alpha(e)}{\alpha(e) + \omega(e)}, \quad \text{as } k \rightarrow \infty.$$

## 4 Long term behavior

Suppose we have any Markov model over  $S_n$ , where we are given or can calculate  $P(G_{k+1}|G_k)$  for all pairs  $(G_{k+1}, G_k)$  in  $S_n \times S_n$ , as in (2) above. Let  $s_m \in S_n$  denote the  $m$ th element of  $S_n$ , for  $m = 1, \dots, M(n)$ . At the  $k$ th time step, let  $\mathbf{w}_k = (w_{k,1}, \dots, w_{k,M(n)})^T$  where  $w_{k,m} = P(G_k = s_m)$ . Then the  $P(G_{k+1}|G_k)$  together determine a transition matrix,  $B$ , such that

$$\mathbf{w}_{k+1} = B\mathbf{w}_k.$$

Under the reasonable assumption of ergodicity, meaning that  $B$  is irreducible, this iteration tends to a unique nonnegative fixed fixed point  $\mathbf{w}^* > 0$  satisfying  $B\mathbf{w}^* = \mathbf{w}^*$ .

Let  $\phi(s)$  be any binary valued feature defined for all  $s \in S_n$ , so that  $\phi = 1$  if the graph has the feature present, and  $\phi = 0$  otherwise. Then as  $k$  tends to infinity the expected value of  $\phi(G_k)$  is given by

$$\langle \phi \rangle = \sum_{m=1}^{M(n)} w_m^* \phi(s_m).$$

In particular each possible edge,  $e$ , is present with a probability

$$p_\infty(e) = \sum_{e \in s_m} w_m^*.$$

Let  $G^*$  denote the *long term expected random graph*, where each edge  $e$  has a probability  $p_\infty(e)$  of being present.

Suppose this model does not use some additional (imposed) knowledge that differentiates between the vertices. Of course range-dependent evolving graphs employ an imposed ordering of the vertices for example; whilst Barabási style aggregative graphs allow vertices to become active in some predefined order, externally imposed. But for evolving graphs having no such vertex discrimination, symmetry demands that  $G^*$  is invariant to any permutations of the vertices. Hence all possible edges in  $G^*$  are equally likely:  $G^*$  is an Erdős-Rényi random graph, with a Poisson distribution of vertex degrees, and no scale free or small world properties. For example, this must be true of the evolving networks depending solely upon (current) vertex degrees or localised clustering coefficients, introduced in the subsections 3.1 and 3.2.

So, if we observe a large evolving graph with long term average behaviour which has a non Poissonian vertex degree distribution, then we know that the dynamics of any assumed underlying Markov model must involve some extra knowledge or imposed information distinguishing the vertices.

## 5 Numerical simulations

In Figure 1 we show the first sixteen adjacency matrices that arise from a degree-based model, as described in subsection 3.1. More precisely, for a pair of vertices  $i$  and  $j$  we used an edge birth probability given by

- $p_a = 0.9$  if  $\min(\deg_i, \deg_j) > 8$ ,
- $p_b = 0.2$  otherwise,

and an edge death probability given by

- $p_c = 0.9$  if  $\min(\deg_i, \deg_j) < 6$ ,
- $p_d = 0.2$  otherwise.

Hence, in this model new edges are favoured between vertices that both have relatively high degree, and existing edges involving at least one low-degree node are penalised. In order to make the adjacency matrices compact for visualisation purposes, we used a relatively small number of vertices,  $n = 16$ . We begin, at time  $t = 0$ , with a sample of an Erdős-Rényi random graph with 38 edges. The vertices are ordered according to their degree in the initial network, from high to low.

After continuing the iteration for 200 more time steps, Figure 2 shows time levels  $t = 216$  to  $t = 231$ .

