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Boundary Element Method for Heat Transfer in a Buried Pipe

by

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Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Elena Panti.

Abstract

In this dissertation we explore the Boundary Element Method for heat transfer in a buried pipe. We are interested in modelling the steady-state heat transfer from buried pipes. We are studying the temperature through Laplace's equation. First, we consider the interior and the exterior problem and then we move on to the full pipe problem.
In the interior problem we solve the problem inside a circle. In the exterior problem we solve the problem outside the bounded domain and because the domain is a circle therefore we solve the problem outside the circle. For the full pipe problem we solve the problem outside the circle in a half plane.
The boundary integral method tells us the value of the temperature on the pipe and on the ground surface. From there we can deduce the temperature anywhere below the ground surface.
The theoretical results are supported by our numerical results.

Contents

Chapter 1

Introduction

This dissertation explores the Boundary Element Method for the Heat Transfer in a buried pipe. Heat transfer occurs due to temperature difference between the pipeline fluid and the ambient fluid which is air or water, overcoming thermal resistances offered by the pipe, coatings and ground.
The state of the fluid (oil, liquid or gas) i.e. the density and the viscosity of the fluid, is defined by the temperature.

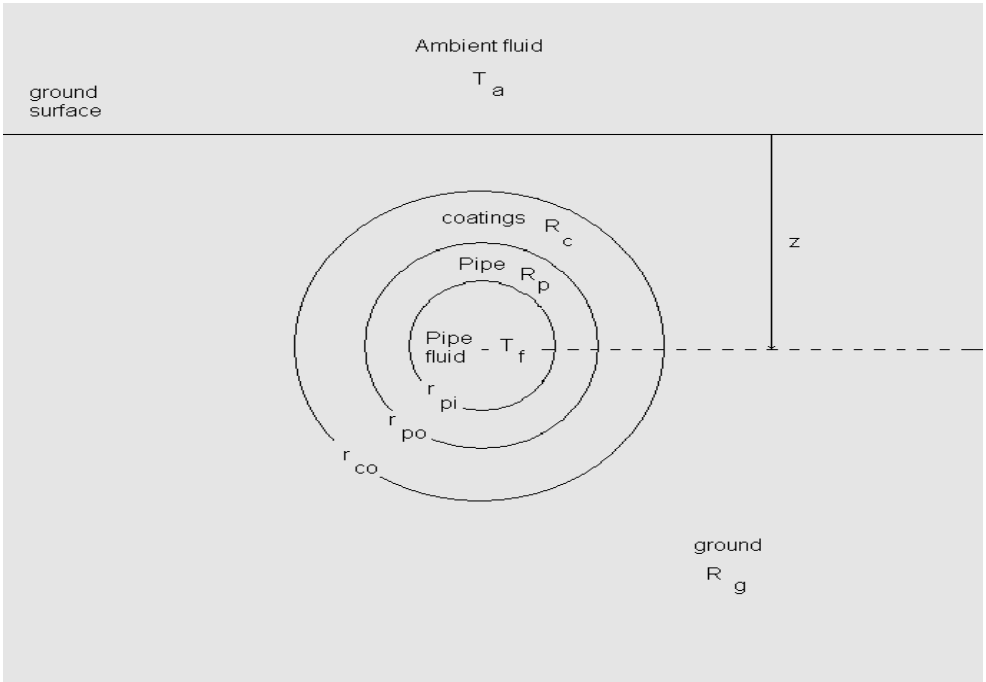


Figure1: Buried pipe (source: Partially Buried Pipe Heat Transfer (Powerpoint), Chuk Ovuworie from Schlumberger Company)

r_{pi} =radius of the pipe inside
r_{po} =radius of the pipe outside
r_{co} = radius of the coating outside
T_a =temperature of the ambient fluid
T_f =temperature of the pipeline fluid
R_p =resistance of the pipe
R_c =resistance of the coatings
R_g =resistance of the ground
z=distance from the centre of the pipe to the ground surface

We can express, at steady state, the rate of heat flow between the pipeline and the ambient fluid as:

$$Q = -2\pi r_{po} U_{po} (T_f - T_a)$$

where

U_{po} = overall heat transfer coefficient

$2\pi r_{po}$ = pipe outside surface area

$T_f - T_a$ = temperature difference between the pipeline fluid and the ambient fluid

At steady state the rate of heat flow (Q) is the same through each of the thermal layers.

We can also write Q across each thermal layer as:

$$Q = \frac{\Delta T}{R}$$

where

ΔT =temperature difference across the layer
R= thermal resistance offered by the layer

The temperature difference between the pipeline fluid and inside wall is:

$$T_f - T_{pi} = -\frac{Q}{2\pi r_{pi} h_{pi}} \tag{1.1}$$

where

h_{pi} = the pipe inside film coefficient

$2\pi r_{pi}$ = pipe inside surface area

T_f =temperature of the pipeline fluid

T_{pi} =temperature of the pipe inside

The temperature difference between the pipe inside and outside walls is given by the following equation:

$$T_{pi} - T_{po} = \frac{\ln(r_{po} / r_{pi})}{2\pi k_p}$$

where

r_{pi} = radius of the pipe inside

r_{po} = radius of the pipe outside

k_p =the pipe thermal conductivity

T_{pi} = the temperature of the pipe inside

T_{po} = the temperature of the pipe outside

The easiest way to approach this problem is to assume radial symmetry.

In this case, simplifying the problem within the pipe to one-dimension, dependent only on r we can easily solve the boundary value problem using traditional techniques. Below are the assumptions made in this analysis:

Assumptions:

- Heat flow, denoted by Q (radial heat flow per length of pipe), is radially symmetric within the pipe and the coatings such that T=T(r).
- Heat flow is in a steady state (dQ/dt = 0)
- Conservation of heat energy reduces down to Laplace's equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = 0 \quad \longrightarrow$$ It is Laplace's equation, but now the $\frac{\partial T}{\partial \theta}$ term has disappeared.

(where r, θ are polar coordinates on the centre of the pipe).

Solving Laplace's equation without the $\frac{\partial T}{\partial \theta}$ term, we obtain the general solution for temperature:

$$T(r) = A \ln r + B \tag{1.2}$$

where A ,B are constants.

