Flux Modelling of Polynyas

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JM

Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully referenced.

Signed . . . . Date . . . .

Abstract

It is common in polynya flux models to take the depth of the pack ice, H, to be a constant value that needs to be predetermined before the polynya is modelled. This is undesirable because this depth influences the polynya width throughout the model time frame not just at the initial time, so by fixing its value beforehand we are losing some freedom in the model. There have been parameterisations that allow H to vary with the frazil ice depth at the polynya edge, we take a different approach to reparameterising the depth on the polynya edge by looking at the pack ice as a sheet of ice which is allowed to diffuse and flow. We start by numerically and analytically evaluating the simple one-dimensional flux model for a latent heat polynya given by H.W. Ou in 1988. This flux model is then coupled to a diffusion PDE governing the pack ice. The diffusion PDE is given by the mass balance equation, with diffusion velocity obtained through Glen's flow law for ice. A depth profile for the pack ice is required for the pack ice as an initial condition. The coupled diffusion and polynya flux model is then modelled through a fixed computational grid and finally a moving grid method is applied to the model.

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Chapter 1

Polynya Introduction

A polynya is an opening of water enclosed in sea ice. Interactions between the atmosphere, ice and the ocean drive formation of polynyas. Polynyas are classified by the mechanisms that force their evolution; latent heat polynyas and sensible heat polynyas. Sensible heat polynyas are thermally driven by warm waters warming the ice and melting it to leave an area of water surrounded by pack ice, these most often occur out in open ocean waters. Latent Heat polynyas are driven by winds or ocean currents. We will be concerned with latent heat coastal polynyas for the remainder of the discussion.

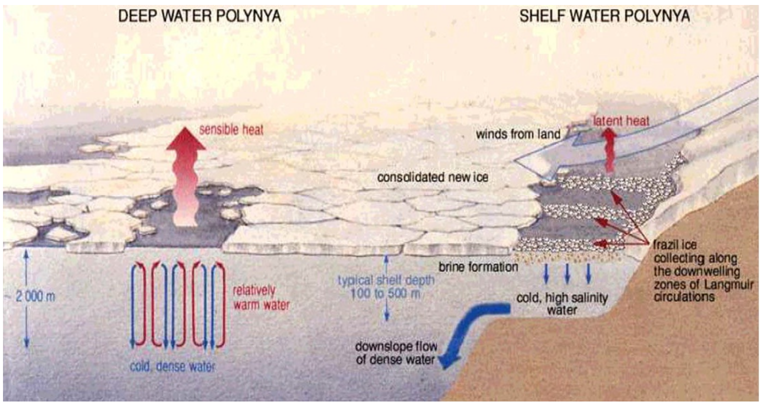

Figure 1.1: Schematic of Sensible Heat polynya and Latent Heat polynya [16]

The polynya opening is formed when some consolidated pack ice is forced by winds away from a coastline, or indeed from another block of rigid ice. Water within the polynya is cold enough that it is able to form ice plates that are suspended in the ocean [10], this ice is known as frazil ice. The production rate of frazil ice is affected by surface wind, air flows being warmed by the relatively warm polynya waters, downwind long wave radiation, evaporation of waters and the motion of tidal waters. This frazil ice produced within the polynya is forced out towards the pack ice by offshore winds.

The velocities of the frazil ice and the pack ice differ although both of these values are dependent on the wind velocity. The pack ice typically has a smaller velocity than that of the frazil ice, which results in there being a build up of frazil ice at the edge of the moving pack ice. This point where the polynya waters meets the pack ice is the polynya edge. Not only is there a difference in the speeds of the two types of ice, there is also a difference in directions that the ice moves. The pack ice is affected by the turning of the earth which is between 23° and 33° to the right (in the northern hemisphere) of the wind velocity [9]. The frazil ice is more susceptible to being affected by water circulations within the polynya waters, such as Langmuir circulations which are vortices that rotate in shallow waters and can vary the direction of the frazil ice flow up to 13° to the right of the wind velocity [9].

The evolution process of polynyas is divided into two phases, opening and closing. A coastal polynya opens when the forcing winds move the consolidated pack ice from the coastline, this will continue until either a steady state is reached or a change in forcing occurs. A steady state is where the polynya size has reached a maximum size and will not grow any further. The closing of a polynya is due to one of a couple of factors. Either there is a change in forcing which now forces the pack ice back to shore as well as the frazil ice in the region, or in *situ ice* formation within the polynya occurs during periods of slack wind, i.e. the polynya freezes over. In the first case the frazil ice will pile up on the coastal boundary instead of at the polynya edge as was the case during the opening of the polynya. [14]

Polynyas frequently occur in the arctic and antarctic over the colder winter months. The frequency of occurrence can vary, and there are instances where polynyas form regularly in the same location sometimes annually. Studying the behaviour of polynyas is a useful means of monitoring climate change. Polynya are sources of heat and moisture transfer from ocean to the atmosphere. They also influence the chemical makeup of the ocean: during the production process of frazil ice a by-product is brine that can alter water flow and circulation routes. Polynyas also affect water-sea flux changes, being sinks for carbon dioxide. In addition to physical and chemical changes in the region, they offer marine life habitats in the otherwise frozen seascape, [10].

There are two prominent methods used when modelling polynyas; General Circulation Models (GCMs) and Flux modelling. Flux models look at the balance between the flux of the frazil ice production and the flux of wind driven offshore divergence of ice, and use this as a guide that governs the position of the polynya edge. GCMs use discretisation of the polynya region into a spatial grid and look at concentrations of ice in each cell, if many neighbouring cells have low concentration of ice, then these cells are interpreted as part of the polynya.

For this dissertation we will focus solely on the flux models which model the opening of a latent heat coastal polynya to a steady state.

Chapter 2

Flux Model Development

Polynya flux models generally rely on a balance of fluxes at the polynya edge. These models, that are discussed below, are based on a paper by a Russian scientist, V.L Lebedev, in 1968 who stipulated that a steady state for a coastal polynya is reached under certain atmopshpheric conditions. Here we examine the derivation of a one-dimensional flux model, starting by looking at the papers that are key in getting to the one-dimensional flux model.

In the one-dimensional case we ignore coastline geometry and directional differences involved in velocities and instead focus on a cross sectional view of the polynya and look at the evolution of a polynya at one point on the coastline.

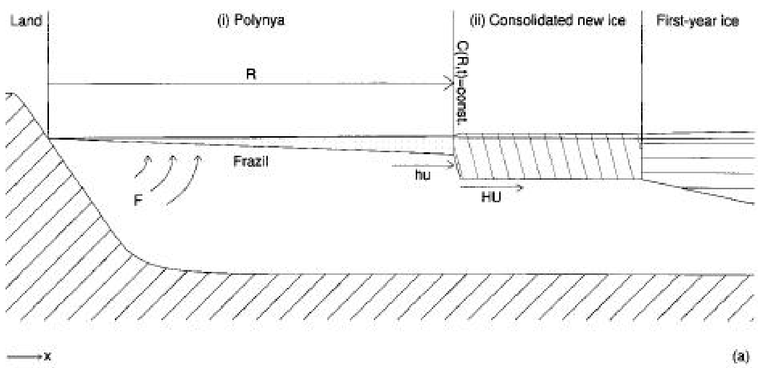

Figure 2.1: One Dimensional model diagram, adapted from [9]

Figure 2.1 shows this cross section that we are considering, . It is divided into three parts, (i) is the polynya - the area of water that is surrounded by sea ice or coastline, (ii) is the consolidated pack ice which is advected away from the coastline by prevailing winds and sea currents, and (iii) further out to sea we have the first year ice which is not included in the flux models we consider below. The point of interest here is the boundary between the polynya and the consolidated ice. As the wind advects the ice away from the coastline, there is production of frazil ice in region (i) which is also advected away from the coast towards region (ii). The balance between the frazil ice production rate and its velocity, and the velocity of the pack ice is the driving idea behind the flux models discussed below.

2.1 Lebedev, 1968

Maximum Size of a Wind-Driven Lead During Sea Freezing

Lebedev stipulated that the dimensions of a wind lead cannot grow indefinitely [8], a lead is a fracture in the ice that is mechanically driven by the motion of ice of sheer forces, [10]. The base model he generates begins with an assumption that the heat loss in the production of ice has three main components.

- rate of turbulent heat loss

$$q_1 = K_1 W (T_w - T_a)$$

- rate of heat loss due to evaporation

$$q_2 = K_2 W (E - e) \rho L$$

- rate of radiation heat loss

$$q_3 = q_{eff}(1 - \beta C) + \delta - (q_d - q_s)(1 - \alpha)(1 - 0.05C)$$

where $(K_{1,2})$ are proportionality constants, (T_w, T_a) are temperatures at the water and at height a respectively, W is wind speed, (e, E) is the pressure of the vapor at height a and saturation vapor at temperature T_w respectively, ρ is the water density, (q_d, q_s) are direct and solar radiation values, and α is the water albedo. C is the degree of cloudiness on appropriate scales, q_{eff} is effective incident radiation with a clear sky at thermal equilibrium of air and water, (β, δ) are correction factors for effective radiation.

These three factors are summed together to get an expression for the rate of change of the frazil ice. This is then integrated over a period of time to where a certain depth of ice is obtained. This time frame obtained for the ice depth to be reached can also be expressed in terms of the polynya width and the drift rate of the surface water and ice in the polynya. By combining these two expressions we get an equivalence for the size of the polynya and the change in ice depth. [8].

$$R = \frac{W\lambda h}{K_3(q_1 + q_2 + q_3)}$$

In this expression, R is the polynya width, K_3 is the drift coefficient, and $q_{1,2,3}$ are as above.

