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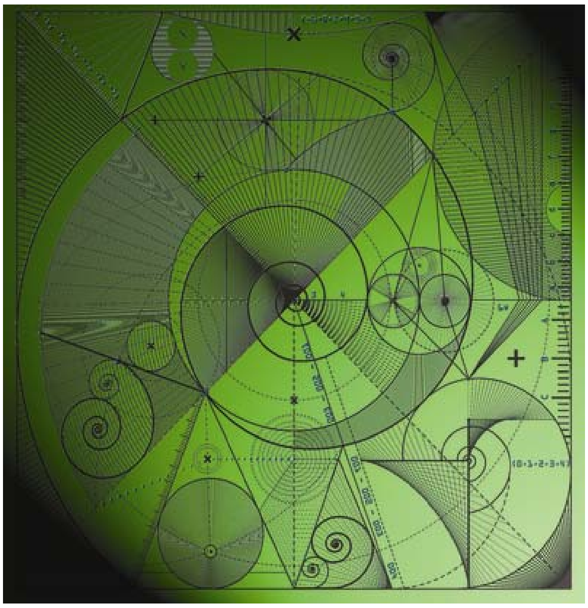

Boundary value problems for the elliptic sine-Gordon equation in a semi-strip

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Abstract

We study boundary value problems posed in a semistrip for the elliptic sine-Gordon equation, which is the paradigm of an elliptic integrable PDE in two variables. We use the method introduced by one of the authors, which provides a substantial generalization of the inverse scattering transform and can be used for the analysis of boundary as opposed to initial-value problems. We first express the solution in terms of a 2×2 matrix Riemann-Hilbert problem formulated in terms of both the Dirichlet and the Neumann boundary values on the boundary of a semistrip. We then concentrate on the case that the prescribed boundary conditions are zero along the unbounded sides of the semistrip and constant along the bounded side; in this particular case we show that the "jump matrices" of the above Riemann-Hilbert problem can be expressed explicitly in terms of the width of the semistrip and the constant value of the solution along the bounded side. This Riemann-Hilbert problem has a unique solution.

1 Introduction

A method for solving initial-boundary value problems for linear and integrable nonlinear PDEs was introduced in [13] and developed by several authors [14]. This method has already been used for:
(a) linear and integrable nonlinear *evolution* PDEs formulated on the half line and on a finite interval [4, 5, 6, 11, 12, 16, 19, 21, 23, 24, 27, 35, 37, 38, 46];
(b) linear and integrable nonlinear *hyperbolic* PDEs [26, 41];
(c) *linear elliptic* PDEs [3, 8, 9, 15, 20, 28, 30, 43, 44, 45].
The aim of this paper is to implement this method in the case of the prototypical integrable *nonlinear elliptic* PDE, namely the celebrated sine-Gordon equation (simple boundary value problems for this equation, using the method of [13], have been analyzed in [40, 42]).
We will consider the sine-Gordon equation in the form

$$q_{xx} + q_{yy} = \sin q, \quad q = q(x, y), \tag{1.1}$$

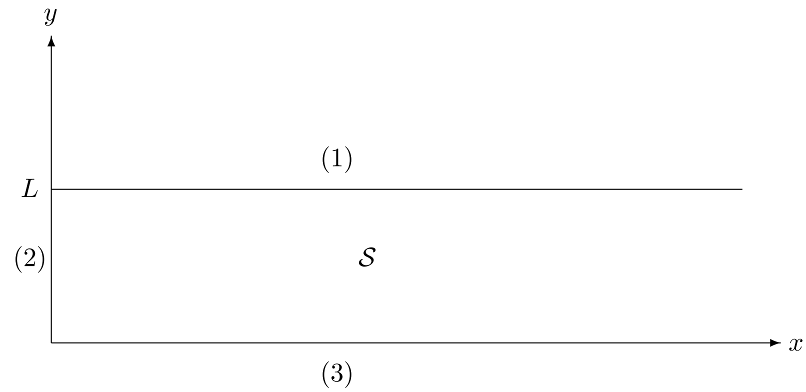

Figure 1: The semistrip S

and we will analyze boundary value problems posed in the semi-infinite strip

$$\mathcal{S} = \{0 < x < \infty, \qquad 0 < y < L\},$$

where L is a positive finite constant. The sides $\{y = L,\ 0 < x < \infty\}$, $\{x = 0,\ 0 < y < L\}$ and $\{y = 0,\ 0 < x < \infty\}$ will be referred to as side (1), (2) and (3) respectively, see figure 1. Suppose that (1.1) is supplemented with appropriate boundary conditions on the boundary of the semistrip $\mathcal{S}$, so that there exists a unique solution $q(x, y)$. It will be shown in section 2 that this solution can be expressed in terms of the solution of a 2×2 matrix Riemann-Hilbert (RH) problem with jumps on the union of the real and imaginary axis of the λ complex plane. The "jump matrices" are expressed in terms of certain functions, called *spectral functions*, which will be denoted by $\{a_j(\lambda), b_j(\lambda)\}$, $j = 1, 2, 3$. These functions can be uniquely characterized via the solution of certain linear Volterra integral equations, in terms of the Dirichlet and Neumann boundary values. Namely, $\{a_1, b_1\}$, $\{a_2, b_2\}$ and $\{a_3, b_3\}$ are uniquely determined in terms of $\{q(x, L), q_y(x, L)\}$, $\{q(0, y), q_x(0, y)\}$ and $\{q(x, 0), q_y(x, 0)\}$ respectively. However, for a well posed problem only a subset of these boundary values are prescribed as boundary conditions. Thus, in order to compute the spectral functions in terms of the given boundary conditions, one must first determine the unknown boundary values, i.e. one must characterize the Dirichlet to Neumann map. The solution of this problem, which makes crucial use of the so-called *global relation*, yields in general a nonlinear map. In the case of integrable nonlinear evolution PDEs, it has been shown in [16, 18, 34, 35] that there exists a particular class of boundary conditions, called linearizable, for which it is possible to avoid the above nonlinear map. The main result of the present paper is the analysis of linearizable boundary conditions for the sine-Gordon equation on the semi-infinite strip. In particular, the following boundary conditions will be investigated in detail:

$$q(x, L) = q(x, 0) = 0, \ \ 0 < x < \infty; \qquad q(0, y) = d, \ \ 0 < y < L, \tag{1.2}$$

where d is a finite constant. It will be shown in section 5 that in this particular case, the "jump matrices" of the associated RH problem can be constructed explicitly in terms of the given constant d and of the width L of the semistrip. This result, as well as the analogous result valid for the elliptic version of the Ernst equation [25], imply that the new method of [13] provides a powerful tool for analyzing effectively a large class of interesting boundary conditions.

2 Spectral analysis under the assumption of existence

In what follows we assume that (1.1) is supplemented with appropriate boundary conditions on the boundary of the semistrip $\mathcal{S}$ so that the existence of a unique solution $q(x,y)$ can be assumed. Furthermore, we assume the following:

$$\begin{aligned} &q(x,L),\ q_y(x,L),\ q(x,0),\ q_y(x,0) \in \mathrm{L}^1(\mathbb{R}^+), \\ &xq(x,L),\ xq_y(x,L),\ xq(x,0),\ xq_y(x,0) \in \mathrm{L}^1(\mathbb{R}^+), \\ &q(0,y),\ q_x(0,y),\ yq(0,y),\ yq_x(0,y) \in \mathrm{L}^1([0,L]). \end{aligned} \tag{2.1}$$

The sine-Gordon equation is the compatibility condition of the following Lax pair for the 2×2 matrix-valued function $\Psi(x,y,\lambda)$, $\lambda \in \mathbb{C}$:

$$\Psi_x + \frac{\Omega(\lambda)}{2}[\sigma_3, \Psi] = Q(x,y,\lambda)\Psi, \tag{2.2}$$

$$\Psi_y + \frac{\omega(\lambda)}{2}[\sigma_3, \Psi] = iQ(x,y,-\lambda)\Psi, \tag{2.3}$$

where

$$\Omega(\lambda) = \frac{1}{2i}\left(\lambda - \frac{1}{\lambda}\right), \qquad \omega(\lambda) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right), \tag{2.4}$$

$$Q(x,y,\lambda) = \frac{i}{4}\begin{pmatrix} \frac{1}{\lambda}(1-\cos q) & q_x - iq_y + \frac{i\sin q}{\lambda} \\ q_x - iq_y - \frac{i\sin q}{\lambda} & -\frac{1}{\lambda}(1-\cos q) \end{pmatrix}, \qquad q = q(x,y). \tag{2.5}$$

Equations (2.2) and (2.3) can be written as the single equation

$$d\left(\mathrm{e}^{(\Omega(\lambda)x+\omega(\lambda)y)\frac{\widehat{\sigma_3}}{2}}\right)\Psi(x,y,\lambda) = W(x,y,\lambda), \tag{2.6}$$

where the differential form W is given by

$$W(x,y,\lambda) = \mathrm{e}^{(\Omega(\lambda)x+\omega(\lambda)y)\frac{\widehat{\sigma_3}}{2}}\left(Q(x,y,\lambda)\Psi(x,y,\lambda)dx + iQ(x,y,-\lambda)\Psi(x,y,\lambda)dy\right), \tag{2.7}$$

and $\widehat{\sigma_3}$ acts on a 2×2 matrix A by

$$\widehat{\sigma_3}A = [\sigma_3, A].$$

Remark 2.1 *Note that*

$$\overline{\Omega(\bar{\lambda})} = -\Omega(\lambda) = \Omega(\frac{1}{\lambda}), \qquad \overline{\omega(\bar{\lambda})} = \omega(\lambda) = \omega(\frac{1}{\lambda}).$$

2.1 Bounded and analytic eigenfunctions

We define three solutions $\Psi_j(x,y,\lambda)$ $j=1,2,3$, of (2.6) by

$$\Psi_j(x,y,\lambda) = I + \int_{(x_j,y_j)}^{(x,y)} \mathrm{e}^{-(\Omega(\lambda)x+\omega(\lambda)y)\frac{\widehat{\sigma_3}}{2}} W(\xi,\eta,\lambda), \tag{2.8}$$

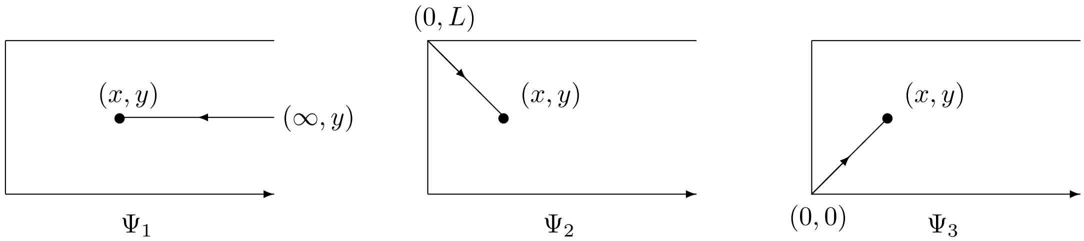

Figure 2: The functions Ψ_i

where

$$(x_1, y_1) = (\infty, y), \quad (x_2, y_2) = (0, L), \quad (x_3, y_3) = (0, 0). \tag{2.9}$$