We do not need to worry that temperature relies on $\ln r$ since we know the temperature in the centre of the pipe already (T_f) and thus we will never have to compute the temperature at r equal to zero, where the solution breaks down.

For the pipe layer, the boundary conditions are:

- $\left.\frac{\partial T}{\partial r}\right|_{r=r_{pi}} = -\frac{h_{pi}\left(T_f - T_{pi}\right)}{k_p}$
- $T(r_{po}) = T_{po}$

We know that $T(r) = A \ln r + B$ .Then by using the boundary condition $T(r_{po}) = T_{po}$ we have :

$$T(r_{po}) = A \ln r_{po} + B = T_{po} \tag{1.3}$$

Now we want to find the constants A and B, using the boundary conditions.

$$T(r) = A \ln r + B \Rightarrow \frac{\partial T}{\partial r} = \frac{A}{r}$$

$$\Rightarrow \left.\frac{\partial T}{\partial r}\right|_{r=r_{pi}} = \frac{A}{r_{pi}}$$

and by using the 1st boundary condition $\left.\frac{\partial T}{\partial r}\right|_{r=r_{pi}} = -\frac{h_{pi}\left(T_f - T_{pi}\right)}{k_p}$

we have:

$$\frac{A}{r_{pi}} = -\frac{h_{pi}(T_f - T_{pi})}{k_p}$$

$$\Rightarrow A = -\frac{r_{pi}h_{pi}(T_f - T_{pi})}{k_p} \tag{1.4}$$

From (1.1) we have:

$$-(T_f - T_{pi})r_{pi}h_{pi} = \frac{Q}{2\pi} \tag{1.5}$$

Substitute (1.5) into (1.4) we have:

$$A = \frac{Q}{2\pi k_p}$$

From (1.3) $B = T_{po} - A\ln r_{po} = T_{po} - \dfrac{Q}{2\pi k_p}\ln r_{po}$

Finally, substitute A and B in (1.2) ,we have the following solution for the pipe layer:

$$T(r) = \frac{Q}{2\pi k_p}\ln\left(\frac{r}{r_{po}}\right) + T_{po}$$

The advantage of this method is that it relies on simple and easy to solve Ordinary Differential Equations. The problem with it, is that it over simplifies the situation, and does not take into account what is happening exterior to the pipe coating.

In particular if we consider the fact that the pipe is buried at a finite depth, or even only partially buried, then the solution will certainly not be radially symmetric. In this dissertation we consider this more complicated case.

The temperature of the ground with increasing depth (discounting the presence of the pipe) is given by:

$T(y) = gy + T_g$ .This is an asymptotic condition as y$\to \infty, x \to \pm\infty$. (1.6)

In this dissertation we begin by introducing the Boundary Element Method and we separate our problem in three stages:

(1) Interior problem (bounded problem)

(2) Exterior problem (unbounded problem)

(3) Full problem.

In the interior problem we will solve the problem inside a circle. In the exterior problem we will solve the problem outside a circle. For the full pipe problem we will solve the problem outside the circle in a half plane.
In this dissertation, we will approach analytical and numerical solutions for each problem.

Chapter 2

Boundary Element Method

The Boundary Element Method (BEM) has been applied to a variety of heat transfer problems in the last thirty years. Initial applications of the method were for steady heat conduction problems described by Laplace's equation [9].
The BEM is a method for solving Partial Differential Equations by reformulating as boundary integral equations and then solving them. Moreover, the boundary element method is derived through the discretisation of an integral equation that is mathematically equivalent to the original partial differential equation. The essential re-formulation of the PDE that underlies the BEM consists of an integral equation that is defined on the boundary of the domain and an integral that relates the boundary solution to the solution at points in the domain.
The Boundary Element Method is often referred to as the Boundary Integral Method (BIM) or Boundary Integral Equation Method.

The advantages in the BEM arise from the fact that only the boundary (or boundaries) of the domain of the PDE requires sub-division. Thus, the dimension of the problem is effectively reduced by one, for example an equation governing a three-dimensional region is transformed into one over its surface.
In cases where the domain is exterior to the boundary the extent of the domain is infinite and hence the advantages of the BEM are even more remarkable; the equation governing the infinite domain is reduced to an equation over the finite boundary.

2.1 Reformulation of a Partial Differential Equation as a Boundary Integral Equation

The main properties of potential functions ($\nabla^2\phi = 0$) can be derived from Gauss' Theorem (divergence theorem) and its corollaries (Green's identities).

(The partial differential operator, ∇^2 or Δ is called the Laplace operator, or just the Laplacian).

Gauss Theorem [9]:

Let V be a region in space bounded by a closed surface S and F be a vector field acting on this region.
The divergence theorem establishes that the total flux of the vector field F across the closed surface S must be equal to the volume integral of the divergence of this vector:

$$\int_S F_i n_i dS = \int_V \frac{\partial F_i}{\partial x_i} dV$$

where n_i=components of the unit vector normal to the surface S

Green's first identity:

By substituting $F_i = \phi \frac{\partial \psi}{\partial x_i}$ into Gauss' Theorem, we have:

$$\int_S \phi \frac{\partial \psi}{\partial x_i} n_i dS = \int_V \frac{\partial}{\partial x_i}\left(\phi \frac{\partial \psi}{\partial x_i} \right) dV \tag{2.1}$$

Then we use the chain rule which give us:

$$\frac{\partial}{\partial x_i}\left(\phi \frac{\partial \psi}{\partial x_i} \right) = \frac{\partial \phi}{\partial x_i}\frac{\partial \psi}{\partial x_i} + \phi \nabla^2 \psi \tag{2.2}$$

Substitute (2.2) into the right hand side of (2.1) we get:

$$\int_S \phi \frac{\partial \psi}{\partial n} dS = \int_V \frac{\partial \phi}{\partial x_i}\frac{\partial \psi}{\partial x_i} dV + \int_V \phi \nabla^2 \psi dV \tag{2.3}$$

This is called the Green's first identity.

Green’s second identity:

The Green’s first identity is also valid when interchanging ϕ and ψ :

$$\int_S \psi \frac{\partial \phi}{\partial n} dS = \int_V \frac{\partial \psi}{\partial x_i}\frac{\partial \phi}{\partial x_i} dV + \int_V \psi \nabla^2 \phi dV \tag{2.4}$$

Subtracting equation (2.4) from (2.3) gives Green’s second identity:

$$\int_S (\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}) dS = \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dV$$

n the expressions of Green’s identities, the functions ϕ and ψ must be differentiable at least to the orders that appear in the integrands.