The equation obtained shows that as wind speed increases, the width of a lead reaches a limit. However, the summarising statements towards the end of the paper imply that the equations obtained need alteration to allow for a retardation in heat transfer by the ice crust.

2.2 Pease, 1987

The Size of Wind-Driven Coastal polynyas

Pease's article expands on the model developed by Lebedev, noting that the ideas presented in his paper were essentially correct but there were some problems with the model formulation [13]. Pease presents a one-dimensional model that describes the evolution of the polynya edge R as

$$RF = H\left(U - \frac{dR}{dt}\right)$$

where U is the advection rate of the consolidated ice, F is the production rate of frazil ice, and H is the depth of the consolidated ice pack. Pease notes that when U, H, and F are linear functions of position or approximately linear then we have a differential equation with solution

$$R = \frac{UH}{F}\left[1 - exp\left(-\frac{tF}{H}\right)\right].$$

For large timescales we have a limit on the size of the polynya, bounded by $R = \frac{UH}{F}$. Pease also remarks that the expression for the maximum polynya size gives access to finding a timescale for the polynya to reach a given proportion of its maximum size, for example a time until the polynya reaches 95% of its maximum size. This timescale is only dependent on the freezing rate scaled by the collection thickness: for a greater freezing rate the time until maximum polynya size is shorter.

Pease gives a formulation for the freezing rate over the polynya. It takes into account the unreflective shortwave radiation, downward and upward long wave radiation, sensible/turbulent heat flux, and the latent heat of evaporation. The latent heat of evaporation is ignored since its contribution is negligible. Similarly the shortwave radiation contribution is zero during winter months and in the summer contribution is also negligible, so these effects are neglected in the model. F is the averaged area production rate and is equivalent to the local production rate $\frac{dH}{dt}$ [13].

$$F = -\frac{\sigma e_a T_a^4 - \sigma e_w T_w^4 + \rho_a C_h C_p V_a (T_a - T_w)}{\rho_i L} \tag{2.1}$$

Here U is a fraction of the wind speed, (normally 3% or 4%), σ is the Boltzmann constant, ρ_i is the new ice density, ρ_a is the cold air density, and L is the latent heat of freezing for salt water. (All other constants/variables have been defined as in Lebedev's paper.)

Pease initially tests the sensitivity of the model by investigating the effects that varying the surface temperature, T_a, the advection rate on the surface, V_a and the maximum polynya size have on the model. The results were that the coastal polynyas reach a stable size within a typical synoptic time scale for low temperatures but not those approaching freezing. [13]

Three experiments in the Bering Sea during 1982, 1983 and 1985, were observed and they provide a means of testing the accuracy of the model. They don't cover a full range of variables contained in the model but were adequate in testing the mid ranges of the variables temperature and wind speed. The conclusion here is that for low wind speeds the model will not be useful since the idea of a collection thickness won't really come into physical being. Similarly for higher temperatures (those approaching freezing point from below) the model won't be appropriate due to lack of model physics [13].

2.3 Ou, 1988

A Time-Dependent Model of a Coastal Polynya

Pease's flux model assumes that the newly produced frazil ice in the polynya is instantaneously deposited at the polynya edge, this could be interprteted as the frazil ice having an infinite velocity towards the polynya edge. This is not the case in the physical world since infinite drift rate is not possible, the frazil ice has a finite drift rate. Ou extends the Pease model by accounting for the drift rate of the frazil ice. The equation for the depth of the frazil ice on the polynya edge is

$$h_t + (hu)_x = F$$

where h is the frazil ice depth, u is the frazil ice velocity, F is the production rate given by (2.1), and the x derivative is the change in position of the frazil ice with $x = 0$ denoting the coastline and $x = R$ denoting the polynya edge as with the Pease model. We have an initial coondition that when $x = 0$, $h = 0$. If we have a steady forcing term, i.e. the wind speed is constant and the frazil ice has constant force applied, then the depth of frazil ice at the polynya edge is

$$h_R = \frac{FR}{u_R} \tag{2.2}$$

where u_R is the velocity of the frazil ice at the polynya edge, similarly h_R denotes the frazil ice thickness at the polynya edge. This results in the flux balance

$$h_R\left(u_R - \frac{dR}{dt}\right) = H\left(U - \frac{dR}{dt}\right). \tag{2.3}$$

Ou's alteration to the Pease model keeps the property of having a steady state solution satisfying

$$R = \frac{HU}{F} \tag{2.4}$$

The model's sensitivity to small perturbations was examined and Ou concludes that high frequency variations in atmospheric conditions do not have a large impact on the polynya edge. By this Ou means that the period of the variations are small in comparison to the transit time of the frazil ice. Varying the ratio between the velocities of the consolidated ice and the frazil ice will affect the response at the polynya edge. Reducing the frazil ice drift speed (with respect to the consolidated ice speed) will lengthen transit time for the frazil ice, but will reduce the inertia of the system and prompt faster response of the polynya edge. Changes in air temperature will affect the production rate and are more effective in producing a response in the polynya edge, [11].

Ou recognises that the model is a very idealised version of the physics involved in the polynya process and ignores some important features namely;

- Although the frazil ice drift speed is assumed directly proportional to the wind speed, in reality we would expect a slight lag to occur between the two, i.e. the response to the wind velocity is not instantaneous
- Although the thickness of the consolidated pack ice, H, is taken to be constant (usually kept to be around $0.1m - 0.2m$) this is mainly due to lack of knowledge of the collection process at the polynya edge

- Since spatial uniformity of the forcing terms has been assumed, it does not take into account strong wind gusts or regional variations across the polynya.

Ou's inclusion of the frazil ice drift results in a two-phase appearance to the process, an initial 'runaway' phase where the consolidated ice pack runs away, and then an 'approach' phase where the polynya edge reaches a steady state when the balance of fluxes reach an equilibrium [11]. When this equilibrium is reached the polynya has reached its maximum size, this is given by (2.4).

In Chapter 3 we look at the Ou model, taking a numerical and an analytical approach to modelling the polynya. Then in Chapter 4 we extend the model by including a new parameterisation for the pack ice.

Chapter 3

One Dimensional Flux Model

The model obtained by Ou shows the balance between the fluxes of the frazil ice and the consolidated pack ice at the polynya edge. This balance governs the position of the polynya edge. We also have that the frazil ice depth is given by a quotient of the production rate over the width of the polynya divided by the velocity of the frazil ice: equations (2.3) and (2.2). By rearranging the flux balance equation (2.3) we obtain

$$\frac{dR}{dt} = \frac{HU - h_R u_R}{H - h_R} \tag{3.1}$$

In the most simplified case we could have, we take H, U and u_R to be all known constants. Similarly, we can take F to be constant by suitably fixing the constants within the parameterisation of the frazil ice production rate (2.1). Individual parameter values are given by Pease and shown in table 3.1, these values were obtained through experimental trials [13].

Variable	Value	Units	Definition
T_a	-20	°C	air temperature
T_w	-1.8	°C	water temperature
V_a	20	ms^{-1}	offshore wind velocity
σ	5.67×10^{-8}	$Wm^{-2}deg^{-4}$	Stefan Boltzmann constant
e_a	0.95		emissivity of the air
e_w	0.98		emissivity of water
ρ_a	1.3	$kg\ m^{-2}$	air density
ρ_i	0.95×10^3	$kg\ m^{-2}$	ice density
C_h	2×10^{-3}		sensible heat coefficient
C_p	1004	$J\ deg^{-1}\ kg^{-1}$	specific heat of air
L	3.34×10^5	$J\ kg_{-1}$	Latent heat of fusion
H	0.1	m	Consolidated ice thickness
U	2% of wind speed	ms^{-1}	consolidated ice speed
u_R	3% of wind speed	ms^{-1}	frazil ice drift

Table 3.1: Model values for standard case, edited from [13]

In order to solve the equation to find the evolution of the polynya, we have to follow some simple steps repetitively to build up the progress of the polynya over a time window;

- calculate the frazil ice depth using(2.2)
- substitute this value into (3.1) and integrate directly with respect to t to find the position of the polynya edge

3.1 Analytic Solution

We have the following separable ODE once we substitute (2.2) into (3.1)

$$\frac{dR}{dt} = \frac{u_R HU - u_R FR}{u_R H - FR}, \tag{3.2}$$

which is separated and integrated to get

$$\int dt = \int \frac{u_R H - FR}{u_R HU - u_R FR} dR.$$

We make substitutions in for the constants to make the algebra easier to follow; by taking

$$\begin{aligned} a &= u_R H \\ b &= -F \\ c &= u_R HU \\ e &= -u_R F \end{aligned}$$

We now have

$$\int dt = \int \frac{a + bR}{c + eR} dR.$$

By separating the right hand integral and rearranging we find

$$\begin{aligned} \int dt &= \int \frac{a}{c + eR} dR + b \int \frac{R}{c + eR} dR \\ &= \int \frac{a}{c + eR} dR + \frac{b}{e} \int dR - \frac{bc}{e} \int \frac{1}{c + eR} dR \\ &= \frac{ae - bc}{e} \int \frac{1}{c + eR} dR - \frac{bc}{e} \int dR \end{aligned}$$

Integrating this directly we find

$$t = \frac{ae - bc}{e^2} ln(c + eR) - \frac{bcR}{e} + \beta.$$

Using the initial condition, $R = 0$ at $t = 0$ we find that the constant of integration is given by

$$\beta = -\frac{ae - bc}{e^2} ln(c)$$

We have

$$\begin{aligned} t &= \frac{ae - bc}{e^2} ln(c + eR) - \frac{ae - bc}{e^2} ln(c) - \frac{bcR}{e} \\ &= \frac{ae - bc}{e^2} \left(ln(c + eR) - ln(c) \right) - \frac{bcR}{e} \end{aligned}$$

Which we can now substitute our original variables into

$$\begin{aligned} t &= \frac{H(U - u_R)}{u_R F} \left(ln(u_R HU - u_R FR) - ln(u_R HU) \right) + \frac{FR}{u_R F} \\ &= \frac{H(U - u_R)}{u_R F} ln \left(\frac{HU - FR}{HU} \right) + \frac{R}{u_R} \end{aligned} \tag{3.3}$$

We note that this is an implicit solution for R. Plotting this function taking a wind speed of $20ms{-}1$, air temperature at $-20°$C, and we use (2.1) for the production rate giving $\approx 0.29 \times 10^{-6} ms^{-1}$. The frazil ice and pack ice velocities are 3% and 2% of the wind speed respectively.