To illustrate long time behaviour on a larger network, Figure 3 shows the results for a case with  $n = 200$  vertices. Here, the initial network, shown in the upper left

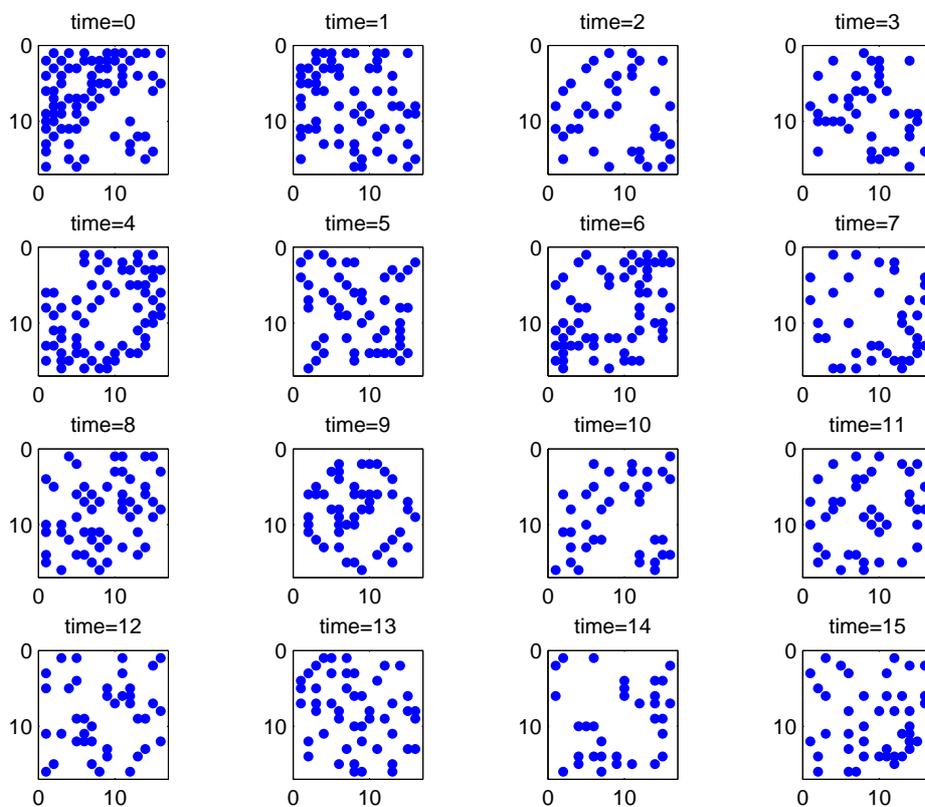


Figure 1: Initial network adjacency matrix and first fifteen iterates for a degree-dependent evolution. Vertices are ordered according to degree at time zero.