2.1.1 Bounded Problem

Before we consider the pipe flow problem, we consider a simpler problem in a bounded domain in order to understand the main ideas.

We consider $\nabla^2 T = 0$ in D and $\dfrac{\partial T}{\partial n}$ is known on C.

where D is a bounded (interior) domain with boundary C.

To begin with we have the following Green second identity:

$$\int_D (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dy = \int_C (\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}) dc$$

The Green’s function is designated as the fundamental solution.

G $(\underline{x}, \underline{y}) = \dfrac{1}{2\pi} \ln \dfrac{1}{|\underline{x} - \underline{y}|}$ where $\underline{x} = (x_1, x_2)$

$\underline{y} = (y_1, y_2)$

then $\nabla^2{}_x G = \delta(\underline{x} - \underline{y})$ for any fixed $\underline{y}$ or $\nabla^2{}_y G = \delta(\underline{x} - \underline{y})$ for any fixed $\underline{x}$,
where δ is the Dirac Delta function.
Strictly speaking it is defined through the integral $\int \delta(x-y) f(x)\,dx = f(y)$

We let $\phi \equiv T$ and $\psi \equiv G$

Where T=temperature, G=Green's function.

Thus we have:

$$\iint_D (T\nabla^2{}_x G - G\nabla^2{}_x T)\,dx_1 dx_2 = \int_C (T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})\,dc \quad .$$

But $\nabla^2 T = 0$ so we have :

$$\iint_D T\delta(\underline{x} - \underline{y})\,dx_1 dx_2 = \int_C (T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})\,dc$$

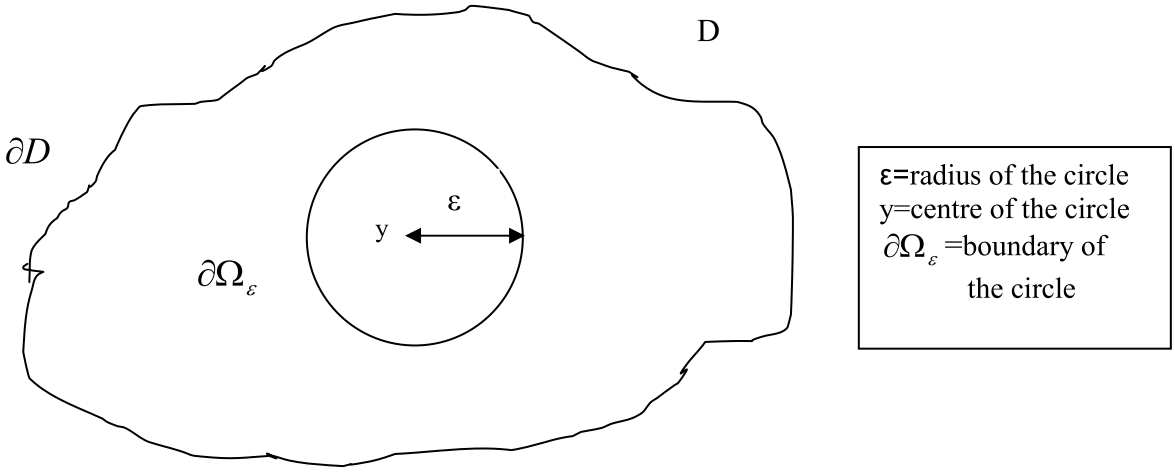
$$\Rightarrow T(y_1, y_2) = \int_C (T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})\,dc \qquad (2.5)$$

in the case that $(y_1, y_2) \in D$

or : $$0 = \int_C (T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})\,dc \quad \text{if} \quad (y_1, y_2) \notin D$$

Suppose ΔG=0 inside a domain D

Suppose $y \in D$



We choose G to be the solution of $\Delta G = 0$ in D/Ω_ε, hence we have:

$$\int_{\partial D_\varepsilon} (T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})d\,\underline{x} = 0 \;\Rightarrow \int_{\partial D} (T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})d\,\underline{x} + \int_{\partial\Omega_\varepsilon} (T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})d\,\underline{x} = 0 \quad (2.6)$$

(from previous section)

We know G $(\underline{x},\underline{y}) = \dfrac{1}{2\pi}\ln\dfrac{1}{|\underline{x}-\underline{y}|}$ so $|\underline{x}-\underline{y}| = |R|$

Therefore,

$$G\,(\underline{x},\underline{y}) = \frac{1}{2\pi}\ln\frac{1}{R} \Rightarrow \frac{\partial G}{\partial n} = \frac{\partial G}{\partial R} = -\frac{1}{2\pi R} = -\frac{1}{2\pi\varepsilon}$$

Let $\varepsilon \to 0$

Then the 2nd part of (2.6) which is $\int_{\partial\Omega_\varepsilon} (T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})d\underline{x}$ can be separated in two parts:

(i) $\int_{\partial\Omega_\varepsilon} T(x)\frac{\partial G}{\partial n}(x)d\underline{x} = -\frac{1}{2\pi\varepsilon}\int_{\partial\Omega_\varepsilon} T(x)d\underline{x}$

$$= -\frac{1}{2\pi\varepsilon}\int_{\partial\Omega_\varepsilon} (T(y) + \varepsilon T'(y) + O(\varepsilon^2))d\underline{x}$$

$$= \left[-\frac{1}{2\pi\varepsilon}T(y) - \frac{T'(y)}{2\pi} + O(\varepsilon)\right]\int_{\partial\Omega_\varepsilon} d\underline{x}$$

$$= \left[-\frac{1}{2\pi\varepsilon}T(y) - \frac{T'(y)}{2\pi} + O(\varepsilon)\right]2\pi\varepsilon$$

$$= -T(y) - \varepsilon T'(y) + O(\varepsilon^2) \longrightarrow -T(y) \text{ as } \varepsilon \to 0$$