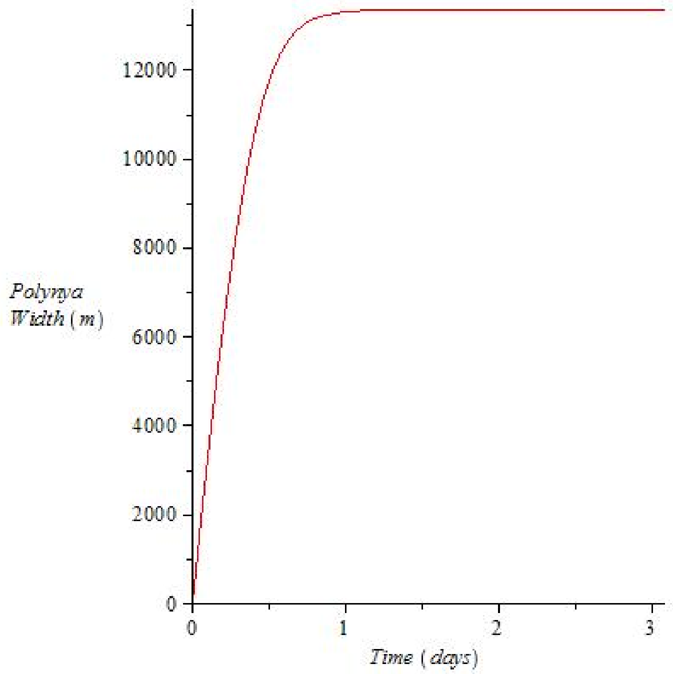

Figure 3.1: Analytical Solution to the steady state of a polynya opening using Ou's 1988 flux model with analytical solution (3.3)

We see that the width of the polynya will be approaching the steady state width (2.4) given to be

$$R = \frac{HU}{F} = 13357.52m.$$

The timeframe and the steady state width is what we expect from the literature, in Ou's concluding section of his paper he said the steady state was reached in less than two days [11]. Here our plot indicates that the steady state is being approached after just over a day from the initial forcing.

3.2 Numerical Solution

We use a Runge Kutta Fourth order scheme (RK4) to numerically integrate (3.1) and to determine the steady state polynya width. The algorithm for the RK4 is given below, the subscript, i, denoting the time at which we are calculating the position R_i which in turn denotes the position of the polynya edge at time t_i.

$$\begin{aligned} R_{i+1} &= R_i + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 &= f(R_i, t_i) \\ k_2 &= f\left(t_i + \frac{\Delta t}{2}, R_i + \frac{k_1 \Delta t}{2}\right) \\ k_3 &= f\left(t_i + \frac{\Delta t}{2}, R_i + \frac{k_2 \Delta t}{2}\right) \\ k_4 &= f\left(t_i + \Delta t, R_i + k_3 \Delta t\right). \end{aligned} \tag{3.4}$$

where the function f on the RHS of this formulation is the right hand side of the polynya flux equation (3.1)

$$f(R) = \frac{HU - h_R u_R}{H - h_R}.$$

with the frazil ice depth at the polynya edge given by

$$h_R = \frac{FR}{u_R}.$$

We see that the model given by Ou approaches the steady state width that we expect using a wind speed of $20ms^{-1}$. We also note that Ou's model shows the time taken for a steady state to be reached is between 1 and 2 days [11], Our numerical solution to the model equation (3.1) agrees with this time frame, and also confirms the analytic solution in Figure 3.1.

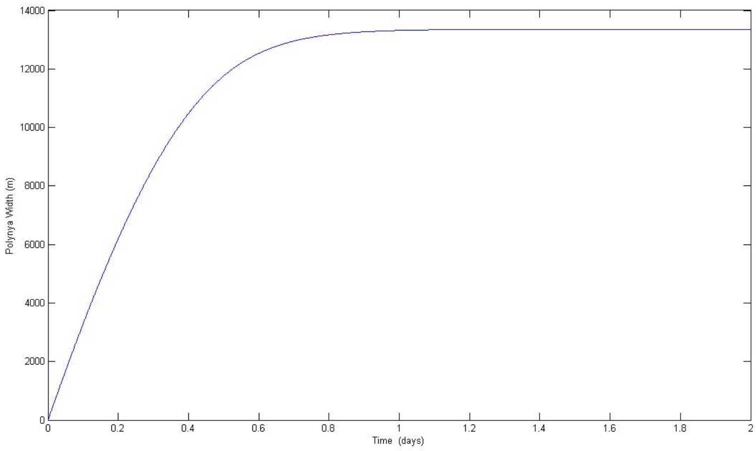

Figure 3.2: Runge-Kutta fourth order

This scheme is fourth order accurate, so that halving the time step should reduce the error between the approximation and the true solution by a factor of 16. We need to check that the solution we obtain for the RK4 method reflects this property. By taking a set time to integrate the RK4 scheme forward to and having a value that we keep as the true solution, successively halving the time step and taking the ratio of the difference between the base solution and the time step solution we will see the a common ratio that illustrates the accuracy of the scheme.

Δt	R (m)	Error	Ratio
100	11462.8772215	0	
200	11462.8772211	0.0000004	
400	11462.8772153	0.0000062	0.06451633
800	11462.8771205	0.0000101	0.06138614

Table 3.2: Accuracy table for RK4

Table 3.2 shows the width of the polynya after 12 hours calculated using successively smaller time steps, the ratio in the final column is roughly $\left(\frac{1}{2}\right)^4$. This is as expected for a fourth order accurate scheme. These values are taken with a wind speed of $20ms^{-1}$.

Figure 3.3 shows by varying the wind speed the size of the steady state width is largely unaffected, see table 3.3 in relation to the steady state width (2.4)). The rate of convergence to this steady state is however affected. The steady state polynya width is in the region of $13.5km$, but by increasing the windspeed the time taken to reach the steady state width is altered.

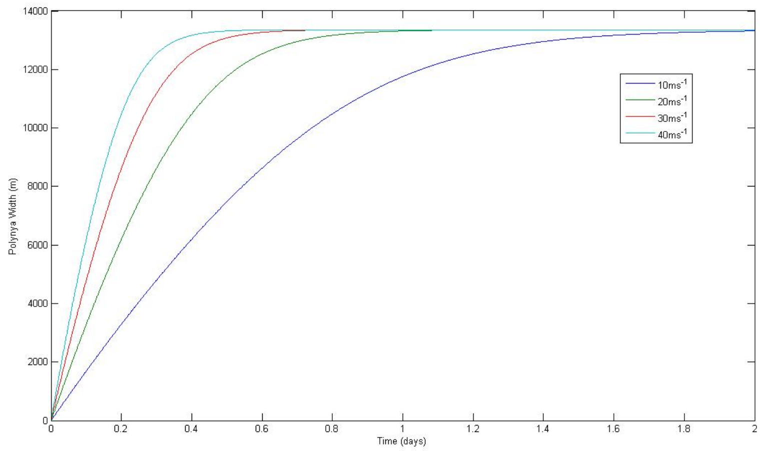

Figure 3.3: Runge-Kutta fourth order, showing the effect that varying the wind speed has on the polynya when temperature is kept constant at -20°C

By changing the air temperature in the production rate formula (2.1) we alter the steady state polynya width. Figure 3.4 illustrates this idea: lower air temperatures the size of the polynya is reduced as the production rate of the frazil ice increases, conversely for higher temperatures the steady state width of the polynya is increased. This would concurr with assessments Pease made about her model that the temperature has a greater influence than wind speed on the polynya response [13].

Wind speed (ms^{-1})	$F(ms^{-1})$	$u_R(ms^{-1})$	$U(ms^{-1})$	$\frac{U}{F}$
10	$1.50 \times 10{-}6$	0.3	0.2	133576.32
20	$2.99 \times 10{-}6$	0.6	0.4	133575.11
30	$4.49 \times 10{-}6$	0.9	0.6	133574.70
40	$5.99 \times 10{-}6$	1.2	0.8	133574.50

Table 3.3: Changes in velocities of the ice and the production rate as the wind increases

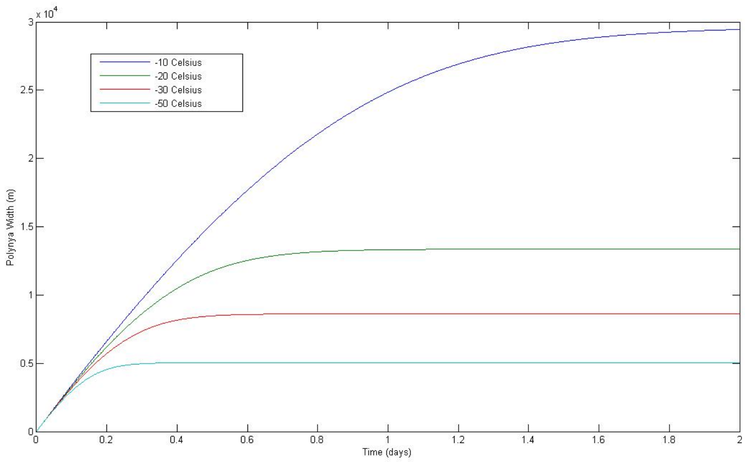

Figure 3.4: Runge-Kutta fourth order, showing the effect that varying the air temperature has on the polynya when wind speeed is constant $20ms^{-1}$

For the remainder of the discussion we keep the wind speed at $20ms^{-1}$, and the air temperature at -20°C.