Since the differential form W is exact, the integral on the right hand side of (2.8) is independent of the path of integration. We choose the particular contours shown in figure 2. This choice implies the following inequalities on the contours:

$$(x_1, y_1) \to (x, y): \quad \xi - x \geq 0.$$

$$(x_2, y_2) \to (x, y): \quad \xi - x \leq 0, \ \eta - y \geq 0.$$

$$(x_3, y_3) \to (x, y): \quad \xi - x \leq 0, \ \eta - y \leq 0.$$

The first inequality above implies that the exponential appearing in the second (first) column of the right hand side of the equation defining Ψ_1 is bounded and analytic for $Im(\lambda) < 0$ ($Im(\lambda) > 0$). Similar considerations are valid for Ψ_2 and Ψ_3. Hence we denote the matrices Ψ_j as follows:

$$\Psi_1 = (\Psi_1^{(12)}, \Psi_1^{(34)}), \qquad \Psi_2 = (\Psi_2^{(4)}, \Psi_2^{(2)}), \qquad \Psi_3 = (\Psi_3^{(3)}, \Psi_3^{(1)}),$$

where the superscript (12) denotes the union of the first and second quadrants of the λ complex plane, and similarly for the other superscripts. The function $\Psi_1^{(12)}$ is analytic for $Im(\lambda) > 0$ and it has essential singularities at $\lambda = \infty$ and $\lambda = 0$; furthermore,

$$\Psi_1^{(12)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty, \quad Im(\lambda) \geq 0. \tag{2.10}$$

Similar considerations are valid for the column vectors $\Psi_1^{(34)}$, $\Psi_3^{(3)}$ and $\Psi_3^{(1)}$. The function Ψ_2 is an analytic function in the entire complex plane, except at $\lambda = \infty$ and $\lambda = 0$, where it has essential singularities. In addition,

$$\Psi_2^{(4)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty, \quad \frac{3\pi}{2} < arg(\lambda) < 2\pi,$$

$$\Psi_2^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty, \quad \frac{\pi}{2} < arg(\lambda) < \pi. \tag{2.11}$$

2.2 Spectral functions

Any two solutions Ψ, $\tilde{\Psi}$ of (2.6) are related by an equation of the form

$$\Psi(x,y,\lambda) = \tilde{\Psi}(x,y,\lambda)\mathrm{e}^{-(\Omega(\lambda)x+\omega(\lambda)y)\frac{\widehat{\sigma_3}}{2}}C(\lambda). \tag{2.12}$$

We introduce the notations

$$S_1(\lambda) = \Psi_1(0,L,\lambda), \quad S_2(\lambda) = \Psi_2(0,0,\lambda), \quad S_3(\lambda) = \Psi_1(0,0,\lambda). \tag{2.13}$$

Then equation (2.12) implies the following equations:

$$\Psi_1(x,y,\lambda) = \Psi_2(x,y,\lambda)\mathrm{e}^{-(\Omega(\lambda)x+\omega(\lambda)y)\frac{\widehat{\sigma_3}}{2}}\mathrm{e}^{\frac{\omega(\lambda)}{2}L\widehat{\sigma_3}}S_1(\lambda), \quad \lambda \in (\mathbb{R}^+,\mathbb{R}^-), \tag{2.14}$$

$$\Psi_2(x,y,\lambda) = \Psi_3(x,y,\lambda)\mathrm{e}^{-(\Omega(\lambda)x+\omega(\lambda)y)\frac{\widehat{\sigma_3}}{2}}S_2(\lambda), \quad \lambda \in (i\mathbb{R}^-, i\mathbb{R}^+), \tag{2.15}$$

$$\Psi_1(x,y,\lambda) = \Psi_3(x,y,\lambda)\mathrm{e}^{-(\Omega(\lambda)x+\omega(\lambda)y)\frac{\widehat{\sigma_3}}{2}}S_3(\lambda), \quad \lambda \in (\mathbb{R}^-,\mathbb{R}^+). \tag{2.16}$$

The notation $\lambda \in (\mathbb{R}^+,\mathbb{R}^-)$ means that the equation for first column vector of (2.14) is valid for $\lambda \in \mathbb{R}^+$, while the equation for the second vector is valid for $\mathbb{R}^-$, and similarly for (2.15), (2.16).

Equations (2.13)-(2.16) suggest the following definitions:

$$\begin{aligned} S_1(\lambda) = \Phi_1(0,\lambda), \qquad & \Phi_1(x,\lambda) = I - \int_x^\infty \mathrm{e}^{\Omega(\lambda)(\xi-x)\frac{\widehat{\sigma_3}}{2}}Q(\xi,L,\lambda)\Phi_1(\xi,\lambda)d\xi, \\ & \lambda \in (\mathbb{C}^+,\mathbb{C}^-), \quad 0 < x < \infty, \end{aligned} \tag{2.17}$$

$$\begin{aligned} S_2(\lambda) = \Phi_2(0,\lambda), \qquad & \Phi_2(y,\lambda) = I - i\int_y^L \mathrm{e}^{\omega(\lambda)(\eta-y)\frac{\widehat{\sigma_3}}{2}}Q(0,\eta,-\lambda)\Phi_2(\eta,\lambda)d\eta, \\ & \lambda \in \mathbb{C}, \quad 0 < y < L, \end{aligned} \tag{2.18}$$

$$\begin{aligned} S_3(\lambda) = \Phi_3(0,\lambda), \qquad & \Phi_3(x,\lambda) = I - \int_x^\infty \mathrm{e}^{\Omega(\lambda)(\xi-x)\frac{\widehat{\sigma_3}}{2}}Q(\xi,0,\lambda)\Phi_3(\xi,\lambda)d\xi, \\ & \lambda \in (\mathbb{C}^+,\mathbb{C}^-), \quad 0 < x < \infty. \end{aligned} \tag{2.19}$$

The matrix Q satisfies the symmetry properties

$$Q(\lambda)_{22} = Q(-\lambda)_{11}, \qquad Q(\lambda)_{12} = Q(-\lambda)_{21}.$$

Hence the matrices Φ_i can be represented in the form

$$\Phi_1 = \begin{pmatrix} A_1(x,\lambda) & B_1(x,-\lambda) \\ B_1(x,\lambda) & A_1(x,-\lambda) \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} A_2(y,\lambda) & B_2(y,-\lambda) \\ B_2(y,\lambda) & A_2(y,-\lambda) \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} A_3(x,\lambda) & B_3(x,-\lambda) \\ B_3(x,\lambda) & A_3(x,-\lambda) \end{pmatrix},$$

and therefore

$$S_i(\lambda) = \begin{pmatrix} a_i(\lambda) & b_i(-\lambda) \\ b_i(\lambda) & a_i(-\lambda) \end{pmatrix}, \qquad i = 1,2,3.$$

The spectral functions $\{a_1(\lambda),\ b_1(\lambda)\}$, $\{a_2(\lambda),\ b_2(\lambda)\}$ and $\{a_3(\lambda),\ b_3(\lambda)\}$ are defined in terms of $\{q(x,L),\ q_y(x,L)\}$, $\{q(0,y),\ q_x(0,y)\}$ and $\{q(x,0),\ q_y(x,0)\}$ respectively, through equations (2.17)-(2.19).

These functions have the following properties:

- $a_1(\lambda)$, $b_1(\lambda)$ are analytic and bounded in $\mathbb{C}^+$.

  $a_1(\lambda)a_1(-\lambda) - b_1(\lambda)b_1(-\lambda) = 1$.

  $a_1(\lambda) = 1 + O\left(\frac{1}{\lambda}\right)$, $b_1(\lambda) = O\left(\frac{1}{\lambda}\right)$ as $\lambda \to \infty,\ Im(\lambda) \geq 0$.

- $a_2(\lambda)$, $b_2(\lambda)$ are analytic functions of λ for all $\lambda \in \mathbb{C}$, except for essential singularities at $\lambda = \infty$ and $\lambda = 0$.

  $a_2(\lambda)a_2(-\lambda) - b_2(\lambda)b_2(-\lambda) = 1$.

  $a_2(\lambda) = 1 + O\left(\frac{1}{\lambda}\right)$, $b_2(\lambda) = O\left(\frac{1}{\lambda}\right)$ as $\lambda \to \infty,\ \frac{3\pi}{2} < arg(\lambda) < 2\pi$.

- $a_3(\lambda)$, $b_3(\lambda)$ are analytic and bounded in $\mathbb{C}^+$.

  $a_3(\lambda)a_3(-\lambda) - b_3(\lambda)b_3(-\lambda) = 1$.

  $a_3(\lambda) = 1 + O\left(\frac{1}{\lambda}\right)$, $b_3(\lambda) = O\left(\frac{1}{\lambda}\right)$ as $\lambda \to \infty,\ Im(\lambda) \geq 0$.

These properties follow from the analogous properties of the matrix-valued functions Φ_j, $j = 1, 2, 3$, from the condition of unit determinant, and from the large λ asymptotics of these functions.

2.3 The global relation

Evaluating equations (2.15) and (2.16) *at* $x = 0$, $y = L$, we find

$$I = \Psi_3(0, L, \lambda)\mathrm{e}^{-\frac{\omega(\lambda)}{2}L\widehat{\sigma_3}}S_2(\lambda)$$

and

$$S_1(\lambda) = \Psi_3(0, L, \lambda)\mathrm{e}^{-\frac{\omega(\lambda)}{2}L\widehat{\sigma_3}}S_3(\lambda).$$

Eliminating $\Psi_3(0, L, \lambda)$ we obtain

$$\mathrm{e}^{\frac{\omega(\lambda)}{2}L\widehat{\sigma_3}}S_1(\lambda) = S_2(\lambda)^{-1}S_3(\lambda). \tag{2.20}$$

The first column vector of this equation yields the following *global relations*:

$$a_1(\lambda) = a_2(-\lambda)a_3(\lambda) - b_2(-\lambda)b_3(\lambda), \qquad \lambda \in \mathbb{C}^+, \tag{2.21}$$

$$b_1(\lambda)\mathrm{e}^{-\omega(\lambda)L} = a_2(\lambda)b_3(\lambda) - a_3(\lambda)b_2(\lambda), \qquad \lambda \in \mathbb{C}^+. \tag{2.22}$$

2.4 The Riemann-Hilbert problem

Equations (2.14)-(2.16),relating the various analytic eigenfunctions, can be rewritten in a form that determines the jump conditions of a 2×2 RH problem, with unitary jump matrices on the real and imaginary axes. This involves tedious but straightforward algebraic manipulations. The final form is

$$\Psi_-(x, y, \lambda) = \Psi_+(x, y, \lambda)J(x, y, \lambda), \lambda \in \mathbb{R} \cup i\mathbb{R}, \tag{2.23}$$

where the matrices $\Psi_\pm$ and J are defined as follows:

$$\begin{aligned}
\Psi_+ &= \left(\Psi_1^{(12)}, \frac{1}{a_3(\lambda)}\Psi_3^{(1)}\right), \quad arg(\lambda) \in [0, \frac{\pi}{2}],\\
\Psi_- &= \left(\ \Psi_1^{(12)}, \tfrac{1}{a_1(\lambda)}\Psi_2^{(2)}\ \right), \quad arg(\lambda) \in [\frac{\pi}{2}, \pi],\\
\Psi_+ &= \left(\frac{1}{a_3(-\lambda)}\Psi_3^{(3)}, \Psi_1^{(34)}\right), \quad arg(\lambda) \in [\pi, \frac{3\pi}{2}],\\
\Psi_- &= \left(\ \tfrac{1}{a_1(-\lambda)}\Psi_2^{(4)}, \Psi_1^{(34)}\ \right), \quad arg(\lambda) \in [\frac{3\pi}{2}, 2\pi],
\end{aligned}$$

$$J(x,y,\lambda) = J^\alpha(x,y,\lambda), \ \text{ if } \ arg(\lambda) = \alpha, \quad \alpha = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \tag{2.24}$$

where, using the global relation, we find

$$J^0 = \begin{pmatrix} \frac{a_2(\lambda)}{a_1(-\lambda)a_3(\lambda)} & \frac{b_3(-\lambda)}{a_3(\lambda)}\mathrm{e}^{-\theta(x,y,\lambda)} \\ -\frac{\mathrm{e}^{-\omega(\lambda)L}b_1(\lambda)}{a_1(-\lambda)}\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix},$$

$$J^{\pi/2} = \begin{pmatrix} 1 & \frac{b_2(-\lambda)}{a_1(\lambda)a_3(\lambda)}\mathrm{e}^{-\theta(x,y,\lambda)} \\ 0 & 1 \end{pmatrix}, \qquad J^{3\pi/2} = \begin{pmatrix} 1 & 0 \\ \frac{b_2(\lambda)}{a_1(-\lambda)a_3(-\lambda)}\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix}$$

and

$$J^\pi = J^{3\pi/2}(J^0)^{-1}J^{\pi/2}, \tag{2.25}$$

where

$$\theta(x,y,\lambda) = \Omega(\lambda)x + \omega(\lambda)y. \tag{2.26}$$

All the matrices J^α have unit determinant: for $J^{\pi/2}$ and $J^{3\pi/2}$ this is immediate, whereas for J^0 we find

$$\det(J^0) = \frac{a_2(\lambda) + \mathrm{e}^{-\omega(\lambda)L}b_1(\lambda)b_3(-\lambda)}{a_1(-\lambda)a_3(\lambda)} = \frac{a_1(-\lambda)a_3(\lambda)}{a_1(-\lambda)a_3(\lambda)} = 1,$$

where we have used the equation

$$a_2(\lambda) = a_1(-\lambda)a_3(\lambda) - b_3(-\lambda)b_1(\lambda)\mathrm{e}^{-\omega(\lambda)L}, \qquad \lambda \in \mathbb{R}. \tag{2.27}$$