(ii) $\int_{\partial\Omega_\varepsilon} G\frac{\partial T}{\partial n}d\underline{x} = \int_{\partial\Omega_\varepsilon} \frac{1}{2\pi}\ln\frac{1}{\varepsilon}\frac{\partial T}{\partial n}d\underline{x}$

$$= -\frac{1}{2\pi}\ln\varepsilon\left[\frac{\partial T}{\partial n}(y) + O(\varepsilon)\right]2\pi\varepsilon$$

$$= -\varepsilon\ln\varepsilon\left[\frac{\partial T}{\partial n}(y) + O(\varepsilon)\right] \to 0 \text{ as } \varepsilon \to 0$$

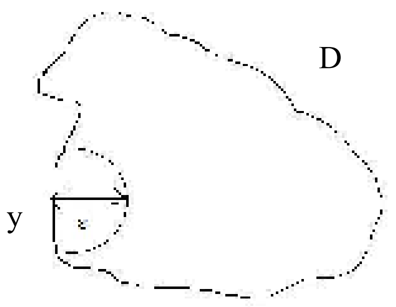
Hence,

(2.6) $$\int_{\partial D}(T\frac{\partial G}{\partial n}-G\frac{\partial T}{\partial n})d\,\underline{x}=-\int_{\partial\Omega_\varepsilon}(T\frac{\partial G}{\partial n}-G\frac{\partial T}{\partial n})d\,\underline{x}=-[-T(y)]=T(y)$$

$y\in D$ as $\varepsilon\to 0$.

which is the same result as (2.5) but is defined in a slightly more rigorous answer.

Suppose now, $y\in\partial D$ (y is on the boundary).In this case, the same procedure as before can be applied with the difference that now we have a semicircle instead of a circle. Therefore the length of the boundary is π instead of 2π in the derivations above.



ε=radius of the circle
y=centre of the circle

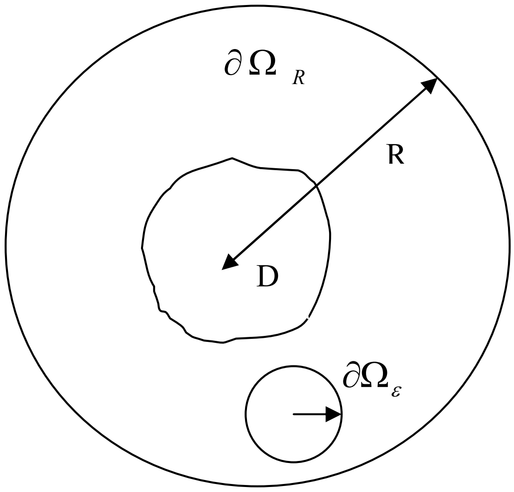
Hence,

$$\int_{\partial D}(T\frac{\partial G}{\partial n}-G\frac{\partial T}{\partial n})d\,\underline{x}=-\int_{\partial\Omega_\varepsilon}(T\frac{\partial G}{\partial n}-G\frac{\partial T}{\partial n})d\,\underline{x}=\frac{1}{2}T(y)\qquad y\in\partial D \qquad (2.7)$$

2.1.2 Unbounded Problem

<u>Suppose ΔT=0 and ΔG=0 are outside the domain D</u> (exterior problem)

Suppose $y \notin D$



ε=radius of the small circle
y=centre of the small circle
$\partial\Omega_\varepsilon$=boundary of the small circle
$\partial\Omega_R$=boundary of the big circle
R=radius of the big circle

The following equation is equal to zero because y is outside the domain as we have mentioned in equation (2.5) when $(y_1, y_2) \notin D$.

$$\int_{\partial D}(T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})ds(x) + \int_{\partial\Omega_\varepsilon}(T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})ds(x) + \int_{\partial\Omega_R}(T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})ds(x) = 0$$
(2.8)

$$\Rightarrow \int_{\partial D}(T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})ds(x) = -\underbrace{\int_{\partial\Omega_\varepsilon}(T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})ds(x)}_{\text{Same as before } = -T(y)} - \underbrace{\int_{\partial\Omega_R}(T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})ds(x)}_{(2.9)} = 0$$

Now, we are going to find (2.9) in the limit as R$\rightarrow\infty$

$$\int_{\partial\Omega_R}(T\frac{\partial G}{\partial n}-G\frac{\partial T}{\partial n})ds(x)=-\int_{\partial\Omega_R}T(x)\frac{1}{2\pi R}ds(x)-\int_{\partial\Omega_R}\frac{1}{2\pi}\ln R\frac{\partial T}{\partial n}ds(x)$$

=0 (If ΔT=0)
(Corollary of Green's 2)

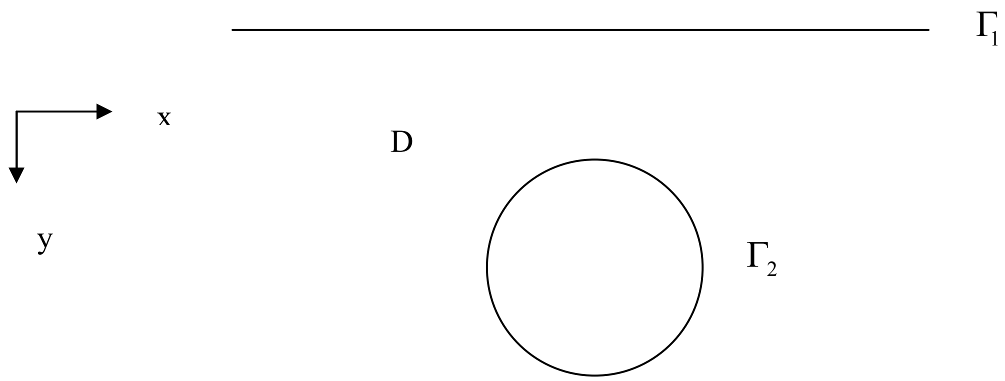
$$=-\frac{1}{2\pi R}\int_{\partial\Omega_R}T(x)ds(x)-0$$

So overall, (2.8)

$$\int_{\partial D}(T\frac{\partial G}{\partial n}-G\frac{\partial T}{\partial n})ds(x)=T(y)-\frac{1}{2\pi R}\int_{\partial\Omega_R}T(x)ds(x) \qquad (2.8)$$

2.1.3 Full pipe Problem

In the following plot the pipe is buried in the ground.



where Γ_1= ground surface

Γ_2 =pipe

D=exterior environment (below Γ_1 and outside Γ_2)

We have the following assumptions:

ΔT=0 in D

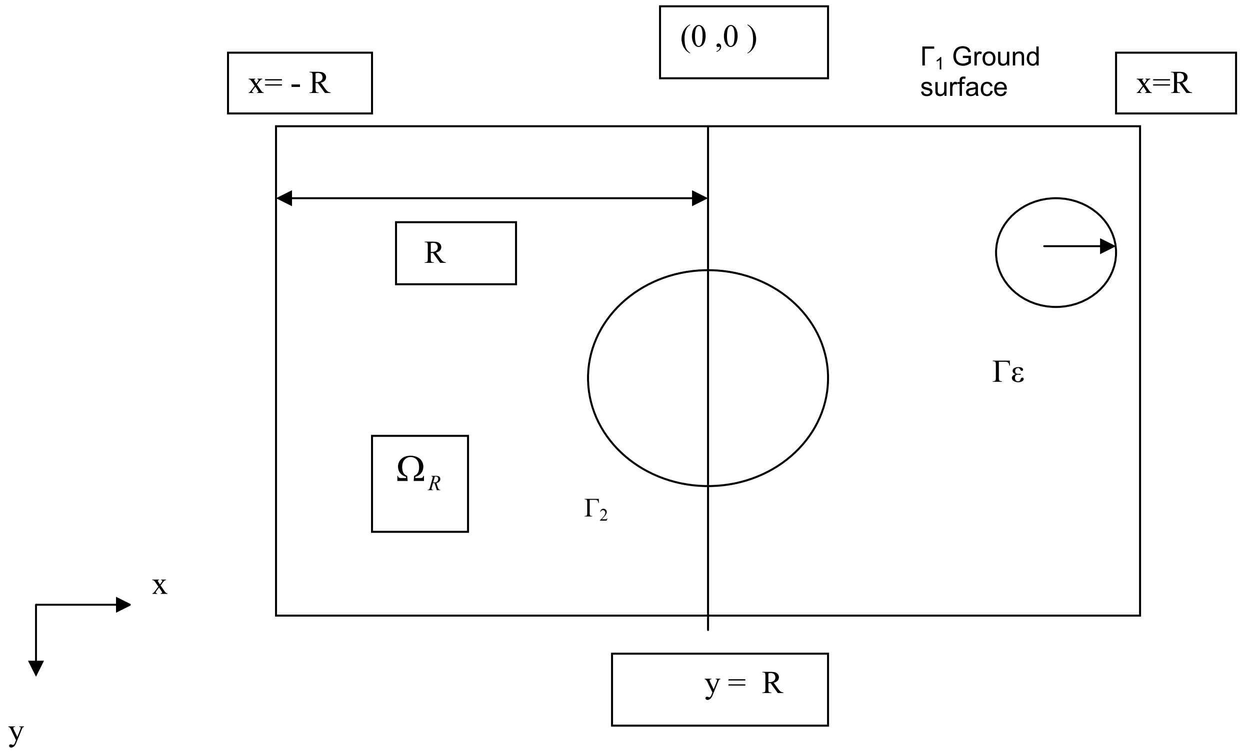
$\frac{\partial T}{\partial n}$ =known on Γ_1=C_1 (a constant)

$\frac{\partial T}{\partial n}$ =known on Γ_2=C_2 (a constant)

The asymptotic condition as $y \to \infty, x \to \pm\infty$
T$\to gy + T_g$

Now, to solve the full pipe problem we will make a rectangular domain as shown below:

2.1.4 Neumann Green's function for a half plane Problem



where Γ_2=pipe

Γ_ε =small circle

ε=radius of small circle

Ω_R =domain

We want to find the integral equation of the domain Ω_R .

<u>In the Domain :</u>

<u>Known:</u>

$$\nabla^2 T = 0$$

$$\nabla^2{}_x G(\underline{x}, \underline{y}) = 0$$

.

We choose $\hat{G}$ such that $\dfrac{\partial \hat{G}}{\partial n} = 0$ on Γ_1. (*)

$$\hat{G}(\underline{x},\underline{y}) = \frac{1}{2\pi}\ln\frac{1}{|\underline{x}-\underline{y}|} + \frac{1}{2\pi}\ln\frac{1}{|\underline{x}-\underline{y}'|}$$

where y' is the reflection of y in the line y = - z and where $\underline{x} = (x_1, x_2)$

$\underline{y} = (y_1, y_2)$

$\hat{G}(\underline{x},\underline{y})$ satisfies $\nabla^2{}_x \hat{G} = \delta(\underline{x}-\underline{y}) + \delta(\underline{x}-\underline{y}')$.

To find the integral equation of the domain we add the integral equations of the pipe, the circle, the ground surface and the lines $x = R, x = -R, y = R$.

$$\int_{\partial\Omega_R} \left(T\frac{\partial \hat{G}}{\partial n} - \hat{G}\frac{\partial T}{\partial n}\right)ds = \int_{\Gamma_2} + \int_{\Gamma_1 \cap [-R,R]} + \int_{x=R} + \int_{x=-R} + \int_{y=R} + \int_{\Gamma\varepsilon} = 0 \qquad (2.10)$$

The equation (2.10) is equal to zero from the earlier Green's function.

Consider the lim (as R$\rightarrow\infty$)

The 1st integral is : $\displaystyle\int_{\Gamma_2} \left(T\frac{\partial \hat{G}}{\partial n} - \hat{G}\frac{\partial T}{\partial n}\right)ds(x) = \int_{\Gamma_2} \left(T\frac{\partial \hat{G}}{\partial n} - \hat{G}C_2\right)ds(x)$

The 2nd integral is : $\displaystyle\int_{\Gamma_1 \cap [-R,R]} \left(T\frac{\partial \hat{G}}{\partial n} - \hat{G}\frac{\partial T}{\partial n}\right)ds(x) = \int_{x=-\infty}^{x=\infty} \left(T\cancel{\frac{\partial \hat{G}}{\partial n}} - \hat{G}C_1\right)ds(x)$

=0 from (*)

The 3rd and 4th integrals are:

$$\int_{x=R, y\in[0,R]} \left(T\frac{\partial \hat{G}}{\partial n} - \hat{G}\frac{\partial T}{\partial n}\right)ds(x) + \int_{x=-R, y\in[0,R]} \left(T\frac{\partial \hat{G}}{\partial n} - \hat{G}\frac{\partial T}{\partial n}\right)ds(x)$$