Chapter 4

Diffusion of Ice

So far we have kept the consolidated ice thickness constant at $0.1m$. In reality this is a big simplification as there are many factors that will alter the depth of the pack ice. The model we have considered thus far (3.1), has kept the value of H constant. By allowing the frazil ice collection depth to evolve we can conceivably reach a point where depth of the frazil ice can exceed that of the consolidated pack ice, $h_R \geq H$. Problems arise in the current model because of a division by zero in (3.1). The motivation in changing the parameterisation of the collection thickness H lies in trying to reduce the number of tunable parameters in the model used to model the polynya.

Some subsequent amendments to the model parameterise the collection thickness H in such a way to keep the relation $h_R < H$, replacing H by a function of h_R, U and u. For example, in 2000 Biggs, Morales-Maqueda & Willmott [4] formulated a parameterisation of H that would be dependent on the frazil ice depth at time t, and the relative velocities of frazil and pack ice at the polynya edge, satisfying

$$H = h_R + c\,(U - u)^2 \tag{4.1}$$

where $c = 0.665m^{-1}s^2$. The parameterisation was used in simulations of a St Lawrence Island polynya, alongside those that used the constant valued ice thickness. Comparing the results they found that they were similar to the simpler constant thickness parameterisation. St lawrence Island lies in the Berring sea to the west of Alaska, polynyas often form here in the winter months and thanks to its recurrent nature it has been used to verify results from various polynya models, [4, 9, 13]. A proposed edit of (4.1) was given in 2004 by Biggs and Willmott [5], and an additional additive term (h_W) was added to account for an effect of wave radiation stress

$$H = h_W + h_R + c\,(U - u)^2 \tag{4.2}$$

This was now a more robust formulation, referring to how this parameterisation discounts possibilities of division by zero in (3.1), that allowed for modelling of unsteady polynya opening.

The formulations above are derived in their respective papers for a two-dimensional model where the velocities of the frazil ice and the pack ice are able to travel in different directions to that of the wind direction. The simplification to a one-dimensional model is stated above so that there is no need for the formulae to have a component which forces the relative velocities to be evaluated in the normal direction to the polynya edge [16].

These parameterisations of the collection thickness are taken at one point, the polynya edge, and is only influenced by the frazil ice in the polynya waters. We now include the mass of the pack ice into a formulation of the ice thickness at the polynya edge by considering ice diffusion.

4.1 Ice Equation

4.1.1 Shallow Ice Approximation

The flux models we have seen so far have all used a simple flux function for the consolidated pack ice, HU. An alternative to this simple flux function is to look at the pack ice as an active block of ice that is allowed to flow. We know that the consolidated ice sheet is shallow compared to its width. This naturally leads to the application of a shallow ice approximation, including diffusion within the ice sheet. We apply the derivation of the mass balance and the diffusion model velocity forice flow equation, given by Partdidge and Baines [12], to the pack ice in our polynya system. This ice flow is achieved by using the mass balance equation

$$\frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x} = m \tag{4.3}$$

where m is the depth averaged ice-equivalent accumulation rate, which is due to either thawing or additional ice from snow or freezing of water; however we will take $m = 0$ in our simple model. Here u denotes the speed of ice diffusion which using Glen's Law for the flow of ice [7] leads to,

$$u = -cH^{n+1}H_x^n. \tag{4.4}$$

Here $c = \frac{2A\rho^n g^n}{n+2}$ is a constant when we assume that the density ρ and temperatures are all constant. A is a function of temperature which is given by Van der Veen in table 4.1 obtained through laboratory experiments and observations [15]

$$A = A_0 exp\left(-\frac{Q}{RT} + \frac{3C}{(T_r - T)^k}\right)$$

Variable	Description	Value
A_0	constant	$9.302 \cdot 10^7 kPa^{-3}yr^{-1}$
Q	activation energy for creep	$78.8kJ/mol$
R	gas constant	$8.321J/(molK)$
C		$0.16612K^k$
T_r	0°C in Kelvin	$273.39K$
k		1.17

Table 4.1: Constants for temperature dependent constant within the flow rate given through Glen's Law

We will take c to be a constant value as well and as such we assume that the ice pack has a spatially invariant temperature. The value of the ice temperature affects the rate of diffusion, higher temperatures giving higher rates of diffusion. We will keep the ice temperature to be around $258.15K$ (-15°C).

We aim to find the evolution of the polynya by explicitly combining (4.3) and(4.4) in order to give an equation for the depth of the consolidated ice sheet at the polynya edge, i.e.

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(cH^{n+2}H_x^n\right). \tag{4.5}$$

This is a 2-point initial boundary value problem, and as such will require boundary conditions on either end. However additional conditions are required at moving boundaries. There will be a Dirichlet boundary condition $H = H_0$ on the left hand boundary, with H_0 specified. We also take a far field boundary condition on the right hand side (within the pack ice) such that $\frac{dH}{dx} = 0$. The boundary condition we specify for H_0 on the left hand boundary of the pack ice is to have the pack ice depth equal to the frazil ice depth, $H = h_R$. The frazil ice depth is calculated using exactly the same calculation as in Chapter 3.

We can take advantage of a rearrangement to get a simplification for the diffusion velocity (4.4). This is obtained through recognising that we have a derivative of a function which is simpler to work with.

$$u = -c(H^{\frac{n+1}{n}}H_x)^n = -c\left(\frac{n}{2n+1}\right)^n\left[(H^{\frac{2n+1}{n}})_x\right]^n \tag{4.6}$$

We always take $n = 3$ for ice flow [12]. Now the equations (4.5) and (4.6) for the diffusion of the ice sheet are given by

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(cH^5H_x^3\right) = \frac{\partial}{\partial x}\left(\alpha H\left(\frac{\partial}{\partial x}H^{\frac{7}{3}}\right)^3\right). \tag{4.7}$$

with $\alpha = \frac{27c}{343}$. We require $\left(H^{\frac{7}{3}}\right)_x > 0$ for there to be a non zero model velocity (4.6).

4.1.2 Application to the polynya situation

The flux model (3.1) is no longer usable after the added dimension given by the diffusion of the pack ice. There is a new velocity for the consolidated pack ice, which is as simple as taking the diffusion velocity away from the velocity given by the forcing wind we used in Chapter 3.

$$U_d = U - cH^5 H_x^3 \tag{4.8}$$

So using this form of the ice velocity we get the following polynya flux balance equation

$$\frac{dR}{dt} = \frac{HU - cH^5 H_x^3 - h_R u_R}{H - h_R}. \tag{4.9}$$

The replacement flux function is now more complicated than the previous one used, under the above assumption we have the flux function for the consolidated ice given by

$$H\left(\frac{dR}{dt} - U + \alpha H^4 H_x^3\right)$$

The boundary conditions are given in Section 4.1.1.

4.2 Steady State Solution

The inclusion of the diffusion term on the pack ice affects the maximum size that the polynya can reach. We expect that this width is now less than that of the simpler Ou model. The equation for the pack ice at the polynya edge is now

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left(HU - cH^5\left(\frac{dH}{dx}\right)^3\right) = 0. \tag{4.10}$$

For a steady state the time variation of the ice depth is zero, so now

$$\frac{\partial}{\partial x}\left(HU - cH^5\left(\frac{dH}{dx}\right)^3\right) = 0$$

which we can integrate directly with respect to x to find

$$HU - cH^5\left(\frac{dH}{dx}\right)^3 = constant. \tag{4.11}$$

We allow $x \to \infty$ to utilise our far field boundary condition within the ice pack which says that the spatial derivative is zero; $\frac{dH}{dx} = 0$. We are left with the constant being

$$H_\infty U \tag{4.12}$$

where the subscript ∞ denotes the pack ice thickness at the point where $\frac{dH}{dx} = 0$. This value of H_∞ is assumed to be known from some source, either from satellite data, physical readings, or even knowledge of physical processes. However at this stage we do not know the exact values for H_∞, so we choose a range of H_∞ values and see how this parameter affects the steady state width.

Now (4.12) can be substituted into (4.11) forming a non-linear ODE,

$$\frac{dH}{dx} = \sqrt[3]{\left(\frac{HU - H_\infty U}{cH^5}\right)}. \tag{4.13}$$

In order to calculate this steady state width for the diffusion model a shooting method will be used. First we set two values for the polynya width R, that will give us the initial values for integrating (4.13) forward using the RK4 scheme. These two values for R will be an upper and lower limits that will be refined through an iterative loop. The initial values are obtained through

$$H = \frac{FR}{U}.$$

The boundary condition on the right hand side is given by the choice of H_∞ We know what the depth of the frazil ice on the polynya edge will be given a certain polynya width R, so integrating forward from this point to find a corresponding H_b value (the depth of the pack ice at its far field boundary) is relatively simple. From here we can adjust the polynya width used to find the frazil ice depth at the polynya edge.Then repeat the process iteratively until the H_b value found lies within a certain distance from H_∞, at this point we have the steady state polynya width which relates to the specified H_∞.