Equation (2.27) is a consequence of equations (2.21) and (2.22) (see also equation (4.19) below).

The function $\Psi(x,y,\lambda)$ solution of this RH problem is a sectionally meromorphic function of λ. The possible poles of this function are generated by the zeros of the function $a_1(\lambda)$ in the region $\{arg(\lambda) \in [\frac{\pi}{2}, \pi]\}$, by the zeros of $a_3(\lambda)$ in the region $\{arg(\lambda) \in [0, \frac{\pi}{2}]\}$, and by the corresponding zeros of $a_1(-\lambda)$, $a_3(-\lambda)$.

We assume

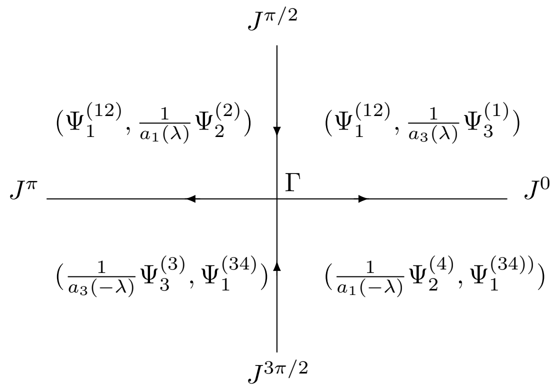

Figure 3: Bounded eigenfunctions and the Riemann-Hilbert problem

- The possible zeros of a_1 in the region $\{arg(\lambda) \in [\frac{\pi}{2}, \pi]\}$ are simple; these zeros are denoted λ_j, $j = 1, .., N_1$

(2.28)

- The possible zeros of a_3 in the region $\{arg(\lambda) \in [0, \frac{\pi}{2}]\}$ are simple; these zeros are denoted ζ_j, $j = 1, .., N_3$

The residues of the function Ψ at the corresponding poles can be computed using equations (2.14)-(2.16). Indeed, equation (2.16) yields

$$\Psi_1^{(12)} = a_3\Psi_3^{(3)} + b_3 \mathrm{e}^{\theta(x,y,\lambda)}\Psi_3^{(1)},$$

hence

$$Res_{\zeta_j}\frac{\Psi_3^{(1)}}{a_3} = \frac{\Psi_3^{(1)}(\zeta_j)}{\dot{a}_3(\zeta_j)} = \frac{\Psi_1^{(12)}(\zeta_j)}{\dot{a}_3(\zeta_j)b_3(\zeta_j)}\mathrm{e}^{-\theta(x,y,\zeta_j)}, \tag{2.29}$$

where $\dot{a}_3(\lambda)$ denotes the derivative of a_3 with respect to λ.
Similarly, using (2.14),

$$Res_{\lambda_j}\frac{\Psi_2^{(2)}}{a_1} = \frac{\Psi_2^{(2)}(\lambda_j)}{\dot{a}_1(\lambda_j)} = \frac{\Psi_1^{(12)}(\lambda_j)}{\dot{a}_1(\lambda_j)b_1(\lambda_j)\mathrm{e}^{-\omega(\lambda_j)L}}\mathrm{e}^{-\theta(x,y,\lambda_j)}. \tag{2.30}$$

Using the notation $[\Psi]_1$ for the first column, $[\Psi]_2$ for the second column for the solution Ψ of the RH problem (2.23), at equations (2.29) and (2.30) imply the following residue conditions:

$$\begin{aligned} Res_{\zeta_j}[\Psi(x,y,\lambda)]_2 &= \frac{\mathrm{e}^{-\theta(x,y,\zeta_j)}}{\dot{a}_3(\zeta_j)b_3(\zeta_j)}[\Psi(x,y,\zeta_j)]_1, \qquad 0 < \arg\lambda < \frac{\pi}{2}, \\ Res_{\lambda_j}[\Psi(x,y,\lambda)]_2 &= \frac{\mathrm{e}^{-\theta(x,y,\zeta_j)}}{\dot{a}_1(\lambda_j)b_1(\lambda_j)\mathrm{e}^{-\omega(\lambda_j)L}}[\Psi(x,y,\lambda_j)]_1, \qquad \frac{\pi}{2} < \arg\lambda < \pi, \end{aligned} \tag{2.31}$$

and similar ones in $\mathbb{C}^-$ by letting $\lambda \to -\lambda$.

The inverse problem

Rewriting the jump condition, we obtain

$$\Psi_+ - \Psi_- = \Psi_+ - \Psi_+ J- = \Psi_+(I - J) \Rightarrow \Psi_+ - \Psi_- = \Psi_+ \tilde{J}, \tag{2.32}$$

where $\tilde{J} = I - J$. The asymptotic conditions (2.10)-(2.11)) imply

$$\Psi(x, y, \lambda) = I + \frac{\Psi^*(x, y)}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \quad |\lambda| \to \infty. \tag{2.33}$$

Equations (2.32) and (2.33) define a Riemann-Hilbert problem.
The solution of this RH problem is given by

$$\Psi(x, y, \lambda) = I + \frac{1}{2\pi i} \int_\Gamma \frac{\Psi_+(x, y, \lambda')\tilde{J}(x, y, \lambda')}{\lambda' - \lambda} d\lambda', \quad \lambda \in \Gamma, \tag{2.34}$$

where

$$\Gamma = \mathbb{R} \cup i\mathbb{R}.$$

Equations (2.33) and (2.34) imply

$$\Psi^* = -\frac{1}{2\pi i} \int_\Gamma \Psi_+(x, y, \lambda)\tilde{J}(x, y, \lambda) d\lambda. \tag{2.35}$$

Using (2.33) in the first ODE in the Lax pair (2.2), we find

$$-\frac{i}{4}[\sigma_3, \Psi^*] = i\frac{q_x - iq_y}{4}\sigma_1 \Rightarrow q_x - iq_y = 2(\Psi^*)_{21} = 2 \lim_{\lambda \to \infty} (\lambda \Psi_{21}), \tag{2.36}$$

(σ_1, σ_3 denote the usual Pauli matrices).
In order to obtain an expression in terms of q rather than its derivatives, we consider the coefficient of the term λ^{-1}. The (1,1) element of this coefficient yields

$$\cos q(x, y) = 1 + 4i(\Psi^*_x)_{11} - 2(\Psi^*)^2_{21}. \tag{2.37}$$

3 Spectral theory assuming the validity of the global relation

3.1 The spectral functions

The above analysis motivates the following definitions for the spectral functions.

The spectral functions at the $y = 0$ and $y = L$ boundaries

Definition 3.1 *Given the functions $q(x, L)$, $q_y(x, L)$ satisfying conditions (2.1), define the map*

$$\mathbb{S}_1 : \{q(x, L), q_y(x, L)\} \to \{a_1(\lambda), b_1(\lambda)\}$$

by

$$\begin{pmatrix} a_1(\lambda) \\ b_1(\lambda) \end{pmatrix} = [\Phi_1(0,L)]_1, \quad \lambda \in \mathbb{C}^+,$$

where $[\Phi_1(x,L)]_1$ *denotes the first column vector of the unique solution* $\Phi_1(x,L)$ *of the Volterra linear integral equation*

$$\Phi(x,\lambda) = I - \int_x^\infty \mathrm{e}^{\Omega(\lambda)(\xi-x)\frac{\widehat{\sigma_3}}{2}} Q(\xi,L,\lambda)\Phi(\xi,\lambda)d\xi, \tag{3.1}$$

and $Q(x,L,\lambda)$ *is given in terms of* $q(x,L)$ *and* $q_y(x,L)$ *by equation (2.5).*

Proposition 3.1 *The spectral functions* $a_1(\lambda)$*,* $b_1(\lambda)$ *have the following properties.*

(i) $a_1(\lambda)$*,* $b_1(\lambda)$ *are continuous and bounded for* $Im(\lambda) \geq 0$*, and analytic for* $Im(\lambda > 0$*.*

(ii) $a_1(\lambda) = 1 + O\left(\frac{1}{\lambda}\right)$*,* $b_1(\lambda) = O\left(\frac{1}{\lambda}\right)$ *as* $\lambda \to \infty,\ Im(\lambda) \geq 0$*.*

(iii) $a_1(\lambda) = \cos\frac{q(x,L)}{2} + O(\lambda)$*,* $b_1(\lambda) = i\sin\frac{q(x,L)}{2} + O(\lambda)$ *as* $\lambda \to 0,\ Im(\lambda) \geq 0$*.*

(iv) $a_1(\lambda)a_1(-\lambda) - b_1(\lambda)b_1(-\lambda) = 1$*,* $Im(\lambda) \geq 0$*.*

(v) The map $\mathbb{Q}_1 : \{a_1, b_1\} \to \{q(x,L)\, q_y(x,L)\}$*, inverse to* $\mathbb{S}_1$*, is given by*

$$\cos q(x,L) = 1 + 4i \lim_{\lambda\to\infty}(\lambda M_x)_{11} + 2\lim_{\lambda\to\infty}(\lambda M)_{21},$$

$$q_y(x,L) = -iq_x(x,L) + 2\lim_{\lambda\to\infty}(\lambda M)_{21},$$

where M *is the solution of the following Riemann-Hilbert problem:*

* *The function*

$$M(x,\lambda) = \begin{cases} M_+(x,\lambda) & \lambda \in \mathbb{C}^+ \\ M_-(x,\lambda) & \lambda \in \mathbb{C}^- \end{cases}$$

is a sectionally meromorphic function of $\lambda \in \mathbb{C}$*.*

* $M = I + O\left(\frac{1}{\lambda}\right)$ *as* $\lambda \to \infty$*, and*

$$M_+(x,\lambda) = M_-(x,\lambda)J_1(x,\lambda), \qquad \lambda \in \mathbb{R},$$

where

$$J_1(x,\lambda) = \begin{pmatrix} 1 & -\frac{b_1(-\lambda)}{a_1(\lambda)}\mathrm{e}^{-\Omega(\lambda)x} \\ \frac{b_1(\lambda)}{a_1(-\lambda)}\mathrm{e}^{\Omega(\lambda)x} & \frac{1}{a_1(\lambda)a_1(-\lambda)} \end{pmatrix}, \qquad \lambda \in \mathbb{R}. \tag{3.2}$$