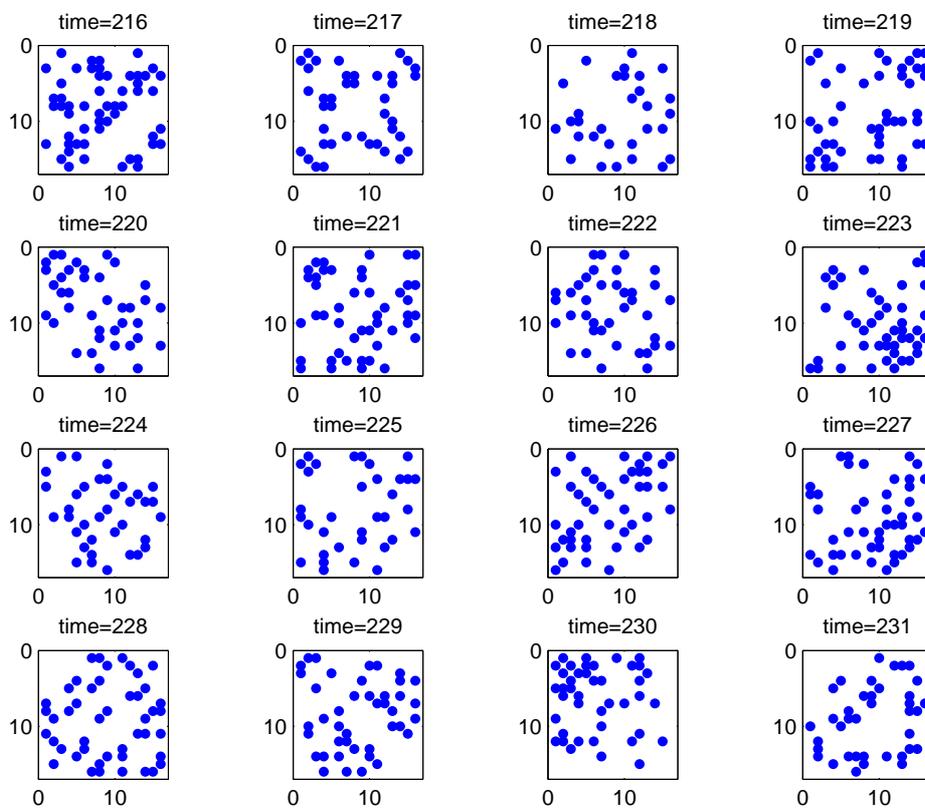


Figure 2: Iterates at time  $t = 216$  to  $t = 231$  for the network sequence initiated in Figure 1.

picture, is chosen from the preferential attachment model [3, 21], as implemented in the function `pref.m` of CONTEST [27]. We used the same degree-based birth and death rate construct as for Figures 1 and 2, with edge birth probabilities rescaled to

- $p_a = 0.3$  if  $\min(\deg_i, \deg_j) > 25$ ,
- $p_b = 0.05$  otherwise,

and an edge death probabilities given by

- $p_c = 0.9$  if  $\min(\deg_i, \deg_j) < 10$ ,
- $p_d = 0.6$  otherwise.

The initial degree distribution, which by construction is scale-free, is shown in the upper right picture.

The lower left picture then shows the adjacency matrix after  $10^4$  time steps, with degree distribution in the lower right. We see that the scale-free pattern of the initial degree distribution is completely lost over time and a Poisson-type distribution has arisen, as predicted in section 4. For confirmation, Figure 4 plots the cumulative degree distribution as circles and superimposes as a solid line the cumulative Poisson distribution with the same mean of 15.3.

Figure 5 shows a different scenario where the edge evolution involves an external factor. Here, we mix together the degree-dependent and range-dependent ideas from subsections 3.1 and 3.3. For the same size  $n = 200$  and the same initial network as in Figure 3, we now use edge birth rates given by

- $p_a = \exp(-|i - j|/10)$  if  $\deg_i + \deg_j > 50$ ,
- $p_b = 0.05$  otherwise,

and edge death rates given by

- $p_c = 0.9$  if  $\max(\deg_i, \deg_j) < 30$ ,
- $p_d = 0.5$  otherwise.

In this case, pairs of vertices are likely to grow a new edge if at least one of them has high degree **and** they are in close proximity. Similarly, an existing edge connecting vertices that have degree below 30 is likely to be removed. We see that the long time network at time  $10^4$  has a small number of vertices that are abundantly connected to near neighbours. The resulting degree distribution looks far from Poisson, as verified in Figure 6.