It would not make sense to integrate backwards from the H_∞ mark to get to the polynya edge because we would be integrating from an unknown point to a known point, integrating from a known condition to find an unknown variable is preferable.

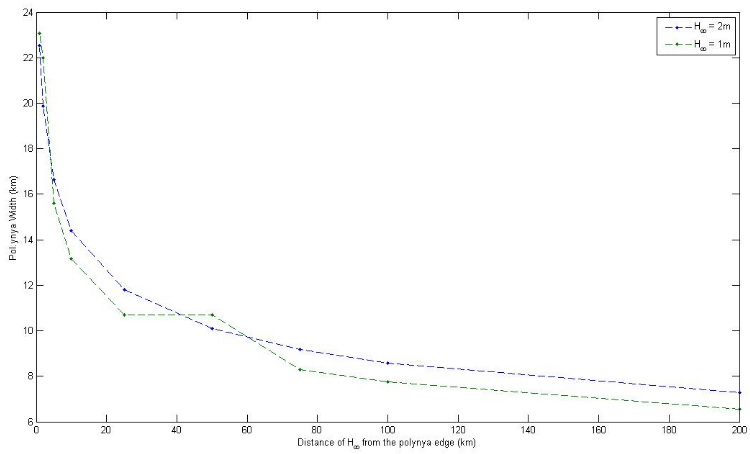

Figure 4.1: Steady state width with distance of H_∞ from the polynya edge

The values used for the constants in this steady state method are wind speed is $20ms^{-1}$, air temperatuere is $-20°$C, and we use (2.1) for the production rate F. The frazil ice and pack ice velocities are 3% and 2% of the wind speed respectively. To see how the steady state is affected we vary H_∞ between $2m$ and $3m$. Also the distance between the polynya edge and the far field boundary is varied between $1km$ and $200km$ to see how the length of the pack ice influences the steady state. The results of using these values are given in Figure 4.1. We see that as we take the width of the pack ice to be larger then the steady state width approaches some limit from above. This is not surprising because the ice will be in a steady state, so in order for the ice thickness to be equal to the frazil ice depth at the polynya edge the ice will need to have enough space to smothly reach a greater thickness towards the interior of the pack ice. Not only is the choice of H_∞ a key factor in determining the steady state width, the distance that we take this value to be from the polynya edge also influences the width.

Using the results from the plot it would appear that for a steady state to coincide with the corresponding one from (3.1) the value for H_∞ would need to be taken at roughly $20km$, ie that the pack ice should be 20km long.

4.3 Numerical Approach

4.3.1 Ice Discretisation

To solve (4.7) numerically we discretise into an explicit finite difference approximation on a fixed computational grid $\{x_i\}$, as in Figure 4.2.

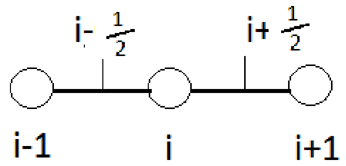

Figure 4.2: Spatial grid for the discretisation

The time derivative is discretised as

$$\frac{H_i^{k+1} - H_i^k}{t^{k+1} - t^k} \tag{4.14}$$

To discretise the spatial derivatives on the RHS of (4.7) we choose to evaluate the interior partial derivative of H on halfway points between the nodes

$$\frac{\partial}{\partial x}\left(cH^5 H_x^3\right) = \frac{\partial\left(f\right)}{\partial x} \tag{4.15}$$

this can be deiscretised to

$$\frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\frac{1}{2}\left(x_{i+1} - x_{i-1}\right)}.$$

now discretising the $f_{i+\frac{1}{2}}$ in 4.15 we have

$$\left(\frac{H_{i+1}^k + H_i^k}{2}\right)^5 \left(\frac{H_{i+1}^k - H_i^k}{x_{i+1}^k - x_i^k}\right)^3$$

similarly for the $f_{i-\frac{1}{2}}$

$$\left(\frac{H_i^k + H_{i-1}^k}{2}\right)^5 \left(\frac{H_i^k - H_{i-1}^k}{x_i^k - x_{i-1}^k}\right)^3$$

substituting these into (4.15) and putting it together with (4.14) we ghave the following discretisation

$$\frac{H_i^{k+1} - H_i^k}{t^{k+1} - t^k} = c\frac{\left(\frac{H_{i+1}^k + H_i^k}{2}\right)^5 \left(\frac{H_{i+1}^k - H_i^k}{x_{i+1}^k - x_i^k}\right)^3 - \left(\frac{H_i^k + H_{i-1}^k}{2}\right)^5 \left(\frac{H_i^k - H_{i-1}^k}{x_i^k - x_{i-1}^k}\right)^3}{\frac{1}{2}\left(x_{i+1} - x_{i-1}\right)}. \tag{4.16}$$

The superscript, k, indicates the time discretisation and the subscript, i, denotes spatial discretisation.

Figure 4.3 shows diffusion in a slab of ice 10m thick after 5000 time intervals of 100 seconds. The initial condition set on the slab of ice, although unrealistic it clearly shows the diffusion process, is a sheer face at the left hand edge with an infinite gradient.

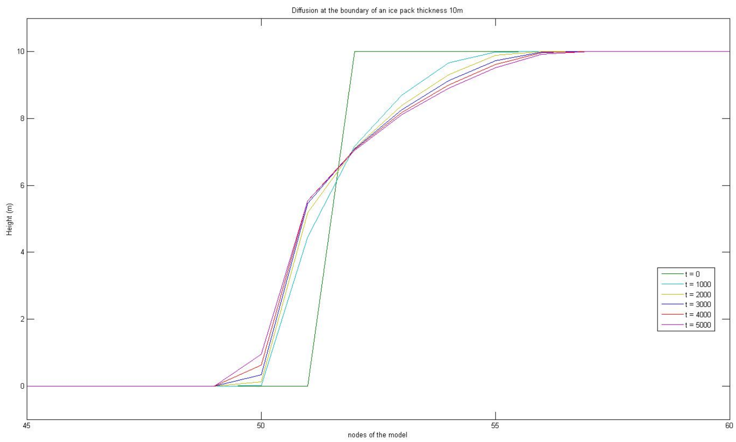

Figure 4.3: Diffusion in a 10m slab of pack ice.

A more realistic shape for the initial conditions is illustrated in Figure 4.4. Here a $2m$ slab of ice is positioned initially at the $500m$ point, with a parabola depicting the front of the ice at this edge, illustrates the form of diffusion we might expect in the thinner pack ice. the initial shape of the pack ice will be important in the evolution of the diffusion. As the ice diffuses it spreads at the base along towards the $0m$ mark.

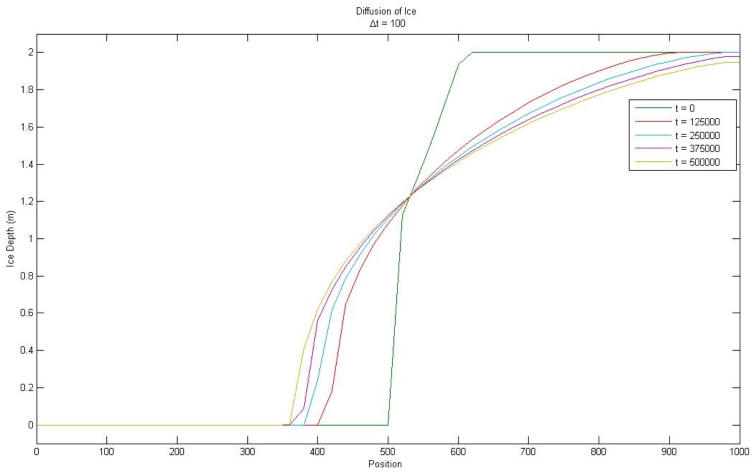

Figure 4.4: Diffusion in a 2m slab of pack ice with a parabolic front

The diffusion we see in the pack ice will be slower than shown in Figure 4.4, due to the thickness of the ice in the polynya model being less than a metre thick at the polynya edge. The two figures above show the expected form of the diffusion but the rate of diffusion will differ in the shallower ice. Also the boundary conditions we see at the polynya edge will mean that there will be a non zero thickness of consolidated ice at the polynya edge, whereas Figures 4.5 and 4.4 show a clear cut off where there is a zero ice thickness.

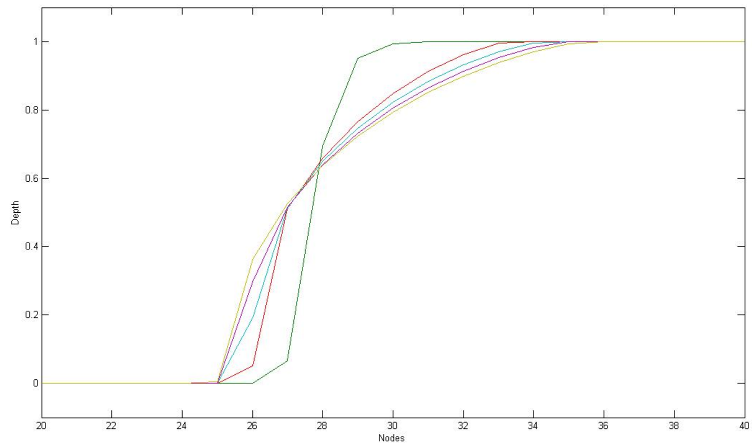

Figure 4.5: Diffusion in a 1m slab of ice which has a tanh curve defining the shape of the ice for its initial conditions

For Figure 4.4 a quadratic term depicts the front of the ice pack, but its shape is not continuous with the interior of the ice pack. A preferable initial condition is to have a scaled *tanh* curve for the ice depth throughout the pack ice at t = 0. This avoids there being discontinuities in the second derivative of the function at the join between the parabola and the constant line. This is shown in Figure 4.5. The diffusion here is similar but without the hidden discontinuous first derivative.