* *The function* $a_1(\lambda)$ *may have* N_1 *simple poles* λ_j *in* $\mathbb{C}^+$*.*

* *Let* $[M]_i$ *denote the i-th column vector of* M*,* $1 = 1,2$*. The possible poles of* M_+ *occur at* λ_j*, and the possible poles of* M_- *occur at* $-\lambda_j$ *in* $\mathbb{C}^-$*, and the associated residues are given by*

$$\begin{aligned} Res_{\lambda_j}[M(x,\lambda)]_2 &= \frac{\mathrm{e}^{-\Omega(\lambda_j)x}}{\dot{a}_1(\lambda_j)b_1(\lambda_j)}[M(x,\lambda_j)]_1, \\ Res_{-\lambda_j}[M(x,\lambda)]_1 &= \frac{\mathrm{e}^{\Omega(\lambda_j)x}}{\dot{a}_1(-\lambda_j)b_1(-\lambda_j)}[M(x,-\lambda_j)]_2. \end{aligned} \tag{3.3}$$

The spectral functions $\{a_3, b_3\}$ satisfy an analogous result:

Definition 3.2 *Given the functions $q(x,0)$, $q_y(x,0)$, satisfying conditions (2.1), define the map*

$$\mathbb{S}_3 : \{q(x,0), q_y(x,0)\} \to \{a_3(\lambda), b_3(\lambda)\}$$

by

$$\begin{pmatrix} a_3(\lambda) \\ b_3(\lambda) \end{pmatrix} = [\Phi_3(0,0)]_1, \quad \lambda \in \mathbb{C}^+,$$

where $[\Phi_3(x,0)]_1$ denotes the first column vector of the unique solution $\Phi_3(x,0)$ of the Volterra linear integral equation

$$\Phi(x,\lambda) = I - \int_x^\infty \mathrm{e}^{\Omega(\lambda)(\xi-x)\frac{\widehat{\sigma_3}}{2}} Q(\xi,0,\lambda)\Phi(\xi,\lambda)d\xi, \tag{3.4}$$

and $Q(x,0,\lambda)$ is given in terms of $q(x,0)$ and $q_y(x,0)$ by equation (2.5).

Proposition 3.2 *The spectral functions $a_3(\lambda)$, $b_3(\lambda)$ have the properties (i)-(v) of proposition (3.1), provided a_1 is replaced by a_3, b_1 is replaced by b_3, $\mathbb{S}_1$ is replaced by $\mathbb{S}_3$ and L in replaced by 0 in all expressions.*

The spectral functions at the $x = 0$ boundary

Definition 3.3 *Given the functions $q(0,y)$, $q_x(0,y)$, satisfying conditions (2.1), define the map*

$$\mathbb{S}_2 : \{q(0,y), q_x(0,y)\} \to \{a_2(\lambda), b_2(\lambda)\}$$

by

$$\begin{pmatrix} a_2(\lambda) \\ b_2(\lambda) \end{pmatrix} = [\Phi_2(0,0)]_1, \quad \lambda \in \mathbb{C}^+,$$

where $[\Phi_2(0,y)]_1$ denotes the first column vector of the unique solution $\Phi_2(0,y)$ of the Volterra linear integral equation

$$\Phi(y,\lambda) = I - i\int_y^L \mathrm{e}^{\omega(\lambda)(\eta-y)\frac{\widehat{\sigma_3}}{2}} Q(0,y,-\lambda)\Phi(\eta,\lambda)d\eta, \tag{3.5}$$

and $Q(0,y,\lambda)$ is given in terms of $q(0,y)$ and $q_x(0,y)$ by equation (2.5).

Proposition 3.3 *The spectral functions $a_2(\lambda)$, $b_2(\lambda)$ have the following properties.*

(i) $a_2(\lambda)$, $b_2(\lambda)$ *are entire functions of λ, except for essential singularities at $\lambda = 0$ and $\lambda = \infty$, bounded for $Re(\lambda) > 0$.*

(ii) $a_2(\lambda) = 1 + O\left(\frac{1}{\lambda}\right)$, $b_2(\lambda) = O\left(\frac{1}{\lambda}\right)$ *as* $\lambda \to \infty$, $Re(\lambda) \geq 0$.

(iii) $a_2(\lambda) = \cos\frac{q(0,y)}{2} + O(\lambda)$, $b_2(\lambda) = i\sin\frac{q(0,y)}{2} + O(\lambda)$ *as* $\lambda \to 0$, $Im(\lambda) \geq 0$.

(iv) $a_2(\lambda)a_2(-\lambda) - b_2(\lambda)b_2(-\lambda) = 1$, $Im(\lambda) \geq 0$.

(v) The map $\mathbb{Q}_2 : \{a_2, b_2\} \to \{q(0,y)\, q_y(0,y)\}$*, inverse to* $\mathbb{S}_2$*, is given by*

$$\cos q(0,y) = 1 + 4i \lim_{\lambda\to\infty} (\lambda M_y)_{11} + 2 \lim_{\lambda\to\infty} (\lambda M)_{21},$$

$$q_x(0,y) = iq_y(0,y) + 2 \lim_{\lambda\to\infty} (\lambda M)_{21},$$

where M *is the solution of the following Riemann-Hilbert problem:*

* *The function*

$$M(y,\lambda) = \begin{cases} M_+(y,\lambda) & Re\lambda \geq 0 \\ M_-(y,\lambda) & Re\lambda \leq 0 \end{cases}$$

is a sectionally meromorphic function of $\lambda \in \mathbb{C}$.

* $M = I + O\left(\frac{1}{\lambda}\right)$ *as* $\lambda \to \infty$*, and*

$$M_+(y,\lambda) = M_-(y,\lambda) J_2(y,\lambda), \qquad \lambda \in i\mathbb{R},$$

where

$$J_2(y,\lambda) = \begin{pmatrix} 1 & -\frac{b_2(-\lambda)}{a_2(\lambda)} \mathrm{e}^{-\omega(\lambda)x} \\ \frac{b_2(\lambda)}{a_2(-\lambda)} \mathrm{e}^{\omega(\lambda)x} & \frac{1}{a_2(\lambda)a_2(-\lambda)} \end{pmatrix}, \qquad \lambda \in i\mathbb{R}.$$

* M *satisfies appropriate residue conditions at the zeros of* $a_2(\lambda)$.

Proof of propositions (3.1)-(3.3)

The proof of properties (i)-(iv) follows from the discussion in Section 2.2. In particular, property (iii) follows from the asymptotic behaviour at $\lambda \to 0$, which can be derived by analysing equations (2.2)-(2.3) (see [40]), and is given by

$$\Psi = \Psi_0 + O(\lambda), \ |\lambda| \to 0, \qquad \Psi_0(x,y) = \begin{pmatrix} \cos \frac{q(x,y)}{2} & i \sin \frac{q(x,y)}{2} \\ i \sin \frac{q(x,y)}{2} & \cos \frac{q(x,y)}{2} \end{pmatrix}. \tag{3.6}$$

To prove (v), we note that the function $\phi_1(x,\lambda)$ is the unique solution of the ODE

$$\begin{aligned} &\phi_x + \frac{\Omega(\lambda)}{2} \widehat{\sigma_3} \phi = Q(x,L,\lambda)\phi(x,\lambda), \\ &\lim_{\lambda\to\infty} \phi(x,\lambda) = I. \end{aligned}$$

Furthermore, $\phi_3(x,\lambda)$ is the solution of the same ODE problem, with $Q(x,L,\lambda)$ replaced by $Q(x,0,\lambda)$.
Similarly, $\phi_2(x,\lambda)$ is the unique solution of the ODE

$$\begin{aligned} &\phi_y + \frac{\omega(\lambda)}{2} \widehat{\sigma_3} \phi = iQ(0,y,-\lambda)\phi(x,\lambda), \\ &\phi(0,L) = I. \end{aligned}$$

The spectral analysis of the above ODEs yields the desired result.

Regarding the rigorous derivation of the above results, we note the following: If $\{q(x,L), q_y(x,L)\}$, $\{q(x,0), q_y(x,0)\}$ and $\{q(y,0), q_x(y,0)\}$ are in $\mathbf{L}^1$, then the Volterra integral equations (3.1), (3.4) and (3.5) respectively, have a unique solution, and hence the spectral functions $\{a_j, b_j\}$, $j = 1, .., 3$, are well defined. Moreover, under the assumption (2.1) the spectral functions belong to $\mathbf{H}^1(\mathbb{R})$, hence the Riemann-Hilbert problems that determine the inverse maps can be characterized through the solutions of a Fredholm integral equation, see [10, 49].
QED

3.2 The Riemann-Hilbert problem

Theorem 3.1 *Suppose that a subset of the boundary values $\{q(x,L), q_y(x,L)\}, \{q(x,0), q_y(x,0)\}$, $0 < x < \infty$, and $\{q(y,0), q_x(y,0)\}$, $0 < y < L$, satisfying (2.1), are prescribed as boundary conditions. Suppose that these prescribed boundary conditions are such that the global relations (2.21) and (2.22) can be used to characterize the remaining boundary values. Define the spectral functions $\{a_j, b_j\}$, $j = 1, .., 3$, by definitions (3.1)-(3.3). Assume that the possible zeros $\{\lambda_j\}_{j=1}^{N_1}$ of $a_1(\lambda)$ and $\{\zeta_j\}_{j=1}^{N_2}$ of $a_3(\lambda)$ are as in assumption 2.28. Define $M(x,y,\lambda)$ as the solution of the following 2×2 matrix Riemann-Hilbert problem:*

- *The function $M(x,y,\lambda)$ is a sectionally meromorphic function of λ away from $\mathbb{R} \cup i\mathbb{R}$.*

- *The possible poles of the second column of M occur at $\lambda = \zeta_j$, $j = 1, ..., N_2$, in the first quadrant and at $\lambda = \lambda_j$, $j = 1, ..., N_1$, in the second quadrant of the complex λ plane.*

  The possible poles of the first column of M occur at $\lambda = -\lambda_j$ $(j = 1, ..., N_1)$ and $\lambda = -\zeta_j$ $(j = 1, ..., N_2)$.

  The associated residue conditions satisfy the relations (2.31).

- *$M = I + O\left(\frac{1}{\lambda}\right)$ as $\lambda \to \infty$, and*

$$M_+(x,y,\lambda) = M_-(x,y,\lambda) J(x,y,\lambda), \qquad \lambda \in \mathbb{R} \cup i\mathbb{R},$$

  where $M = M_+$ for λ in the first or third quadrant, and $M = M_-$ for λ in the second or fourth quadrant of the complex λ plane, and J is defined in terms of $\{a_j, b_j\}$ by equations (2.25).

Then M exists and is unique, provided that the $\mathbf{H}^1$ norm of of the spectral functions is sufficiently small.
Define $q(x,y)$ is terms of $M(x,y,\lambda)$ by

$$q_x - iq_y = 2 \lim_{\lambda \to \infty} (\lambda M)_{21}, \tag{3.7}$$

$$\cos q(x,y) = 1 + 4i\left(\lim_{\lambda \to \infty} (\lambda M_x)_{11}\right) - 2\left(\lim_{\lambda \to \infty} (\lambda M)_{21}\right)^2. \tag{3.8}$$

Then $q(x,y)$ solves (1.1). Furthermore, $q(x,y)$ evaluated at the boundary, yields the functions used for the computation of the spectral functions.