We consider the asymptotic condition as $y \to \infty, x \pm \infty$

$$T \to gy + T_g \ . \tag{2.11}$$

Therefore,

$$\frac{\partial T}{\partial x} = 0 \ \text{ and } \ \frac{\partial T}{\partial y} = g$$

We solve for $\quad u = T - gy - T_g$ as $y \to \infty, x \pm \infty$. (2.12)

Hence if we substitute (2.11) into (2.12) we have :

$u = T - gy - T_g = gy + T_g - gy - T_g = 0$ .Thus $u \to 0$ as $y \to \infty, x \pm \infty$

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \to 0$$

Thus the 3rd and 4th integrals are:

$$\int_{x=R, y\in[0,R]} (T \underset{=0}{\cancel{\frac{\partial \hat{G}}{\partial n}}} - \hat{G} \underset{=0}{\cancel{\frac{\partial T}{\partial n}}}) ds(x) + \int_{x=-R, y\in[0,R]} (T \underset{=0}{\cancel{\frac{\partial \hat{G}}{\partial n}}} - \hat{G} \underset{=0}{\cancel{\frac{\partial T}{\partial n}}}) ds(x) = 0$$

Therefore, the result of the addition of the 3rd and 4th integral is zero as $R \to \infty$.

In the 5th integral we consider:

$$\hat{G}(\underline{x}, \underline{y}) = \frac{1}{2\pi} \ln \frac{1}{|\underline{x} - \underline{y}|} + \frac{1}{2\pi} \ln \frac{1}{|\underline{x} - \underline{y}'|}$$

where $\Phi(\underline{x}, \underline{y}) = \dfrac{1}{2\pi} \ln \dfrac{1}{|\underline{x} - \underline{y}|}$, $\ \Phi(\underline{x}, \underline{y}') = \dfrac{1}{2\pi} \ln \dfrac{1}{|\underline{x} - \underline{y}'|}$

$$\frac{\partial \Phi}{\partial n(y)}(\underset{-}{x},\underset{-}{y}) = -\frac{\partial \Phi}{\partial y_2} = -\frac{\partial}{\partial y_2}\left(\frac{1}{2\pi}\ln\frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}}\right)$$

$$\frac{\partial \Phi}{\partial n(y)}(\underset{-}{x},\underset{-}{y}) = -\frac{1}{2\pi}\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\left(-\frac{1}{2}\right)[(x_1-y_1)^2+(x_2-y_2)^2]^{-\frac{3}{2}}2(x_2-y_2)$$

$$= \frac{1}{2\pi}\frac{(x_2-y_2)}{[(x_1-y_1)^2+(x_2-y_2)^2]}$$

and in the same way

$$\frac{\partial \Phi}{\partial n(y)}(\underset{-}{x},\underset{-}{y'}) = -\frac{1}{2\pi}\frac{(x_2+y_2)}{[(x_1-y_1)^2+(x_2+y_2)^2]}$$

Therefore, if we add $\frac{\partial \Phi}{\partial n}(\underset{-}{x},\underset{-}{y}), \frac{\partial \Phi}{\partial n}(\underset{-}{x},\underset{-}{y'})$ and $y_2 = 0$ then the result is zero:

$$\left[\frac{\partial \Phi}{\partial n}(\underset{-}{x},\underset{-}{y}) + \frac{\partial \Phi}{\partial n}(\underset{-}{x},\underset{-}{y}')\right]_{y_2=0} = 0$$

For that reason , the 5th integral is zero.

So from (2.10)

-

$$0 = \int_{\Gamma_2}(T\frac{\partial \hat{G}}{\partial n} - \hat{G}C_2)ds(x) + \int_{\Gamma_1}(-\hat{G}C_1)ds(x) + \int_{\Gamma\varepsilon}(T\frac{\partial \hat{G}}{\partial n} - \hat{G}\frac{\partial T}{\partial n})ds(x)$$

$$0 = \int_{\Gamma_2}(T\frac{\partial \hat{G}}{\partial n} - \hat{G}C_2)ds(x) + \int_{\Gamma_1}(-\hat{G}C_1)ds(x) + T(y_1,y_2)$$

Therefore,

$$-T(y_1,y_2) = \int_{\Gamma_2}(T\frac{\partial \hat{G}}{\partial n} - \hat{G}C_2)ds(x) + \int_{\Gamma_1}(-\hat{G}C_1)ds(x)$$

$$-T(y_1, y_2) - \int_{\Gamma_2} \frac{\partial \hat{G}}{\partial n} T(x) ds(x) = \int_{\Gamma_2} (-\hat{G}C_2) ds(x) + \int_{\Gamma_1} (-\hat{G}C_1) ds(x)$$

We set

$$\int_{\Gamma_2} (-\hat{G}C_2) ds(x) + \int_{\Gamma_1} (-\hat{G}C_1) ds(x) = F(y)$$

Hence,

$$-T(y_1, y_2) - \int_{\Gamma_2} \frac{\partial \hat{G}}{\partial n} T(x) ds(x) = F(y)$$

We are going to solve this integral equation numerically. The general integral equation is of the following form:

$$u(x) + \int_{\Gamma} \mathrm{K}(x, y) u(y) dy = f(x)$$

Chapter 3

Methods for solving a single integral equation

We study certain Fredholm integral equations of the 2nd kind,

$$u(x)+\int_{\Gamma}\mathrm{K}(x,y)u(y)dy=f(x) \qquad (3.1)$$

Where :

Kernel= $\mathrm{K}(x,y)$ is known

Γ is some closed boundary , $\Gamma=(\gamma_1(\varsigma),\gamma_2(\varsigma))$ where $\varsigma\in[0,2\pi]$ and (γ_1,γ_2) are periodic functions

$f(x)$ is known

u is not known $\longrightarrow$ u is what we have to find

We can solve equation (3.1) by three methods:

(i) Galerkin method

(ii) Collocation method

(iii) Nyström method

The Galerkin method used for analysis, but the other two methods are easier for programming.

In this project we will use the Nyström and the Collocation method.