4.3.2 Polynya Model with Ice Discretisation

Polynya Diffusion Model

In order to model the 1D polynya which incorporates the diffusion equation for the pack ice as described in Section 4.1.2, we use the following algorithm at each time step. All the calculations are carried out using quantities found at the previous time step.

First we will calculate the frazil ice depth at the polynya edge using equation (2.2). This depth will be set as the values in the artificial region into which the ice pack will be allowed to diffuse using the discretisation (4.16). Then an averaged value over the first few nodes of the ice pack will be used to model the polynya opening using the RK4 scheme (3.4). The reason for the averaging is to pass a depth value to the flux model (4.9) and without a local average on the polynya edge the depth would be very close to zero because the diffusion velocity is very slow. This also allows us to evaluate the gradient of the ice at the boundary, which is required in the new velocity of the polynya edge. The new flux equation (4.9) will be used in the RK4 subroutine when the $k_{1,2,3,4}$ are called, which will advance the polynya edge forward.

This method has two way feedback between the polynya width and the diffusion of the pack ice The frazil ice depth, which is dependent on the polynya width, affects the diffusion in the form of boundary term on the polynya edge although it isn't a boundary it represents one because it occupies the artificial domain of the computational grid. In turn the diffusion rate affects the width of the polynya by reducing the velocity of the pack ice from U to $U_d = U - cH^5 H_x^3$.

Polynya Diffusion Results

The parameters that we keep constant are the velocity of the frazil ice, u_R, the production rate, F, and the velocity of the pack ice U. In addition the constants in table 4.1 for the parameterisation of the diffusion velocity prescribed through Glen's law for the flow of ice will also remain constant.

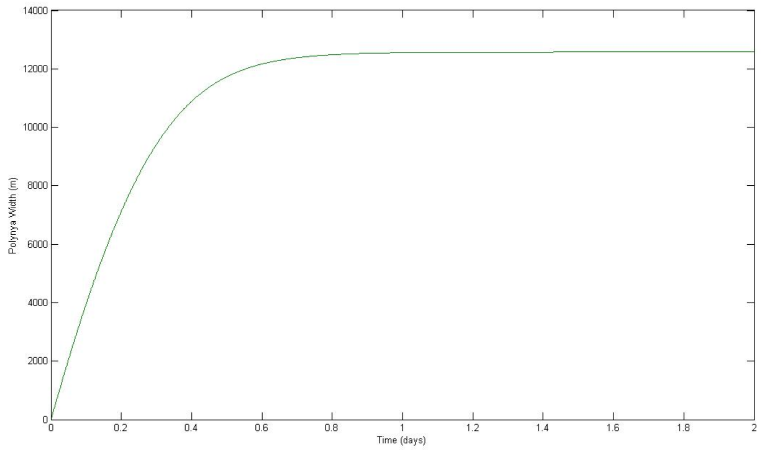

Figure 4.6: Polynya opening with diffusion in the pack ice, wind speed is $20ms^{-1}$

Figure 4.6 shows that with the inclusion of the ice equation modelling diffusion in the pack ice, the polynya opens up to a steady state, as expected. The time that the polynya takes to reach this steady state is similar to that in the simpler model with constant ice thickness, this time frame is between one and two days. The steady state width in this instance is just short of 12.5km. If we plot this against the polynya model with constant ice thickness we see how the ice diffusion affects the steady state width.

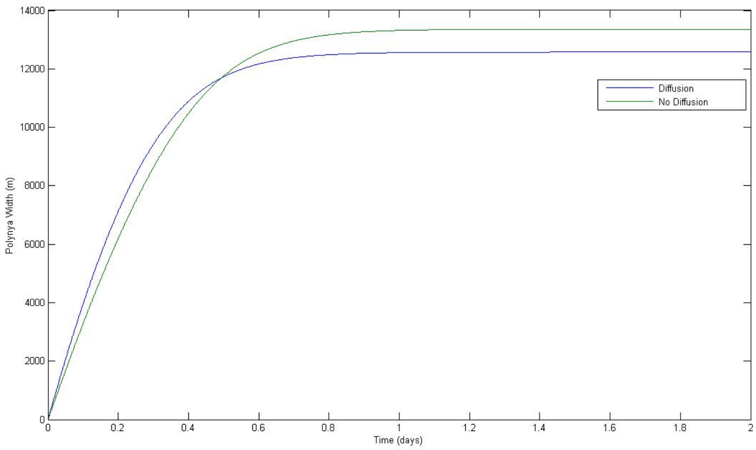

Figure 4.7: Comparative opening phase for a polynya with and without the ice diffusion included in the parameterisation, wind speed is $20ms^{-1}$

Over a time period of 2 days the diffusion of the ice has reduced the steady state of the polynya by a few hundred meters, Figure 4.7. This is intuitive since the pack ice is diffusing into the polynya waters, and in so doing it is reducing the size of the polynya.

An alternative to modeling the polynya evolution with a fixed grid for the pack ice is to model the diffusion of the pack ice using a moving mesh model with the velocity of the nodes defined by the pack ice wind speed U.

Chapter 5

A Moving Mesh Method

In the previous model of the diffusion equations we used a fixed grid discretisation. This is computationally simple although it stores artificial nodes outside the ice region into which the ice is able to diffuse. We now consider a moving mesh method which has moving boundarioes on both ends of the pack ice. This has the advantage of keeping storage to a minimum and will also compute the polynya boundary evolution as well as the diffusion of the ice. We use the velocity of the diffusion-polynya system (4.8) as a velocity for the moving mesh model. This method is described by Baines, Hubbard and Jimack in [1–3] using the normalised conservation moving mesh priciple we advance the diffusion and the polyna edge, defined here to be the edge of the ice sheet.

From (4.10) the thickness H of the pack ice satisfies

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left(HU - cH^5\left(\frac{dH}{dt}\right)^3\right) = 0 \tag{5.1}$$

Using (4.1) our boundary conditions on the left hand side of the consolidated ice pack on the polynya edge are

$$\begin{aligned} H &= h_R + 0.665(U - u_R)^2 \\ \frac{dx}{dt} &= \frac{h_R(U - u_R)}{0.665(U_d - u_R)^2} + U \end{aligned}$$

The second of these is a condition which gives a flux balance of the frazil ice and the consolidated ice. This is obtained by substituting the pack ice depth on the boundary given by (4.1) into the Ou flux balance equation (3.1). With an alteration to the (4.1) parameterisation to account for the diffusion velocity of the ice,

$$U_d = U - cH^5 H_x^3$$

On the right hand boundary of the pack ice we have the boundary conditions,

$$\begin{aligned} \frac{dH}{dx} &= 0 \\ \frac{dx}{dt} &= U. \end{aligned}$$

The total mass within the ice block is

$$\theta(t) = \int_{a(t)}^{b(t)} H(x,t)dx \tag{5.2}$$

where a(t) and b(t) are the polynya edge and the far field boundary of the pack ice, respectively. Differentiating equation (5.2) with respect to time using Leibniz integral rule we have

$$\begin{aligned} \dot{\theta} &= \frac{d}{dt}\left(\int_{a(t)}^{b(t)} H(x,t)dx\right) \\ &= \int_{a(t)}^{b(t)} \frac{dH}{dt}dx + \left[H\frac{dx}{dt}\right]_{a(t)}^{b(t)} \\ &= \int_{a(t)}^{b(t)} \frac{d}{dx}\left(cH^5\left(\frac{dH}{dx}\right)^3\right)dx + \left[H\frac{dx}{dt}\right]_{a(t)}^{b(t)} \\ &= \left[cH^5\left(\frac{dH}{dx}\right)^3 + H\frac{dx}{dt}\right]_{a(t)}^{b(t)} \\ &= H_{b(t)}U - \left[cH^5H_x^3 + H\left(\frac{h_R(U-ur)}{0.665(U_d-u_R)^2} + U\right)\right]_{a(t)} \\ &= H_{b(t)}U - \left[cH^5H_x^3 + \frac{Hh_R(U-uR)}{0.665(U_d-u_R)^2} + HU\right]_{a(t)} \\ &= U\left(H_{b(t)} - H\right) - cH^5H_x^3 + \frac{Hh_R(U-u_R)}{0.665(U_d-u_R)^2} \qquad (5.3) \\ &\neq 0 \end{aligned}$$

We drop the subscript $a(t)$ now and we have H evaluated at the polynya edge unless otherwise stated.

We use a normalised conservation moving mesh principle, defining $x(t)$ by

$$\frac{1}{\theta(t)} \int_{x(t)}^{b(t)} H dx = \gamma(x, b), \tag{5.4}$$

taken to be constant in time and therefore determined by the initial condition. $\dot{\theta}$ is as defined in (5.3) so that if we have $x(t) = a(t)$ then $\gamma(a, b) = 1$. This $\gamma(x, b)$, represents a fraction of the mass that lies between the x node and the far field boundary, it is constant in time.