Proof: Under the assumptions (2.1), the spectral functions are in $\mathbf{H}^1$.

In the case when $a_1(\lambda)$ and $a_3(\lambda)$ have no zeros, the Riemann-Hilbert problem is regular and it is equivalent to a Fredholm integral equation. However, we have *not* been able to establish a vanishing lemma, hence we require a small norm assumption for solvability.
If $a_1(\lambda)$ and $a_3(\lambda)$ have zeros, the singular RH problem can be mapped to a regular one coupled with a system of algebraic equations [22]. Moreover, it follows from standard arguments, using the dressing method [47, 48], that if M solves the above RH problem and $q(x, y)$ is defined by (3.7)-(3.8), then $q(x, y)$ solves equation (1.1). The proof that q evaluated at the boundary yields the functions used for the computation of the spectral functions follows arguments similar to the ones used in [24].
QED

4 Linearizable boundary conditions

We now concentrate on the particular boundary conditions (1.2).
In this case, equations (2.17)-(2.19) simplify as follows:

$$\begin{aligned}
\begin{pmatrix} A_1(x,\lambda) \\ B_1(x,\lambda) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{4}\int_x^{\infty} \begin{pmatrix} q_y(\xi,L)A_1(\xi,\lambda) \\ \mathrm{e}^{\Omega(\lambda)(x-\xi)}q_y(\xi,L)B_1(\xi,\lambda) \end{pmatrix} d\xi, \\
& \quad 0<x<\infty, \qquad Im(\lambda)\geq 0, \qquad (4.1) \\
\begin{pmatrix} A_2(y,\lambda) \\ B_2(y,\lambda) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{4}\int_y^{L} \begin{pmatrix} -\frac{(1-\cos d)}{\lambda}A_2(\eta,\lambda) + [q_x(0,y) - i\frac{\sin d}{\lambda}]B_2(\eta,\lambda) \\ \mathrm{e}^{\omega(\lambda)(y-\eta)}\Big[q_x(0,y) + i\frac{\sin d}{\lambda}]A_2(\eta,\lambda) + \frac{(1-\cos d)}{\lambda}B_2(\eta,\lambda)\Big] \end{pmatrix} d\eta, \\
& \quad 0<y<L, \qquad \lambda\in\mathbb{C}, \qquad (4.2) \\
\begin{pmatrix} A_3(x,\lambda) \\ B_3(x,\lambda) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{4}\int_x^{\infty} \begin{pmatrix} q_y(\xi,0)A_3(\xi,\lambda) \\ \mathrm{e}^{\Omega(\lambda)(x-\xi)}q_y(\xi,0)B_3(\xi,\lambda) \end{pmatrix} d\xi, \\
& \quad 0<x<\infty, \qquad Im(\lambda)\geq 0. \qquad (4.3)
\end{aligned}$$

In equations (4.1) and (4.3), the only dependence on λ is through $\Omega(\lambda)$. Thus, since $\Omega(-\frac{1}{\lambda}) = \Omega(\lambda)$, it follows that the vector functions (A_1, B_1) and (A_3, B_3) satisfy the same symmetry properties. Hence,

$$a_j(-\frac{1}{\lambda}) = a_j(\lambda), \quad b_j(-\frac{1}{\lambda}) = b_j(\lambda), \qquad j=1,3, \quad Im(\lambda)\geq 0. \qquad (4.4)$$

It turns out that the vector function (A_2, B_2) also satisfies a certain symmetry condition, as stated in the following proposition.

Proposition 4.1 *Let $q_x(0, y)$ be a sufficiently smooth function. Then the vector solution of the linear Volterra integral equation (4.2) satisfies the following symmetry conditions (where we do not indicate the explicit dependence of A_2, B_2 on y):*

$$\begin{aligned}
A_2(\frac{1}{\lambda}) &= \frac{1}{1-F(\lambda)^2}[A_2(\lambda) - F(\lambda)B_2(\lambda) + F(\lambda)\mathrm{e}^{\omega(\lambda)(y-L)}B_2(-\lambda) - F(\lambda)^2\mathrm{e}^{\omega(\lambda)(y-L)}A_2(-\lambda)], \\
B_2(\frac{1}{\lambda}) &= \frac{1}{1-F(\lambda)^2}[B_2(\lambda) - F(\lambda)A_2(\lambda) + F(\lambda)\mathrm{e}^{\omega(\lambda)(y-L)}A_2(-\lambda) - F(\lambda)^2\mathrm{e}^{\omega(\lambda)(y-L)}B_2(-\lambda)], \\
& \qquad 0<y<L, \quad \lambda\in\mathbb{C}, \qquad (4.5)
\end{aligned}$$

where the function $F(\lambda)$ is given by

$$F(\lambda) = i\frac{1-\lambda^2}{1+\lambda^2}\tan\frac{d}{2}. \tag{4.6}$$

Proof: Let the 2×2 matrix valued function $\Phi_2(y, \lambda)$ be defined by

$$\Phi_2(y,\lambda) = \begin{pmatrix} A_2(y,\lambda) & B_2(y,-\lambda) \\ B_2(y,\lambda) & A_2(y,-\lambda) \end{pmatrix}, \qquad 0 < y < L, \quad \lambda \in \mathbb{C}. \tag{4.7}$$

Then Φ_2 satisfies the ODE

$$\begin{aligned} &(\Phi_2)_y + \frac{\omega(\lambda)}{2}[\sigma_3, \Phi_2] = iQ(0, y, -\lambda)\Phi_2, \qquad 0 < y < L, \\ &\Phi_2(L, \lambda) = I, \end{aligned} \tag{4.8}$$

where $Q(x, y, \lambda)$ is defined in (2.5), and $q(0, y) = d$.
Letting

$$\Phi_2(y,\lambda) = \phi_2(y,\lambda)\mathrm{e}^{\frac{\omega(\lambda)}{2}\sigma_3(y-L)} \tag{4.9}$$

it follows that ϕ_2 satisfies the ODE

$$\begin{aligned} &(\phi_2)_y = V\phi_2, \\ &\phi_2(L,\lambda) = I, \qquad 0 < y < L, \quad Re(\lambda) \geq 0, \end{aligned} \tag{4.10}$$

where

$$V(y,\lambda) = \frac{1}{4}\begin{pmatrix} -(\lambda + \frac{\cos d}{\lambda}) & -q_x(0,y) + \frac{i\sin d}{\lambda} \\ -q_x(0,y) - \frac{i\sin d}{\lambda} & \lambda + \frac{\cos d}{\lambda} \end{pmatrix}. \tag{4.11}$$

We seek a non singular matrix $R(\lambda)$, independent of y, such that

$$V(y, \frac{1}{\lambda}) = R(\lambda)V(y,\lambda)R(\lambda)^{-1}. \tag{4.12}$$

It can be verified that such a matrix is given by

$$R(\lambda) = \begin{pmatrix} 1 & -F(\lambda) \\ -F(\lambda) & 1 \end{pmatrix}, \tag{4.13}$$

where F is defined by (4.6).
Replacing in equation (4.10) λ by $\frac{1}{\lambda}$, and using (4.12), we find the following equation:

$$\left(R(\lambda)^{-1}\phi_2(y, \frac{1}{\lambda})\right)_y = V(y,\lambda)\left(R(\lambda)^{-1}\phi_2(y, \frac{1}{\lambda})\right),$$

hence

$$R(\lambda)^{-1}\phi_2(y, \frac{1}{\lambda}) = \phi_2(y,\lambda)C(\lambda),$$

where C is a y-independent matrix. Using the second of equations (4.8), it follows that $C = R^{-1}$, and therefore

$$\phi_2(y, \frac{1}{\lambda}) = R(\lambda)\phi_2(y,\lambda)R(\lambda)^{-1}.$$

This equation and equation (4.9) imply

$$\Phi_2(y, \frac{1}{\lambda}) = R(\lambda)\Phi_2(y,\lambda)\left(\mathrm{e}^{\omega(\lambda)\frac{\widehat{\sigma_3}}{2}(y-L)}R(\lambda)^{-1}\right). \tag{4.14}$$

The first column vector of this equation implies (4.5).
QED

Remark 4.1 *Recalling that $a_2(\lambda) = A_2(0,\lambda)$, and $b_2(\lambda) = B_2(0,\lambda)$, equations (4.5) immediately imply the following important relations:*

$$\begin{aligned}
&a_2(\frac{1}{\lambda}) = \frac{1}{1-F(\lambda)^2}[a_2(\lambda) - F(\lambda)b_2(\lambda) + F(\lambda)\mathrm{e}^{-\omega(\lambda)L}b_2(-\lambda) - F(\lambda)^2\mathrm{e}^{-\omega(\lambda)L}a_2(-\lambda)],\\
&b_2(\frac{1}{\lambda} = \frac{1}{1-F(\lambda)^2}[b_2(\lambda) - F(\lambda)a_2(\lambda) + F(\lambda)\mathrm{e}^{-\omega(\lambda)L}a_2(-\lambda) - F(\lambda)^2\mathrm{e}^{-\omega(\lambda)L}b_2(-\lambda)],\\
&Im(\lambda) \geq 0.
\end{aligned} \tag{4.15}$$

In summary, the basic equations characterizing the spectral functions are:

(a) the symmetry relations (4.4) and (4.15);

(b) the global relations (2.21) and (2.22);

(c) the conditions of unit determinant.

It turns out that, using these equations, it is possible to provide an explicit characterization of all the spectral functions in terms of the given constant d.

Proposition 4.2 *Assume that the functions $\{a_j(\lambda), b_j(\lambda)\}$, $j = 1,2,3$ satisfy the symmetry relations (4.4) and (4.15), the global relations (2.21) and (2.22) and the "unit determinant" conditions*

$$a_j(\lambda)a_j(-\lambda) - b_j(\lambda)b_j(-\lambda) = 1, \qquad j = 1,2,3. \tag{4.16}$$

Then the following relations are valid:

$$a_1(\lambda)b_1(-\lambda) - a_1(-\lambda)b_1(\lambda) = G(\lambda), \qquad \lambda \in \mathbb{R}, \tag{4.17}$$

$$a_3(\lambda)b_3(-\lambda) - a_3(-\lambda)b_3(\lambda) = -G(\lambda), \qquad \lambda \in \mathbb{R}, \tag{4.18}$$

$$a_2(\lambda) = a_1(-\lambda)a_3(\lambda) - \mathrm{e}^{-\omega(\lambda)L}b_1(\lambda)b_3(-\lambda), \qquad \lambda \in \mathbb{C}, \tag{4.19}$$

$$b_2(\lambda) = a_1(-\lambda)b_3(\lambda) - \mathrm{e}^{-\omega(\lambda)L}b_1(\lambda)a_3(-\lambda), \qquad \lambda \in \mathbb{C}. \tag{4.20}$$

where

$$G(\lambda) = \frac{i(1-\lambda^2)}{1+\lambda^2}\frac{\mathrm{e}^{\omega(\lambda)L} + \mathrm{e}^{-\omega(\lambda)L} - 2}{\mathrm{e}^{\omega(\lambda)L} - \mathrm{e}^{-\omega(\lambda)L}}\tan\frac{d}{2}. \tag{4.21}$$

Remark 4.2 *The two relations (4.19) and (4.20) are a direct consequence of the global relation and of the conditions of unit determinant. On the other hand, equations (4.17) and (4.18) depend on the particular symmetry properties.*