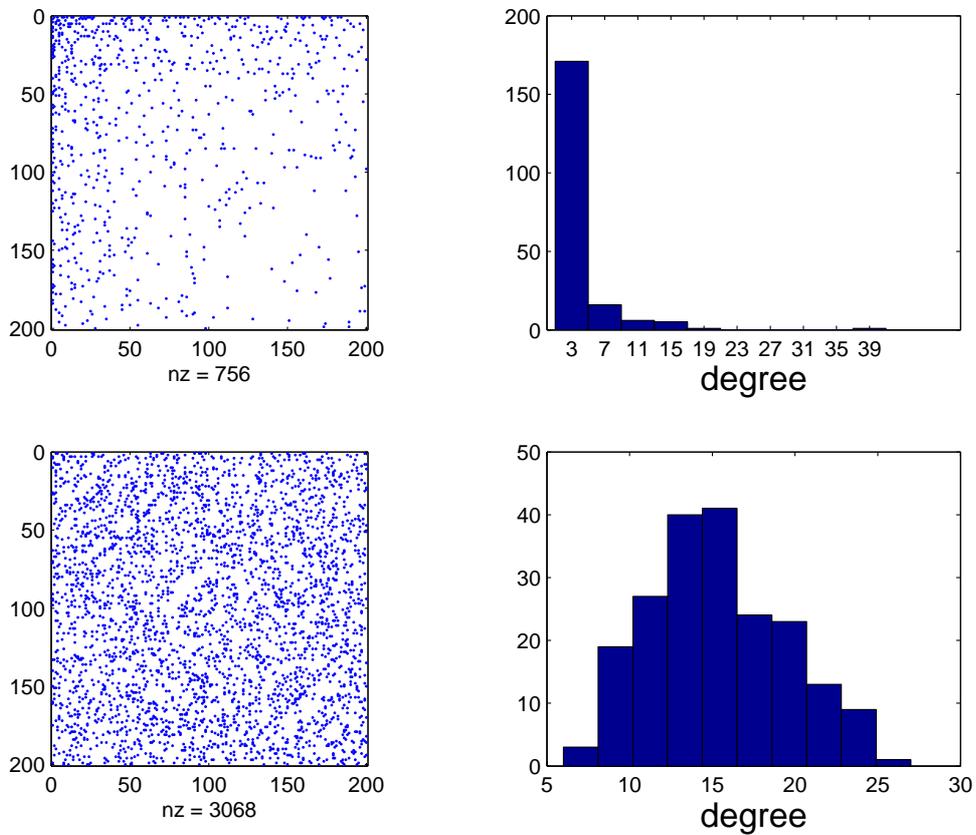


Figure 3: An evolving network with degree based evolution. Top left: initial network. Top right: initial degree distribution. Bottom left: network at time  $t = 10^4$ . Bottom right: degree distribution at time  $t = 10^4$ .

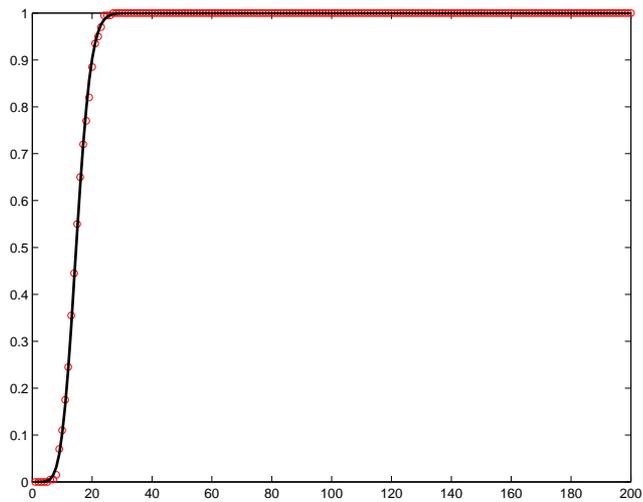


Figure 4: Circles: cumulative degree distribution for time  $t = 10^4$  network shown in Figure 3. Solid line: interpolant through cumulative degree distribution for a Poisson random variable with matching mean of 15.3.