Then, differentiating (5.4) with respect to t we have

$$\begin{aligned} \frac{d}{dt}\left(\gamma(x,b)\theta(t)\right) = \gamma(x,b)\dot{\theta} &= \frac{d}{dt}\int_{x(t)}^{b(t)} H dx = \int_{x(t)}^{b(t)} \frac{\partial H}{\partial t} dx + \left[H \frac{dx}{dt}\right]_{x(t)}^{b(t)} \\ &= \left[cH^5 \left(\frac{\partial H}{\partial x}\right)^3 + H\frac{dx}{dt}\right]_{x(t)}^{b(t)} \\ &= [HU]_{b(t)} - \left[cH^5 H_x^3\right]_{x(t)} - \left[H\frac{dx}{dt}\right]_{x(t)} \end{aligned}$$

this leaves us with

$$\gamma(x,t)\dot{\theta} = [HU]_{b(t)} - \left[cH^5 H_x^3\right]_{x(t)} - \left[H\frac{dx}{dt}\right]_{x(t)} \tag{5.5}$$

By rearranging equation (5.5), we find an explicit expression for the spatial derivative,

$$\frac{dx}{dt} = \frac{H_b U - \gamma(x,b)\dot{\theta}}{H} - cH^4 H_x^3 \tag{5.6}$$

The discretisation of this will use the simplification (4.6) to get a simpler form

$$H^4 H_x^3 = \left(H^{\frac{7}{3}} H_x\right)^3 = \frac{27}{343}\left(H^{\frac{7}{3}}\right)_x^3$$

to get the new spatial positions by systematically finding

$$X_i^k = X_i^{k-1} - \Delta t \left(-\frac{H_b U - \gamma(x_i, b)\dot{\theta}}{H_i} + \frac{27c}{343}\left(\frac{H_i^{\frac{7}{3}} - H_{i-1}^{\frac{7}{3}}}{X_i - X_{i-1}}\right)^3 \right) \tag{5.7}$$

The k superscript denotes the time step. We need to update the mass θ in the system because (5.3) is non-zero, so we use

$$\theta^{k+1} = \theta^k + dt * \dot{\theta}.$$

This ensures that when we calculate the depth of the pack ice we keep the ratio of the mass in the cells to the total mass in the ice sheet constant. The depth (H_i) at the nodes (X_i) will be calculated after their positions have been found, the depths are found by using a discretisation of the conservation principle (5.4), in this case

$$H_i = \frac{\theta^{k+1}}{\theta^0} \frac{\gamma(x_i, b)}{(X_{i+1} - X_{i-1})} \tag{5.8}$$

θ^0 represents the mass of the ice pack at the initial time. The ratio in the θ quantities scales the γ ratio to give the correct depth values such that the mass in the nodes remains at the same fraction of the total mass in the system.

The algorithm that we will follow to calculate the new position and depth of the ice at the nodes is as follows. First calculate the values for the $\gamma(x_i, b)$ constants at each node at time $t = 0$ using (5.4). Then calculate new positions (5.7) and finally we calculate the depth of the ice using the mid-point rule (5.8).

Using this method we model the polynya evolution by calculating the distance the first node of the ice pack has travelled, and it will also calculate the diffusion within the ice sheet. We have two way feedback in this model, just as was the case witgh the fixed grid model. The diffusion of the pack ice affects the rate of growth for the polynya, and in turn this affects the diffusion by means of the frazil ice depth providing conditions for the diffusion at the polynya edge.

5.1 Initial Moving Mesh Results

For the moving grid model we expect the polynya edge to be moving with the same behaviour as in the fixed grid model. We expect the diffusion in the pack ice to behave in a similar manner to the results seen in Figures 4.4,4.5.

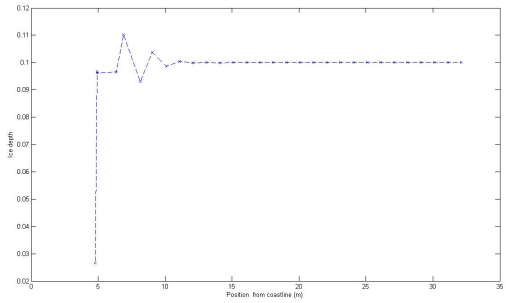

Figure 5.1: 12 second Movving grid instability

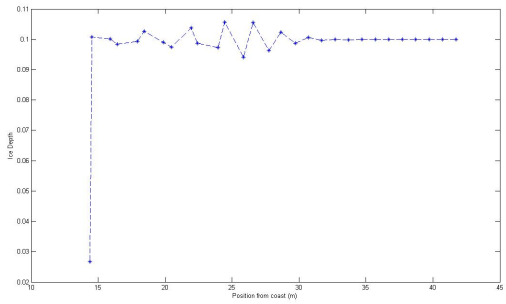

Figure 5.2: 36 second Movving grid instability

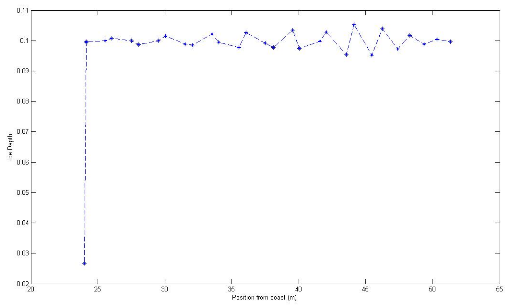

Figure 5.3: 60second Movving grid instability

These initial steps in the moving grid model show that there is instability in the diffusion, with $\Delta t = 0.01s$. This could be due to the discretisation grid being too coarse. This could be investigated by looking at the CFL condition, in order to control the size of these steps. The oscilations in the depth of the pack ice is quite noticable as the pack ice moves away from the coastline, Figures 5.1, 5.2, 5.3 show the increased oscoilations .

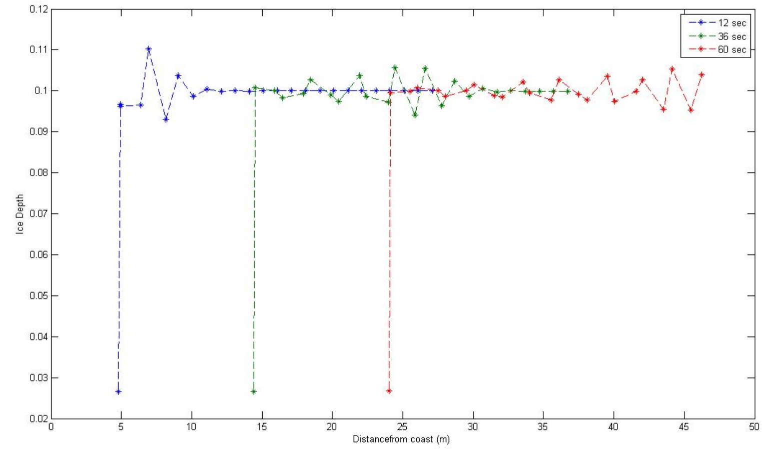

Figure 5.4: Initial sixty seconds of the moving grid model, illustrating the instability.

The behaviour of the polynya edge seems to be what we expect at this stage, the pack ice is moving away from the coastline at a rate which is the same as the ice pack velocity $0.4ms^{-1}$ for a wind speed of $20ms^{-1}$. The depth of the frazil ice is not great enough to slow the advancing polynya edge.

The oscilations in the pack ice are going to influence the positions of the nodes, through 5.7 so the issue of the instability will need to be resolved before the steady state width reached. Neither are we going to see the correct diffusion profile of the ice until the oscilations at this early stage are resolved.

5.2 A change in Boundary Condition for the fixed grid model

While formulating the moving grid approach it became apparent that the boundary condition imposed on the polynya edge for the depth of the pack ice incurred an enforced division by zero in the flux balance equation used to determine the polynya edge position. To avoid this disasterous result we change the boundary condition in Chapter 4, such that we conform with the parameterisation (4.1) of the ice thickness given by Biggs et al [4].

$$H = h_R + 0.665(U - u_R)^2$$

This gives a new flux balance equation when we substitute the new H into (3.1)

$$\begin{aligned} \frac{dR}{dt} &= \frac{(h_R + c(U_d - u_R)^2)U - h_R u_R}{h_R + c(U_d - u_R)^2) - h_R} \\ &= \frac{h_R(U - u_R)}{c(U_d - u_R)^2} + U \end{aligned} \tag{5.9}$$

We are not expecting a substantial change from the results shown in Figures 4.6, 4.7 since the conclusions about the parameterisations say the new thickness parameterisation give results which are very similar to those found when a constant ice thickness is assumed [4,5]. We would hope that the same would apply here when we use the parameterisation in the diffusion model (5.9).

For windspeeds of $20ms^{-1}$ the change in parameterisation will add $0.026m$ to the ice thickness on the polynya edge, this shouldn't make a drastic impact on the diffusion. Although for higher windspeeds the difference between the two velocities will be greater so the effect on the thickness of the ice will be greater, i.e. for $30ms^{-1}$ the changed boundary condition will add $0.059m$ to the ice thickness on the polynya edge. We now use this boundary condtion for H on the polynya edge of the pack ice and compare them to those in Section 4.3.2.

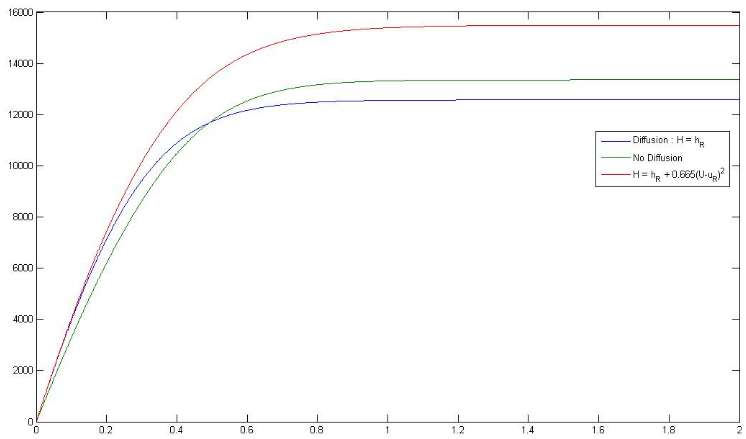

Figure 5.5: Comparative opening phase for a polynya with and without the altered boundary condition, wind speed is $20ms^{-1}$

Figure 5.5 illustrates how the new boundary condition for the diffusion equation affects the polynya's steady state, rather than the polynya opening to just under $12.5km$ the polynya opens to just over $15km$, which is not as close to the simple polynya flux solutions presented in Chapter 3.