Proof: for simplicity, we will use the notations

$$f = f(\lambda), \qquad \hat{f} = f(-\lambda). \tag{4.22}$$

Replacing λ with $-\lambda$ in (2.21) and solving the resulting equation and equation (2.22) for $a_2(\lambda)$ and $b_2(\lambda)$ we find equations (4.19) and (4.20).
It can be verified directly that if $\{a_2, b_2\}$ are defined by equations (4.19) and (4.20) and $\{a_j, b_j\}$, $j = 1, 3$ satisfy the determinant condition (4.16), then $\{a_2, b_2\}$ also satisfies the determinant condition.
Replacing in the global relations(2.21) and (2.22) λ by $-\frac{1}{\lambda}$, and using in the resulting equations the symmetry relations (4.4), we find

$$a_1(\lambda) = a_2(\frac{1}{\lambda})a_3(\lambda) - b_2(\frac{1}{\lambda})b_3(\lambda), \tag{4.23}$$

$$b_1(\lambda)\mathrm{e}^{\omega(\lambda)L} = a_2(-\frac{1}{\lambda})b_3(\lambda) - b_2(-\frac{1}{\lambda})a_3(\lambda). \tag{4.24}$$

Replacing in these equations $a_2(\pm\frac{1}{\lambda})$ and $b_2(\pm\frac{1}{\lambda})$ by the right hand side of the symmetry relation (4.15), as well as by the right hand side of the equation obtained from equations (4.15) under the transformation $\lambda \to -\lambda$, after extensive simplifications we find the following equations, valid for $\lambda \in \mathbb{R}$:

$$\frac{(1-F^2)a_1}{b_1 - Fa_1} = (a_3^2 - b_3^2)\frac{\hat{a}_1 - F\hat{b}_1}{b_1 - Fa_1} + \Theta, \tag{4.25}$$

$$\frac{(1-F^2)b_1}{a_1 - Fb_1} = (a_3^2 - b_3^2)\frac{\hat{b}_1 - F\hat{a}_1}{a_1 - Fb_1} + \Theta, \tag{4.26}$$

where

$$\Theta = \mathrm{e}^{-\omega(\lambda)L}(b_3\hat{a}_3 - \hat{b}_3 a_3) + \mathrm{e}^{-\omega(\lambda)L}F, \qquad \lambda \in \mathbb{R}.$$

Hence

$$\frac{(1-F^2)a_1 + \Delta(\hat{a}_1 - F\hat{b}_1)}{b_1 - Fa_1} = \frac{(1-F^2)b_1 + \Delta(\hat{b}_1 - F\hat{a}_1)}{a_1 - Fb_1} = \Theta, \qquad \Delta = b_3^2 - a_3^2. \tag{4.27}$$

Multiplying the numerator and denominator of the first fraction in (4.27) by F and adding the resulting expression to the second fraction, as well as multiplying the numerator and denominator of the second fraction in (4.27) by F and adding the resulting expression to the first fraction, we find

$$\frac{Fa_1 + b_1 + \Delta\hat{b}_1}{a_1} = \frac{a_1 + Fb_1 + \Delta\hat{a}_1}{b_1} = \Theta. \tag{4.28}$$

Using equation (4.16) with $j = 1$, the left hand side of (4.28) implies

$$b_1^2 - a_1^2 = b_3^2 - a_3^2.$$

Then, the second equation of (4.27) implies

$$\frac{a_1}{b_1} + \frac{\hat{a}_1(b_1^2 - a_1^2)}{b_1} = \mathrm{e}^{-\omega(\lambda)L}(b_3\hat{a}_3 - \hat{b}_3 a_3) + \mathrm{e}^{-\omega(\lambda)L}F - F. \tag{4.29}$$

The left hand side of (4.29), using (4.16) with $j = 1$, simplifies as follows:

$$\frac{a_1}{b_1} + \hat{a}_1 b_1 - \frac{a_1}{b_1}(1 + b_1 \hat{b}_1) = \hat{a}_1 b_1 - a_1 \hat{b}_1.$$

Thus, equation (4.29) becomes

$$b_1 \hat{a}_1 - \hat{b}_1 a_1 = \mathrm{e}^{-\omega(\lambda)L}(b_3 \hat{a}_3 - \hat{b}_3 a_3) + \mathrm{e}^{-\omega(\lambda)L} F - F. \tag{4.30}$$

Replacing in this equation λ by $-\lambda$ yields

$$-(b_1 \hat{a}_1 - \hat{b}_1 a_1) = -\mathrm{e}^{\omega(\lambda)L}(b_3 \hat{a}_3 - \hat{b}_3 a_3) + \mathrm{e}^{\omega(\lambda)L} F - F. \tag{4.31}$$

Equations (4.30) and (4.31), taking into account the definition (4.6) of F, yield equations (4.17) and (4.18).
QED

Remark 4.3 The determinant condition (4.16) with $j = 1$ and equation (4.17) imply

$$\left[(a_1(\lambda)^2 - b_1(\lambda)^2)\right]\left[(a_1(-\lambda)^2 - b_1(-\lambda)^2)\right] = 1 - G(\lambda)^2, \quad \lambda \in \mathbb{R}. \tag{4.32}$$

Indeed, equation (4.16) with $j = 1$ implies the identity

$$\left[(a_1^2 - b_1^2)\right]\left[(\hat{a}_1^2 - \hat{b}_1^2)\right] = 1 - (a_1 \hat{b}_1 - \hat{a}_1 b_1)^2, \quad \lambda \in \mathbb{R}. \tag{4.33}$$

Then equation (4.17) implies (4.32).

Equation (4.32) defines the jump relation of a scalar RH problem for the sectional analytic functions defined by

$$\{(a_1(\lambda)^2 - b_1(\lambda)^2,\ \lambda \in \mathbb{C}^+ \quad a_1(-\lambda)^2 - b_1(-\lambda)^2,\ \lambda \in \mathbb{C}^-\}.$$

Taking into consideration that $a_1(\lambda) \neq b_1(\lambda)$ for $\lambda \in \mathbb{C}^+$ (otherwise equation (4.16) with $j = 1$ is violated) it follows that the above Riemann-Hilbert problem has a unique solution

$$a_1(\lambda)^2 - b_1(\lambda)^2 = h(\lambda), \qquad \lambda \in \mathbb{C}^+, \tag{4.34}$$

where

$$h(\lambda) = \mathrm{e}^{H(\lambda)}, \quad H(\lambda) = \frac{1}{2\pi i}\int_{\mathbb{R}} \ln[1 - G^2(\lambda')]\frac{d\lambda'}{\lambda' - \lambda}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.35}$$

Using the fact that $G(\lambda)$ is an odd function, it follows that $H(\lambda)$ is also an odd function, hence $h(-\lambda) = \mathrm{e}^{-H(\lambda)}$. This implies that the function $h(\lambda)$ defined by (4.35) satisfies the jump condition (4.32).

Remark 4.4 Equation (4.16) with $j = 1$, and equations (4.17) and (4.34) imply

$$a_1(-\lambda) = \frac{1}{h(\lambda)}\left(a_1(\lambda) + G(\lambda) b_1(\lambda)\right), \quad b_1(-\lambda) = \frac{1}{h(\lambda)}\left(b_1(\lambda) + G(\lambda) a_1(\lambda)\right), \quad \lambda \in \mathbb{R}. \tag{4.36}$$

Indeed, equation (4.17) yields

$$\hat{b}_1 = \frac{1}{a_1}(G + \hat{a}_1 b_1).$$

Replacing $\hat{b}_1$ in equation (4.16) with $j = 1$ by the above expression, and making use of (4.34), we find the first of equations (4.36). The second of equations (4.36) can be obtained in a similar way by eliminating $\hat{a}_1$ instead of $\hat{b}_1$.

Remark 4.5 The equations satisfied by a_3 and b_3 can be obtained from equations (4.36) by replacing $G(\lambda)$ by $G(-\lambda)$. Hence

$$a_3(-\lambda) = \frac{1}{h(\lambda)}\left(a_3(\lambda) - G(\lambda)b_3(\lambda)\right), \quad b_3(-\lambda) = \frac{1}{h(\lambda)}\left(b_3(\lambda) - G(\lambda)a_3(\lambda)\right), \quad \lambda \in \mathbb{R}, \tag{4.37}$$

where G is given by (4.21) and $h(\lambda)$ is given by (4.35).

Remark 4.6 The function G is an entire function, thus each of equations (4.36) defines the jump condition of a scalar RH problem. However, it will be shown in section 5 that equations (4.36) and (4.37) are sufficient to determine the jump matrix (2.25).

Remark 4.7 Equations (4.17)-(4.20) imply the following identity:

$$\mathrm{e}^{\omega(\lambda)L}[a_1(\lambda)^2 - b_1(\lambda)^2] + \mathrm{e}^{-\omega(\lambda)L}[a_1(-\lambda)^2 - b_1(-\lambda)^2] = (\mathrm{e}^{\omega(\lambda)L} + \mathrm{e}^{-\omega(\lambda)L})(1 - F^2) + 2F^2, \quad \lambda \in \mathbb{R}. \tag{4.38}$$

Indeed, equations (4.19)-(4.20) imply

$$\mathrm{e}^{\omega(\lambda)L}(a_2^2 - b_2^2) = \mathrm{e}^{\omega(\lambda)L}\hat{a}_1^2(a_3^2 - b_3^2) - \mathrm{e}^{-\omega(\lambda)L}b_1^2(\hat{a}_3^2 - \hat{b}_3^2) - 2\hat{a}_1 b_1(a_3\hat{b}_3 - \hat{a}_3 b_3) \tag{4.39}$$

Replacing in this equation λ by $-\lambda$, adding the resulting equation to equation (4.39) and using equation (4.30) we find

$$\mathrm{e}^{\omega(\lambda)L}(a_2^2 - b_2^2) + \mathrm{e}^{-\omega(\lambda)L}(\hat{a}_2^2 - \hat{b}_2^2) = (\mathrm{e}^{\omega(\lambda)L} + \mathrm{e}^{-\omega(\lambda)L})(a_1^2 - b_1^2)(\hat{a}_1^2 - \hat{b}_1^2) + 2(a_1\hat{b}_1 - \hat{a}_1 b_1)(a_3\hat{b}_3 - \hat{a}_3 b_3).) \tag{4.40}$$

Using (4.32), the right hand side of (4.40) equals the following expression:

$$(\mathrm{e}^{\omega(\lambda)L} + \mathrm{e}^{-\omega(\lambda)L}) - (a_1\hat{b}_1 - \hat{a}_1 b_1)\left[(\mathrm{e}^{\omega(\lambda)L} + \mathrm{e}^{-\omega(\lambda)L})(a_1\hat{b}_1 - \hat{a}_1 b_1) - 2(a_3\hat{b}_3 - \hat{a}_3 b_3)\right].$$

Using equations (4.17) and (4.18) the last expression becomes the right hand side of (4.38).