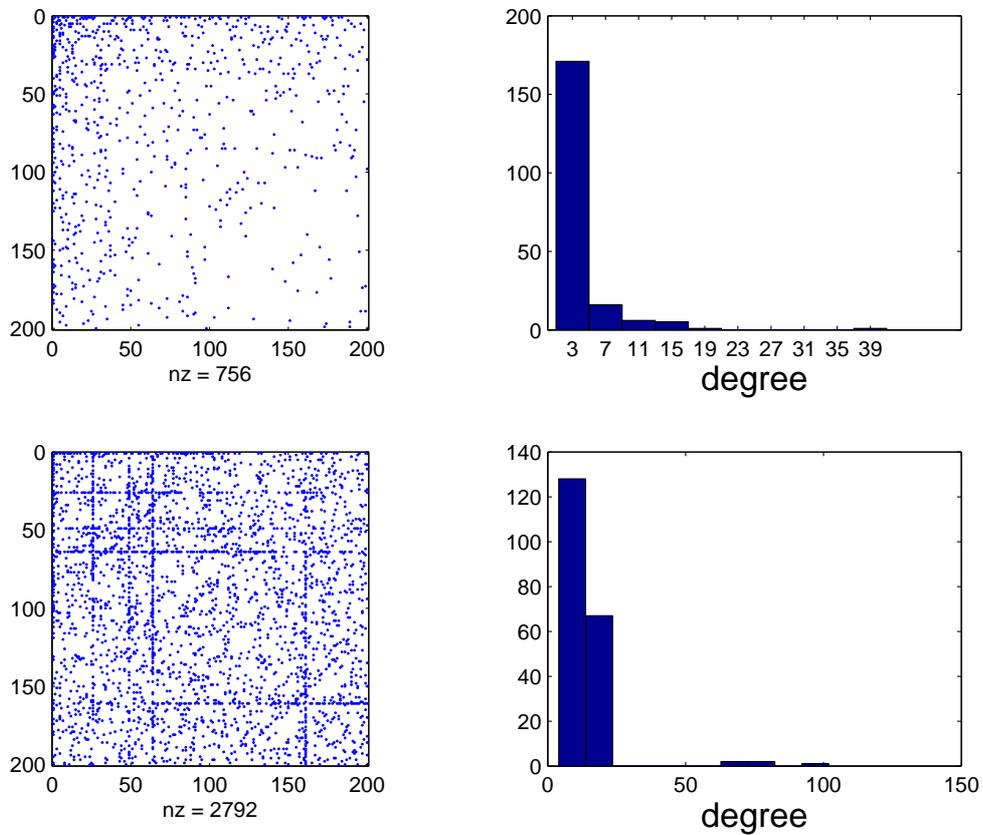


Figure 5: An evolving network with evolution based on degree and edge range. Top left: initial network. Top right: initial degree distribution. Bottom left: network at time  $t = 10^4$ . Bottom right: degree distribution at time  $t = 10^4$ .

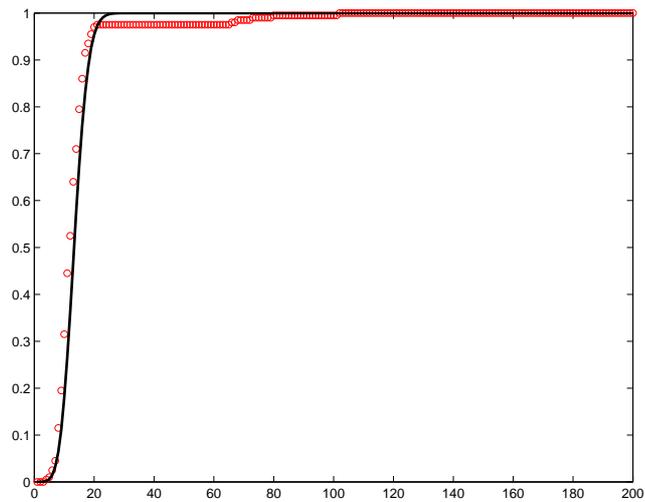


Figure 6: Circles: cumulative degree distribution for time  $t = 10^4$  network shown in Figure 5. Solid line: interpolant through cumulative degree distribution for a Poisson random variable with matching mean of 14.

As a final computational test, we consider an evolving network from [5]. This data comes from a “Reality Mining” study that used mobile phones to follow 106 subjects at MIT over the course of the 2004–2005 academic year. Pairwise calls, SMS activity and proximity information were recorded. Here we consider just the voice call component of the data, summarized into weekly activity. So a link between nodes  $i$  and  $j$  in the  $k$ th adjacency matrix indicates that at least one phone call took place between subjects  $i$  and  $j$  in week  $k$ . This represents an an evolving network over 52 time points. This network sequence was also used in [13], a visualization of the complete data set can be found there. Of course, understanding the mechanisms that drive this type of dynamic network has immediate benefits for designing mobile phone contracts, identifying and marketing to specific customer groups, and predicting future patterns of network usage.