3.1 Nyström Method

To solve

$$u(x)+\int_{0}^{2\pi}K(x,y)u(y)dy=f(x)$$

We replace the integral by a quadrature rule. The easiest way is to use the trapezium rule.

$$\int_0^{2\pi} g(y)dy \approx \frac{h}{2}[g(0)+2g(h)+2g(2h)+......+2g((n-1)h)+g(2\pi)]$$
$$= h[g(0)+g(h)+......+g((n-1)h)]$$ [2π periodic function $\Rightarrow g(0)=g(2\pi)$].

Where $h=\dfrac{2\pi}{n}$ (n =quadrature parameter)

In our case we replace u by u_n.

Therefore, we have:

$$u_n(x)+h[K(x,0)u_n(0)+K(x,h)u_n(h)+....+K(x,((n-1)h)u_n((n-1)h)]=f(x)$$

$\forall x$

We take $x=0,h,2h,3h,.....,(n-1)h$

1st equation when $x=0$:

$$u_n(0)+h[K(0,0)u_n(0)+K(0,h)u_n(h)+....+K(0,((n-1)h)u_n((n-1)h)]=f(0)$$

2nd equation when $x=h$:

$$u_n(h)+h[K(h,0)u_n(0)+K(h,h)u_n(h)+....+K(h,((n-1)h)u_n((n-1)h)]=f(h)$$

3rd equation when $x=2h$:

$$u_n(2h)+h[K(2h,0)u_n(0)+K(2h,h)u_n(h)+....+K(2h,((n-1)h)u_n((n-1)h)]=f(2h)$$

Last equation when $x=(n-1)h$:

$$u_n((n-1)h)+h[K((n-1)h,0)u_n(0)+K((n-1)h,h)u_n(h)+....+$$
$$+K((n-1)h,((n-1)h)u_n((n-1)h)]=f((n-1)h)$$

We have n equations with n unknowns:

$$u_n(0), u_n(h), u_n(2h), u_n(3h), ..., u_n((n-1)h)$$

We write it as a matrix $Ax = b$

$$A = \begin{bmatrix} 1 + hK(0,0) & hK(0,h) & hK(0,2h) & .... & hK(0,(n-1)h) \\ hK(h,0) & 1 + hK(h,h) & hK(h,2h) & .... & hK(h,(n-1)h) \\ hK(2h,0) & hk(2h,h) & 1 + hK(2h,2h) & .... & hK(2h,(n-1)h) \\ .... & .... & .... & .... & .... \\ h[K(n-1)h,0) & h[K(n-1)h,h] & .... & .... & 1 + h[K(n-1)h,(n-1)h] \end{bmatrix}$$

$$x = \begin{pmatrix} u_n(0) \\ u_n(h) \\ u_n(2h) \\ u_n(3h) \\ ...... \\ u_n((n-1)h) \end{pmatrix}$$

$$b = \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ f(3h) \\ .... \\ f((n-1)h) \end{pmatrix}$$

- The conventional Nyström method is a simple and efficient mechanism for discretizing integral equations with non-singular kernels (K(x,y)).

- With a high-order quadrature rule, the solution one obtains by this method is a high-order approximation to the exact solution.

In the Nyström method we could use midpoint rule, Gaussian quadrature,Simpson's rule and trapezoidal rule.

Question:

For a general integral equation in [a,b] of a general function which is the best quadrature?

Answer:

If you have [0,2π] and a periodic function and if the function is C^{∞},then the trapezoidal rule is exponentially accurate and also equivalent to replacing u by its trigonometric interpolating polynomial and collocating at mesh points.

3.1.1 Example of a single periodic integral equation

In order to test our method we derive analytical solution:

Analytical solution:

Our example is of the form:

$$u(x) + \int_{\Gamma} \mathrm{K}(x, y)u(y)dy = f(x)$$

Where the Kernel=K(x, y) = $\dfrac{1}{2\pi}\sin(2x + y)$

Γ is a closed boundary from 0 to 2π

u(x)=cos(x) , u(y)=cos(y)

Thus,

$$\cos(x) + \frac{1}{2\pi}\int_0^{2\pi} \sin(2x + y)\cos(y)dy = f(x)$$

$$\Rightarrow \cos(x) + \frac{1}{2\pi}\int_0^{2\pi} [(\sin(2x)\cos(y) + \sin(y)\cos(2x)]\cos(y)dy = f(x)$$

$$\Rightarrow \cos(x) + \frac{1}{2\pi}\int_0^{2\pi} (\sin(2x)\cos^2(y) + \sin(y)\cos(2x)\cos(y))dy = f(x)$$

$$\Rightarrow \cos(x) + \frac{1}{2\pi}\int_0^{2\pi} (\sin(2x)\frac{\cos(2y)+1}{2} + \cos(2x)\frac{\sin(2y)}{2})dy = f(x)$$

$$\Rightarrow \cos(x) + \frac{1}{2\pi}\left[\sin(2x)\frac{\sin(2y)}{4} + \frac{y}{2}\sin(2x) - \frac{\cos(2x)\sin(2y)}{4}\right]_0^{2\pi} = f(x)$$

$$\Rightarrow \cos(x) + \frac{1}{2\pi}(0 + \pi\sin(2x)) = f(x)$$

Hence,

$$f(x) = \cos(x) + \frac{1}{2}\sin(2x)$$

We program this example of a single periodic integral equation in Matlab and we have the following results for the $\|u - uexact\|_{L2}$

where u_{exact}=cos(x) and u is computed with the Nyström Method.

(Note: where $\|u - u_{exact}\|_{L2} = \left[\int_0^{2\pi} (u - u_{exact})^2\right]^{\frac{1}{2}}$)

TABLE 1

n	$\|u - u_{exact}\|_{L2}$
2	3.9356e-016
4	2.8353e-016
8	3.6854e-016
16	5.2234e-016
32	7.5510e-016

We just use mesh points to evaluate the $\|u - u_{exact}\|_{L2}$. We always get zero to machine precision.

We have used $$\|u - u_{exact}\|_{L2} = \left[\int_0^{2\pi} (u - u_{exact})^2\right]^{\frac{1}{2}}) \qquad (3.2)$$

$$= \left[\hat{h}\sum_{j=1}^{n-1}[u(j\hat{h}) - u_{exact}(j\hat{h})]^2\right]^{\frac{1}{2}}$$

with $\hat{h} = h$ and then we had zero. However this is not an accurate approximation to the error.