5 Spectral theory in the linearisable case

In the case of the linearisable boundary conditions (1.2), it is possible to express $q(x, y)$ in terms of the solution of a RH problem whose jump matrices are computed explicitly in terms of the given constant d. Indeed, recall that the jump matrices of the basic RH problem of section 2.4 are defined as follows:

$$J^{\pi/2} = \begin{pmatrix} 1 & I(\lambda)\mathrm{e}^{-\theta(x,y,\lambda)} \\ 0 & 1 \end{pmatrix}, \quad J^{3\pi/2} = \begin{pmatrix} 1 & 0 \\ I(-\lambda)\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix}, \quad I(\lambda) = \frac{b_2(-\lambda)}{a_1(\lambda)a_3(\lambda)},$$

$$J^0 = \begin{pmatrix} R(\lambda) & \frac{b_3(-\lambda)}{a_3(\lambda)} \mathrm{e}^{-\theta(x,y,\lambda)} \\ -\frac{\mathrm{e}^{-\omega(\lambda)L} b_1(\lambda)}{a_1(-\lambda)} \mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix}, \qquad R(\lambda) = \frac{a_2(\lambda)}{a_1(-\lambda)a_3(\lambda)},$$

and

$$J^\pi = J^{3\pi/2}(J^0)^{-1}J^{\pi/2}, \tag{5.1}$$

Equation (4.19) and (4.20) imply that

$$R(\lambda) = 1 - \mathrm{e}^{-\omega(\lambda)L} \frac{\hat{b}_3}{a_3}\frac{b_1}{\hat{a}_1}, \quad I(\lambda) = \frac{\hat{b}_3}{a_3} - \mathrm{e}^{\omega(\lambda)L}\frac{\hat{b}_1}{a_1}. \tag{5.2}$$

Thus in the linearisable case, the jump matrices involve only the rations $\frac{\hat{b}_3}{a_3}$ and $\frac{\hat{b}_1}{a_1}$, evaluated at λ and at $-\lambda$. Equations (4.35) and (4.34) imply that these rations are given by

$$\frac{\hat{b}_3}{a_3} = -\frac{G}{h} + \frac{b_3}{a_3 h}, \qquad \frac{\hat{b}_1}{a_1} = \frac{G}{h} + \frac{b_1}{a_1 h}. \tag{5.3}$$

Hence the jump matrices depend on the known function $\frac{G}{h}$ as well as on the unknown functions $\frac{b_1}{a_1 h}$ and $\frac{b_3}{a_3 h}$. Using the fact that these unknown functions are bounded and analytic in $\mathbb{C}^+$, it is possible to formulate a RH problem, equivalent to the basic one defined by (5.1), in terms of the known function $\frac{G}{h}$ only. This new RH problem is therefore defined by the following jump matrices:

$$\tilde{J}^{\pi/2} = \begin{pmatrix} 1 & \tilde{I}(\lambda)\mathrm{e}^{-\theta(x,y,\lambda)} \\ 0 & 1 \end{pmatrix}, \quad \tilde{J}^{3\pi/2} = \begin{pmatrix} 1 & 0 \\ \tilde{I}(-\lambda)\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix}, \qquad \tilde{I}(\lambda) = -\frac{G}{h}(1+\mathrm{e}^{-\omega(\lambda)L}),$$

$$\tilde{J}^0 = \begin{pmatrix} \tilde{R}(\lambda) & -\frac{G}{h}\mathrm{e}^{-\theta(x,y,\lambda)} \\ \frac{\mathrm{e}^{-\omega(\lambda)L}G}{\hat{h}}\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix}, \qquad \tilde{R}(\lambda) = 1 - \frac{G^2}{h\hat{h}}\mathrm{e}^{-\omega(\lambda)L},$$

and

$$\tilde{J}^\pi = \tilde{J}^{3\pi/2}(\tilde{J}^0)^{-1}\tilde{J}^{\pi/2}, \tag{5.4}$$

Theorem 5.1 *Let $q(x,y)$ satisfy equation (1.1) and the boundary conditions (1.2). Then $q(x,y)$ is given by equations (2.36)-(2.37) with Ψ replaced by $\tilde{\Psi}$, where $\tilde{\Psi}$ is the solution of the Riemann-Hilbert problem (2.23) with the jump matrix J replaced by the matrix $\tilde{J}$ defined as follows:*

$$\tilde{J}(x,y,\lambda) = \tilde{J}^\alpha(x,y,\lambda), \text{ if } arg(\lambda) = \alpha, \quad \alpha = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \tag{5.5}$$

where

$$\tilde{J}^0 = \begin{pmatrix} 1 - \frac{G^2(\lambda)}{h(\lambda)h(-\lambda)}\mathrm{e}^{-\omega(\lambda)L} & -\frac{G(\lambda)}{h(\lambda)}\mathrm{e}^{-\theta(x,y,\lambda)} \\ \mathrm{e}^{-\omega(\lambda)L}\frac{G(\lambda)}{h(-\lambda)}\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix},$$

$$\tilde{J}^{\pi/2} = \begin{pmatrix} 1 & -\frac{G(\lambda)}{h(\lambda)}\left(1+\mathrm{e}^{\omega(\lambda)L}\right)\mathrm{e}^{-\theta(x,y,\lambda)} \\ \\ 0 & 1 \end{pmatrix}, \qquad \tilde{J}^{3\pi/2} = \begin{pmatrix} 1 & 0 \\ \\ \frac{G(\lambda)}{h(-\lambda)}\left(1+\mathrm{e}^{-\omega(\lambda)L}\right)\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix}$$

and

$$\tilde{J}^{\pi} = \tilde{J}^{3\pi/2}(\tilde{J}^{0})^{-1}\tilde{J}^{\pi/2}, \tag{5.6}$$

where $\theta(x,y,\lambda)$ is given by (2.26), while $G(\lambda)$ and $h(\lambda)$ are defined in terms of the given constant d by equations (4.21) and (4.35).
This Riemann-Hilbert problem is regular and, if $d \in \mathbb{R}$, it has a unique solution.

Proof: The solution $\tilde{\Psi}$ of the new RH problem satisfies the jump relation

$$\tilde{\Psi}_{-}(x,y,\lambda) = \tilde{\Psi}_{+}(x,y,\lambda)\tilde{J}(x,y,\lambda), \lambda \in \mathbb{R}\cup i\mathbb{R}, \tag{5.7}$$

where the jump matrix $\tilde{J}$ is given by equations (5.5) and (5.6) (the latter are identical to equations (5.4)). Let Ψ_j and $\tilde{\Psi}_j$, $j = 1, ..., 4$, denote Ψ and $\tilde{\Psi}$ in the j-th quadrant of the complex λ plane, respectively. We seek matrices A_j, $j = 1, ..., 4$ which are analytic and bounded in the j-th quadrant of the complex λ plane, respectively, and such that

$$\tilde{\Psi}_j(x,y,\lambda) = \Psi_j(x,y,\lambda)A_j(x,y,\lambda), \quad (j-1)\frac{\pi}{2} < arg(\lambda) < j\frac{\pi}{2}, \qquad j = 1, ..., 4. \tag{5.8}$$

Let

$$A_1 = \begin{pmatrix} 1 & \alpha_1(\lambda)\mathrm{e}^{-\theta(x,y,\lambda)} \\ \\ 0 & 1 \end{pmatrix}, \qquad 0 < arg(\lambda) < \frac{\pi}{2} \tag{5.9}$$

$$A_2 = \begin{pmatrix} 1 & \alpha_2(\lambda)\mathrm{e}^{-\theta(x,y,\lambda)} \\ \\ 0 & 1 \end{pmatrix}, \qquad \frac{\pi}{2} < arg(\lambda) < \pi \tag{5.10}$$

$$A_3 = \begin{pmatrix} 1 & 0 \\ \\ \alpha_3(\lambda)\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix}, \qquad \pi < arg(\lambda) < \frac{3\pi}{2} \tag{5.11}$$

$$A_4 = \begin{pmatrix} 1 & 0 \\ \\ \alpha_4(\lambda)\mathrm{e}^{\theta(x,y,\lambda)} & 1 \end{pmatrix}, \qquad \frac{3\pi}{2} < arg(\lambda) < 2\pi. \tag{5.12}$$

Equations (2.23) and (5.7) imply the following relations:

$$J^{\pi/2}A_2 = A_1\tilde{J}^{\pi/2}, \qquad J^{3\pi/2}A_4 = A_3\tilde{J}^{3\pi/2}, \qquad J^0 A_4 = A_1\tilde{J}^0. \tag{5.13}$$

The first two equations (5.13) imply

$$\alpha_2(\lambda) + I(\lambda) = \alpha_1(\lambda) + \tilde{I}(\lambda), \qquad \alpha_4(\lambda) + I(-\lambda) = \alpha_3(\lambda) + \tilde{I}(-\lambda). \tag{5.14}$$

Using the definitions of $I(\lambda)$ and $\tilde{I}(\lambda)$ (see equations (5.2) and (5.4)) as well as (5.3), equations (5.14) become

$$\alpha_2 + \frac{b_3}{a_3 h} - \mathrm{e}^{\omega(\lambda)L}\frac{b_1}{a_1 h} = \alpha_1, \qquad \alpha_4 + \frac{\hat{b}_3}{\hat{a}_3\hat{h}} - \mathrm{e}^{-\omega(\lambda)L}\frac{\hat{b}_1}{\hat{a}_1\hat{h}} = \alpha_3.$$

The simplest solution of these equations that satisfy the requirement that the functions $\alpha_j(\lambda)$ are bounded and analytic in the j-th quadrant of the complex λ-plane, $j = 1, ..., 4$, are the following:

$$\alpha_1 = \frac{b_3}{a_3 h}, \qquad \alpha_2 = \frac{b_1}{a_1 h} \mathrm{e}^{\omega(\lambda)L},$$
$$\alpha_3 = \frac{\hat{b}_3}{\hat{a}_3 \hat{h}}, \qquad \alpha_4 = \frac{\hat{b}_1}{\hat{a}_1 \hat{h}} \mathrm{e}^{-\omega(\lambda)L}. \tag{5.15}$$

The third of equations (5.13) yields the following relations:

$$\frac{\hat{b}_3}{a_3} = \alpha_1 - \frac{G}{h}, \qquad \frac{b_1}{\hat{a}_1} \mathrm{e}^{-\omega(\lambda)L} = \alpha_4 - \frac{G}{\hat{h}} \mathrm{e}^{-\omega(\lambda)L},$$
$$R + \alpha_4 \frac{\hat{b}_3}{a_3} = \tilde{R} + \alpha_1 \mathrm{e}^{-\omega(\lambda)L} \frac{G}{\hat{h}}. \tag{5.16}$$

Using the definitions of $R(\lambda)$, $\tilde{R}(\lambda)$ (equations (5.1), (5.4)) the definitions of α_j , $j = 1, .., 4$ (equations (5.15)) and and equations (5.3) it can be verified that equations (5.16) are satisfied identically.

The eigenfunctions $\tilde{\Psi}_j$, $j = 1, ..., 4$ are given explicitly in terms of the eigenfunctions Ψ_j by equations (5.8). This yields the following expressions:

$$\tilde{\Psi}_1 = (\Psi_1^{(12)}, \frac{b_3}{a_3 h} \Psi_1^{(12)} \mathrm{e}^{-\theta(x,y,\lambda)} + \frac{1}{a_3} \Psi_3^{(1)}), \quad \tilde{\Psi}_2 = (\Psi_1^{(12)}, \frac{b_1}{a_1 h} \mathrm{e}^{\omega(\lambda)L} \Psi_1^{(12)} \mathrm{e}^{-\theta(x,y,\lambda)} + \frac{1}{a_1} \Psi_2^{(2)}), \tag{5.17}$$
$$\tilde{\Psi}_3 = (\frac{1}{\hat{a}_3} \Psi_3^{(3)} + \frac{\hat{b}_3}{\hat{a}_3 \hat{h}} \Psi_1^{(34)} \mathrm{e}^{\theta(x,y,\lambda)}, \Psi_1^{(34)}), \quad \tilde{\Psi}_4 = (\frac{1}{\hat{a}_1} \Psi_2^{(4)} + \frac{\hat{b}_1}{\hat{a}_1 \hat{h}} \mathrm{e}^{-\omega(\lambda)L} \Psi_1^{(34)} \mathrm{e}^{\theta(x,y,\lambda)}, \Psi_1^{(34)}).$$