We attempted to calibrate this evolving network based on a range-dependent model. Following [11], we formed a Laplacian matrix based on the cumulative edge data, and used the Fiedler vector to infer an ordering of the nodes. Then for each pair of nodes  $i$  and  $j$  we estimated the birth and death probabilities (4) based on their observed frequency. However, in this context it is not appropriate to use a simple frequency count for the edge death,  $s/N$ , where

- $s$  is the number of edge deaths observed, and
- $N$  is the total number of time points between the first and penultimate time on which an edge existed.

This is because  $N = 0$  for many pairs  $i$  and  $j$ . For this reason, we used Laplace’s Law of Succession [20] to give  $(s + 1)/(N + 2)$  as our estimate for the edge death probability. Similarly, the edge birth probability was estimated as  $(s' + 1)/(N' + 2)$ , where

- $s'$  is the number of edge births observed, and
- $N'$  is the total number of time points between the first and penultimate time on which no edge existed.

Figures 7 and 8 indicate the estimated birth and death probabilities, respectively, for the node ordering given by the Fiedler vector. There are two main observations to be made.

1. In both figures, the probabilities are not uniform across pairs  $i$  and  $j$ . Following the discussion in section 4, this rules out the possibility that the data comes from the steady state of an evolving network model like those discussed in subsections 3.1 and 3.2, where nodes are not differentiated by some external property.
2. The nodes have been reordered in an attempt to reveal range-dependency, so that, in this new ordering, birth and death rates depend solely on  $|i - j|$ . In Figures 7 and

8 this would correspond to a Toeplitz structure (common values along each super-diagonal). No such obvious pattern is observed, although there is an indication that some nodes are very active, having relatively high edge birth and death probabilities, and in many cases this activity is localised to neighbours who are close in the new ordering. There is also evidence of clusters of high activity in blocks along the diagonal, where near-neighbours in this newly-discovered ordering have strong mutual affinity. This behaviour is in broad agreement with the mixed ‘range and degree dependence’ model that was used for Figure 5.

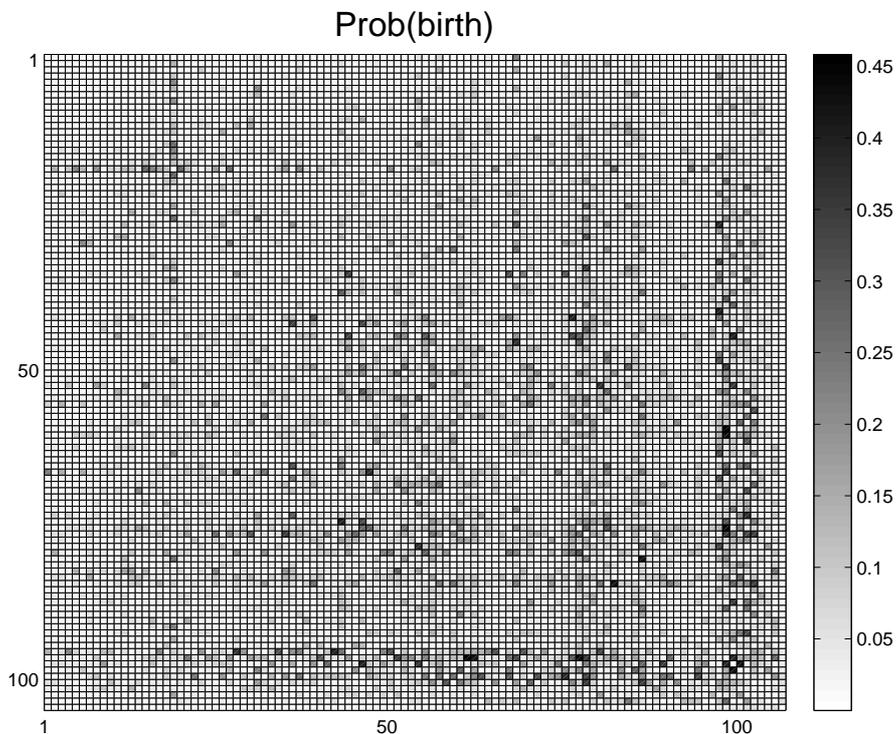


Figure 7: Symmetric matrix whose  $i, j$  element shows the estimated birth probability in (4) for an edge between nodes  $i$  and  $j$ , assuming range-dependent activity. The underlying data came from an evolving network arising in telecommunication. Nodes are ordered via a Fiedler vector in an attempt to reveal range-dependency, as described in [11].