Chapter 6

Conclusions

In Chapter 2 we reviewed the literature regarding one dimensional polynya flux models. The model presented by Ou in 1988 was then evaluated both analytically and numerically in Chapter 3. A Runge Kutta fourth order scheme was used to numerically evolve the opening of a polynya to its steady state, and the results here conformed to the expected results given in the literature. In Ou's paper he observed that the model had a few shortcomings that were due to lack of understanding; the collection thickness of the pack ice was kept constant for this model, the wind speed was kept spatially invariant and also there is no lag between the forcing and the velocities of the ice. The last two of these simplifications were left alone. However the first of the shortcomings described in Ou's conclusions about his model was approached in Chapter 4 where we looked at a different parameterisation for the collection thickness H which modelled diffusion in the pack ice.

The shallow ice approximation was used to model the consolidated ice pack allowing for diffusion, where the diffusion velocity was taken from Glen's law for the flow of ice. The diffusion model was discretised explicitly and solved for a couple of simple sets of initial cnditions for the shape of the ice pack. The change in initial conditions enabled the behaviour of the diffusion to be examined to see how changing initial conditions for the ice pack influenced the form of the diffusion: this is something that would need to be investigated in order to ensure that the results obtained correlate with the physical world. For simplicity, the initial conditions applied to the pack ice modelled by a parabola near the left hand edge of the ice, which flattened out similar to a constant. The discontinuity in the first derivative was avoided by using a *tanh* curve.

The ice equation described in [12] was applied to the pack ice in the polynya model, thus changing the flux balance equation that Ou had presented, this new model was numerically modelled in Chapter 4. The velocity of the pack ice at the polynya edge is reduced from U, given in the Ou model, to a reduced velocity; $U_d = U - cH^5 H_x^3$. The resulting change in the polynya evolution is that the steady state width is reduced. In addition to the changed flux equation, new boundary conditions were applied to the pack ice. Following from parameterisations of H given by Biggs *et al* in 2000 and 2004, the pack ice depth on the polynya edge was set to be $H = h_R + 0.665(U - u_R)^2$.

This change in flux model lead to a different steady state width for the polynya, Section 4.2. However, due to lack of knowledge of the ice thickness far away from the polynya edge, we were not able to produce an exact steady state width for the polynya. A range of values for the steady state width was caculated given a range of depths of the pack ice at the far field boundary of the pack ice. These widths are dependent on how far from the pack ice the far field boundary is taken to be, Figure 4.1 illustrates that by increasing the length of the pack ice we see a reduction in the steady state polynya width.

Numerical modelling the new polynya model with the inclusion of the diffusion parameterisation for the consolidated ice thickness was approached in two ways, using a fixed grid and using a moving grid. The fixed grid method was applied in Section 4.3.2, which involved inclusion of an artificial region outside of the pack ice that would cater for the diffusion of the ice. The boundary condition that was applied required an alteration in how the polynya evolution is modelled. Instead of taking the pack ice depth at the polynya edge, an averaged value for H over a small area of the pack ice on the boundary was used, which avoided the division by zero that would have undoubtedly spoiled the results. This is not however a perfect fix for this problem since, by allowing a possibility of a division by zero the boundary condtion itself is flawed. Another issue also arose in that how far in to the pack ice do you take an averaged thickness for the depth on the polynya edge in order to run the polynya model.

The results from running the diffusive model with a fixed grid showed that the polynya evolution was slowed down and the steady state width was reduced by a couple of hundred meters. This is not unexpected since the diffusion will be acting in the opposite way to the motion of the polynya edge. In our diffusion model, width of the polynya is dependent the averaged depth of the pack ice at the polynya edge, this means that the size of the area that we take the average on affects the polynya width.

In Chapter 5 a different approach was used to model the polynya evolution. Rather than keeping the nodes in the numerical shceme fixed in place, the positions of the nodes were allowed to vary. The velocity for the moving mesh model was obtained through the new velocity of the pack ice at the polynya edge, U_d. By using a normalised conservation moving mesh principle we kept the mass in between the nodes to be time invariant. This approach doesn't look at the polynya edge primarily as the important piece of information: it is involved in calculating the position of the first node of the pack ice but the majority of the work involved modelled the diffusion of the ice on the moving grid.

The moving mesh model will need to be investigated further, since the results obtained from it are not what we would expect. The reasons for the unsatisfactory results could be from using a time step which is too large, this can be investigated by looking at the CFL condition for the model. The velocity of the polynya edge appears to be consistent with the results form the fixed grid model, however with the instability in the depth calculations this needs to be resolved fully before a steady state is reached.

By considering diffusion in the pack ice, an alternative flux balance model was produced and used to model the evolution of a Latent heat coastal polynya to its steady state. The effect that the added diffusion parameterizatin had on this steady state width is hard to interpret because of the difficulties in having initial starting values for the shape of the pack ice. Also the effect that the change in the shape of the pack ice would have on the bouyancy of the pack ice has been ignored for the purpose of this model; the Archimedean property could be a factor in the depth of the ice at the polynya edge when diffusion is parameterised.

Chapter 7

Further Work

In addition to investigating the current instabilities with the moving grid model there are natural extensions and further work that could be looked at, some of these are detailed below along with possible methods as to how they could be approached.

7.1 Seasonality

With the ice thickness parameterisation we could add in a seasonality term to add an effect that time of year has on the polynya modelling process. There are various ways we can do this to affect the diffusion rate. An additive constant dependent on time would be one way we could affect the diffusion. A scaled *sine* or *cosine* curve would be a sensible starting point to explore how the diffusion reacts, with the zeros on the curve indicating the start and end of winter and summer.

However the inclusion of the seasonality effect on the model may be largely unproductive since the lifespan of the opening phase of a coastal polynya is less than a few days and having a factor that changes due to effects that vary in term of months rather than days would not make a great deal of difference. Also since polynyas are predominantly a winter phenomenon.

Including a variable to account for weather patterns could be included to allow for the effects that a snow storm or warmer weather would have on the ice equivalent accumulation rate in equation (4.3). This would affect the temperature of the ice in the constant affecting the diffusion rate in Glen's Law (4.4).

7.2 Application to polynya closing models

Thus far the models we have been concerned with are all modelling the polynya opening to a steady state, we could apply the ice equation (4.5) to polynya closing models such as the one presented by Teal, Willmott, Biggs & Morales Maqueda [14]. This could be done in a similar manner to how we have done with the opening models. In this case we would expect the diffusion to aid the process of closing the polynya.

7.3 Extension to Two Dimensions

In the One-Dimensional models we considered a cross sectional view of the polynyas, but we could extend the diffusion term added to the pack ice into models which look at the two-dimensional polynya process.

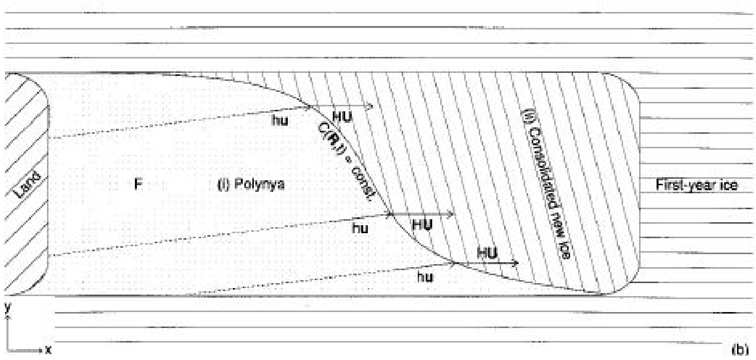

Figure 7.1: Two-dimensional polynya model diagram, adapted from [9]

This would allow us to look at the shape and the area of the polynya rather than the width at one point. A 2D time dependent flux model is presented in 2000 by Morales-Maqueda & Willmott (MMW) [9] that requires specification of the coastal boundary, time varying surface wind, shortwave radiation, air temperature and relative humidity. This is developed from previous two dimensional models given by Darby, Willmott and Somerville in 1995 [6] and Willmott, Morales Maqueda and Darby in 1997 [17]. This is a generalisation of the 1D model set out in the 1988 Ou paper: the generalisation to two dimensions given by

$$\nabla C \cdot \frac{H\mathbf{U} - h_C \mathbf{u}_C}{H - h_C} + \frac{\partial C}{\partial t} = 0$$

Now **U**= (U,V) is the velocity of the pack ice, and **u** = (u,v) is the velocity of the frazil ice. C is the curve C(**R**,t) where **R** = (X,Y), is the curve of the polynya edge. MMW take the initial position of the polynya to be the time when the polynya first opens, ie C depicts the coastline at time $t = 0$. The characteristic curves of this equation satisfy

$$\frac{d\mathbf{R}}{dt} = \frac{H\mathbf{U} - h_C \mathbf{u}_C}{H - h_C} \tag{7.1}$$

This model is solved considering the characteristic curves of the frazil ice and the pack ice separately, these differ because the directions of motion of the two is now dependent on more factors than wind speed. The RK4 scheme could be used to integrate forward along these characteristics, updating the positions along the frazil ice trajectory and the pack ice trajectory. Applying the diffusion model to the two-dimensional polynya model would be difficult since the polynya edge is now a curve rather than a point, and the diffusion would need to be in a normal direction to this curve. So modelling the diffusion would require modeling of the surface of the pack ice in two-dimensions before applying it to a polynya model.

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