The above equations can be simplified as follows:

$$\tilde{\Psi}_1 = (\Psi_1^{(12)}, \frac{b_3}{h} \Psi_3^{(3)} \mathrm{e}^{-\theta(x,y,\lambda)} + \frac{a_3}{h} \Psi_3^{(1)}), \quad \tilde{\Psi}_2 = (\Psi_1^{(12)}, \frac{b_1 \mathrm{e}^{\omega(\lambda)L}}{h} \Psi_2^{(4)} \mathrm{e}^{-\theta(x,y,\lambda)} + \frac{a_1}{h} \Psi_2^{(2)}), \tag{5.18}$$
$$\tilde{\Psi}_3 = (\frac{\hat{a}_3}{\hat{h}} \Psi_3^{(3)} + \frac{\hat{b}_3}{\hat{h}} \Psi_3^{(1)} \mathrm{e}^{\theta(x,y,\lambda)}, \Psi_1^{(34)}), \quad \tilde{\Psi}_4 = (\frac{\hat{a}_1}{\hat{h}} \Psi_2^{(4)} + \frac{\hat{b}_1 \mathrm{e}^{-\omega(\lambda)L}}{\hat{h}} \Psi_2^{(2)} \mathrm{e}^{\theta(x,y,\lambda)}, \Psi_1^{(34)}).$$

Indeed, using equation (2.16) we find

$$\frac{b_3}{a_3 h} \Psi_1^{(12)} \mathrm{e}^{-\theta(x,y,\lambda)} + \frac{1}{a_3} \Psi_3^{(1)} = \frac{b_3}{a_3 h} (a_3 \Psi_3^{(3)} + b_3 \Psi_3^{(1)} \mathrm{e}^{\theta(x,y,\lambda)}) \mathrm{e}^{-\theta(x,y,\lambda)} + \frac{1}{a_3} \Psi_3^{(1)}$$

Using $a_3^2 - b_3^2 = h$ the above expression is equal to the regular function

$$\frac{b_3}{h} \Psi_3^{(3)} \mathrm{e}^{-\theta(x,y,\lambda)} + \frac{a_3}{h} \Psi_3^{(1)}.$$

A similar computation yields the result for the other eigenfunctions.

Thus the above Riemann-Hilbert problem (5.7) is regular.

We now prove that the Riemann-Hilbert problem defined by (5.7) is uniquely solvable. It can be verified that when $d \in \mathbb{R}$, then $h(\lambda) = \overline{h(\bar{\lambda})}$. In this case, the jump matrices $\tilde{J}^{(\alpha)}$ satisfy the following conditions: the matrices are Schwarz invariant on the imaginary axis and have zero real part on the real axis of the complex λ plane. Under these assumptions, it follows from general results (see e.g. [10, 29, 49]) that the so-called "vanishing lemma" holds. This guarantees the existence of a unique solution.
QED

6 Conclusions

We have studied boundary value problems for the elliptic sine-Gordon posed on a semistrip. In particular we have shown that if the prescribed boundary conditions are zero along the unbounded sides of the semistrip and constant on the bounded side, then it is possible to obtain the solution in terms of a Riemann-Hilbert problem which is uniquely defined in terms of the width L of the semistrip and the constant value d of the solution along the $x = 0$ boundary. Indeed, the "jump matrices" of this Riemann-Hilbert problem are defined in terms of the two functions $G(\lambda)$ and $h(\lambda)$ which are explicitly defined, by equations (4.21) and (4.35) respectively, in terms of L and d. Furthermore, this Riemann-Hilbert problem has a unique solution, see theorem 5.1.

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References

[1] M.J Ablowitz and A.S. Fokas, *Introduction and applications of complex variables*, Cambridge University Press, 2nd ed. 2003.

[2] Y.Antipov and A.S. Fokas, The modified Helmholtz equation in a semi-strip, Math. Proc. Camb. Phil Soc. **138**, 339-365 (2005).

[3] D. ben-Avraham and A.S. Fokas, The Modified Helmholtz Equation in a Triangular Domain and an Application to Diffusion-limited Coalescence, Phys. Rev. E **64**, 016114-6 (2001).

[4] J.L. Bona, and A.S. Fokas, Initial-Boundary-Value Problems for Linear and Integrable Nonlinear Dispersive Partial Differential Equations, Nonlinearity **21**, 10, T195–T203 (2008).

[5] A. Boutet de Monvel, A.S. Fokas and D. Shepelsky, The Modified KdV Equation on the Half-Line, J. of the Inst. of Math. Jussieu **3**, 139-164 (2004).

[6] A. Boutet de Monvel, A.S. Fokas and D. Shepelsky, The Analysis of the Global Relation for the Nonlinear Schrödinger Equation on the Half-Line, Lett. Math. Phys. **65**, 199-212 (2003).

[7] A.B. Borisov and V.V. Kiseliev, Inverse problems for an elliptic sine-Gordon equation with an asymptotic behaviour of the cnoidal type, *Inv. Prob.* **5**, 959-982 (1989).

[8] D. Crowdy and A.S. Fokas, Explicit Integral Solutions for the Plane Elastostatic Semi-Strip, Proc. R. Soc. London A **460**, 1285-1309 (2004).

[9] G. Dassios and A.S. Fokas, The Basic Elliptic Equations in an Equilateral Triangle, Proc. R.Soc. Lond. A **461**, 2721-2748 (2005).

[10] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert approach*, American Mathematical Society (2000).

[11] G.M. Dujardin, Asymptotics of Linear Initial Boundary Value Problems with Periodic Boundary Data on the Half-Line and Finite Intervals, Proc. R. Soc. Lond. A **465** 3341-3360 (2009).

[12] N. Flyer and A.S. Fokas, A Hybrid Analytical Numerical Method for Solving Evolution Partial Differential Equations. I. The Half-Line, Proc. R. Soc. **464**, 1823-1849 (2008).

[13] A.S. Fokas, A unified transform method for solving linear and certain nonlinear PDE's, *Proc. Royal Soc. Series A* **453**, 1411–1443 (1997).

[14] A.S. Fokas, *A Unified Approach to Boundary Value Problems*, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM (2008).

[15] A.S. Fokas, Two dimensional linear PDE's in a convex polygon, *Proc. Royal Soc. Lond. A* **457**, 371–393 (2001).

[16] A.S. Fokas, Nonlinear evolution equations on the half line, *Comm. Math. Phys.* **230**, 1-39 (2002).

[17] A.S. Fokas, The generalized Dirichlet-to-Neumann map for certain nonlinear evolution PDEs, *Comm. Pure Appl. Math.* **LVIII**, 639-670 (2005).

[18] A.S. Fokas, Linearizable initial boundary value problems for the sine-Gordon equation on the half line, Nonlin. **17**, 1521-1534 (2004).

[19] A.S. Fokas, A New Transform Method for Evolution PDEs, IMA J. Appl. Math. **67**, 1-32 (2002).

[20] A.S. Fokas and K. Kalimeris, Eigenvalues of the Laplace Operator in the Interior of an Equilateral Triangle (submitted).

[21] A.S. Fokas and A.R. Its, An Initial-Boundary Value Problem for the sine-Gordon Equation in Laboratory Coordinates, Theor. Math. Phys. **92**, 387-403 (1992).

[22] A.S. Fokas and A.R. Its, The linearization of the initial-boundary value problem of the nonlinear Schrödinger equation, SIAM J. Math. Anal. **27** 738-764 (1996).

[23] A.S. Fokas and A.R. Its, The nonlinear Schrödinger equation on the interval, J. Phys. A. **37**, 6091-6114 (2004).

[24] A.S. Fokas, A.R. Its and L.Y. Sung, The Nonlinear Schrödinger Equation on the Half-Line, Nonlinearity **18**, 1771-1822 (2005).

[25] A.S. Fokas and J. Lennels, Boundary Value Problems for the Stationary Axisymmetric Einstein Equations: A Rotating Disk (preprint).

[26] A.S. Fokas and C. Menyuk, Integrability and Similarity in Transient Stimulated Raman Scattering, J. Nonl. Sci. **9**, 1-31 (1999).

[27] A.S. Fokas and B. Pelloni, A transform method for linear evolution PDEs on a finite interval, *IMA J. of Appl. Math.* **70**, 1-24 (2005).

[28] A.S. Fokas and E.A. Spence, Novel Analytical and Numerical Methods for Elliptic Boundary Value Problems, in "Highly Oscillatory Problems", Cambridge University Press (2009).

[29] A.S. Fokas and X. Zhou, On the solvability of Painleve II and IV, Comm. Math. Phys. **144**, 601-6222 (1992).

[30] A.S. Fokas and M. Zyskin, The Fundamental Differential Form and Boundary Value Problems, Quart. J. Mech. Appl. Math. **55**, 457-479 (2002).

[31] F.D.Gakhov, *Boundary value problems*, Pergamon Press (1966).

[32] E.S. Gutshabash and V.D.Lipovskii, Boundary value problem for the two-dimensional elliptic sine-Gordon equation and its applications to the theory of the stationary Josephson effect, *J. Math. Sciences* **68**, 197-201 (1994).

[33] P. D. Lax, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21**, 467-490 (1968).

[34] J. Lenells and A.S. Fokas, An integrable generalisation of the nonlinear Schrödinger equation on the half-line and solitons, Inv. Prob. **25** (2009)

[35] J. Lenells and A.S. Fokas, Explicit Soliton Asymptotics for the Korteweg-de-Vries Equation on the Half-Line (submitted).

[36] V.D. Lipovskii and S.S. Nikulichev, Vestn. LGU Ser. Fiz. Khim. 4, 61-34 (1988).

[37] B. Pelloni, The spectral representation of two-point boundary value problems for linear evolution equations, *Proc. R. Soc. A* (2005) **461**, 2965-2984.

[38] B. Pelloni, Well posed boundary value problems for linear evolution equations in finite intervals, *Math. Proc. Camb. Phil. Soc.* (2004), **136**, 361-382.

[39] B. Pelloni. The asymptotic behaviour of the solution of boundary value problems for the sine-Gordon equation on a finite interval, *J. Nonlin. Math. Phys.* **12**(4), 518-529 (2005).

[40] B. Pelloni, Spectral analysis of the elliptic sine-Gordon equation in the quarter plane, *Theor. Math. Phys.* (in press)

[41] B. Pelloni and D.A. Pinotsis, The Klein-Gordon Equation on the Half Line: a Riemann-Hilbert Approach, J. Nonlinear Math. Phys. **15**, 334 - 344 (2008).

[42] B. Pelloni and D.A. Pinotsis, The elliptic sine-Gordon equation in a half plane *Nonlin*, (in press).

[43] S.A. Smitheman, E.A. Spence and A.S. Fokas, A Spectral Collocation Method for the Laplace and Modified Helmholtz Equations in a Convex Polygon, IMA J. Num. Anal. (to appear).

[44] E.A. Spence, A.S. Fokas, A New Transform Method I: Domain Dependent Fundamental Solutions and Integral Representations. submitted to Proc. Roy. Soc. A.

[45] E.A. Spence, A.S. Fokas, A New Transform Method II: the Global Relation, and Boundary Value Problems in Polar Co-ordinates. submitted to Proc. Roy. Soc. A.

[46] P.A. Treharne and A.S. Fokas, The Generalized Dirichlet to Neumann map for the KdV Equation on the Half-Line, J. Nonlinear Science **18**, 191-217 (2008).

[47] V.E. Zakharov and A.B.Shabat, A scheme for integrating the nonlinear equations of numerical physics by the method of the inverse scattering problem I, Funct. Anal. Appl. **8** 226-235 (1974)

[48] V.E. Zakharov and A.B.Shabat, A scheme for integrating the nonlinear equations of numerical physics by the method of the inverse scattering problem II, Funct. Anal. Appl. **13** 166-174 (1979)

[49] X. Zhou, The Riemann-Hilbert problem and inverse scattering, SIAM J. Math. Anal. **20** (4) 966-986 (1989).