## 6 Discussion

Ideas from network science have proved to be useful in a range of disciplines, but we feel there is great value to be had from moving attention away from fixed topological structures. Many applications, notably in telecommunications, social networking, on-line trading and utility consumption, give rise to a sequence of network “snapshots” from an

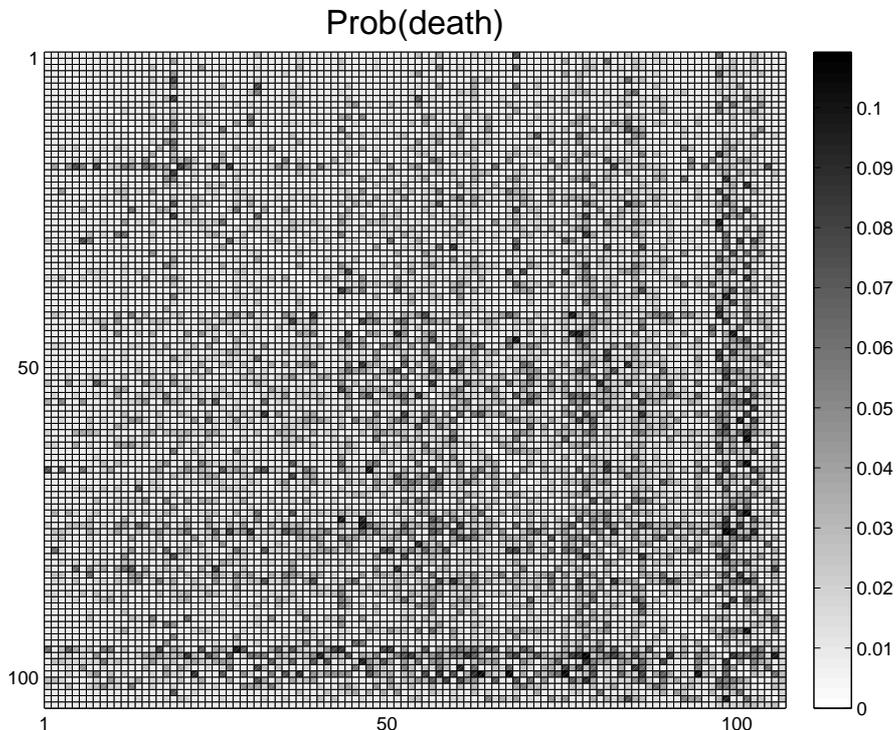


Figure 8: Analogue of Figure 7 for edge death probabilities.

evolving system. Summarizing and quantifying the mechanisms that drive the network evolution has a clear potential to help with decision making issues faced by professionals in areas such as on-line marketing, telecommunications and business development.

New challenges arise in this time-dependent context. Here, we focussed on modelling aspects—what mechanisms might govern the changes in topology? We extended the framework in [11], showed that broad conclusions may be drawn about steady state behaviour, and tested these ideas on real mobile phone data.

We are currently in a DRIP (data rich, information poor) era where there are huge potential benefits to be had from smarter, more strategic use of evolving network data. Among the key challenges *calibration and model selection* stand out immediately—comparing models that have a tractable number of parameters in terms of their explanatory and predictive power on real data sets. An accurate, well-tuned model would not only offer a high-level summary of the nature of the interactions, but would also give a powerful quantitative tool to predict future evolution and study the response of the network under ‘what-if’ scenarios. We hope that the framework outlined here sets the scene for a systematic modelling approach.

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