Instead, we need to work out integral (3.2) in a better way. We can work out u_{exact} everywhere because we have exact formula to work out an approximation to u which valid anywhere. So we can use the following formula where $P_n u(x)$ is called the trigonometric interpolating polynomial [10].

$$P_n u(x) = \sum_{j=0}^{2m-1} u(jh)\left[\frac{1}{2m}\left(1 + 2\sum_{k=1}^{m-1}\cos(k(x - jh)) + \cos(m(x - jh))\right)\right]$$

Where $m = \dfrac{n}{2}$

3.1.2 Example of a single non-periodic integral equation

<u>Analytical solution:</u>

$$u(x)+\int_{\Gamma}\mathrm{K}(x,y)u(y)dy=f(x)$$

Where

the Kernel=K(x, y) = x^2y^2

Γ is a closed boundary from 0 to 1

u(x)=$1+\dfrac{5x^2}{12}$, u(y)= $1+\dfrac{5y^2}{12}$

So,

$$\left(1+\frac{5x^2}{12}\right)+\int_0^1(1+\frac{5y^2}{12})x^2y^2dy=f(x)$$

$$\Rightarrow\left(1+\frac{5x^2}{12}\right)+\int_0^1(x^2y^2+\frac{5x^2y^4}{12})dy=f(x)$$

$$\Rightarrow\left(1+\frac{5x^2}{12}\right)+\left[\frac{1}{3}x^2y^3+\frac{1}{12}x^2y^5\right]_0^1=f(x)$$

$$\Rightarrow\left(1+\frac{5x^2}{12}\right)+x^2\left[\frac{1}{3}y^3+\frac{1}{12}y^5\right]_0^1=f(x)$$

$$\Rightarrow\left(1+\frac{5x^2}{12}\right)+x^2\left(\frac{1}{3}+\frac{1}{12}\right)=f(x)$$

$$\Rightarrow 1+x^2\left(\frac{5}{12}+\frac{1}{3}+\frac{1}{12}\right)=f(x)$$

Hence,

$$f(x) = 1 + \frac{5}{6}x^2$$

We program this example of a single periodic integral equation in Matlab and we have the following results for the $\|u - uexact\|_{L2}$.

We have used $$\|u - u_{exact}\|_{L2} = \left[\int_0^{2\pi} (u - u_{exact})^2\right]^{\frac{1}{2}}) \qquad (3.2)$$

$$= \left[\hat{h}\sum_{j=1}^{n-1}[u(j\hat{h}) - u_{exact}(j\hat{h})]^2\right]^{\frac{1}{2}}$$

TABLE 2

n	$\|u - u_{exact}\|_{L2}$
2	0.5000
4	0.1221
8	0.0443
16	0.0191
32	0.0089

We computed the error at the mesh points.
The error appears to half as we double the value of n.

If we compare our two examples, the periodic function with the non-periodic function, we can see that the periodic function is faster.

3.2 Collocation Method

The idea is to choose a finite-dimensional space of candidate solutions and a number of points in the domain (called *collocation points*), and to select that solution which satisfies the given equation at the collocation points.

To solve

$$u(x)+\int_{0}^{2\pi}K(x,y)u(y)dy=f(x) \tag{3.3}$$

We seek an approximation u_n of the form:

$$u_n=\sum_{j=0}^{n-1}\phi_j(x)u(x_j) \qquad \text{where } \phi_j = \text{basis functions}$$

Substitute u into (3.3)

$$\sum_{j=0}^{n-1}\phi_j(x)u(x_j)+\int_{0}^{2\pi}K(x,y)\sum_{j=0}^{n-1}\phi_j(y)u(x_j)dy=f(x)$$

$$\Rightarrow \sum_{j=0}^{n-1}[\phi_j(x)+\int_{0}^{2\pi}K(x,y)\phi_j(y)dy]u(x_j)=f(x) \tag{3.4}$$

So we have one equation with n unknowns $\rightarrow$ the values of $u(x_j)$.

To get n equations we fix (3.4) to hold at n points i.e. take $x=\hat{x}_1,.....,\hat{x}_n$ and then we will have n equations with n unknowns.

If we choose $\hat{x}_1,.....,\hat{x}_n$ to be the same points as x_j

$$\Rightarrow \sum_{j=0}^{n-1}[\phi_j(x_m)+\int_{0}^{2\pi}K(x_m,y)\phi_j(y)dy]u(x_j)=f(x_m)$$

We replace the integral by a quadrature rule as in Nyström Method. The easiest way is to use the trapezium rule.

$$\int_0^{2\pi} g(y)dy \approx \frac{h}{2}[g(0)+2g(h)+2g(2h)+......+2g((n-1)h)+g(2\pi)]$$

$$= h[g(0)+g(h)+......+g((n-1)h)] \qquad \text{(periodic function)}$$

Where $h = \frac{2\pi}{n}$ (n =quadrature parameter)

In our case we replace ϕ by ϕ_n .

Therefore, we have:

$$\phi_n(x) + h[K(x,0)\phi_n(0) + K(x,h)\phi_n(h) + ... + K(x,((n-1)h)\phi_n((n-1)h)] \cdot u(x_j) = f(x)$$

$\forall x$

We take $x = 0, h, 2h, 3h, ..., (n-1)h$
We define $u(x_j) = u_n$

We have n equations with n unknowns:

$$u_n(0), u_n(h), u_n(2h), u_n(3h), ..., u_n((n-1)h)$$

We use the trapezoidal rule in this method and we have exactly the same matrix as for the Nyström Method.

Hence the Nyström Method is exact at mesh points.

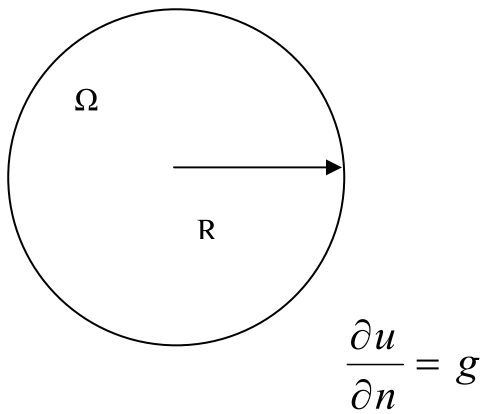
Chapter 4

Interior Problem for Laplace's equation

Consider $\Delta u = 0$,

Ω is a circle with radius R

$$\frac{\partial u}{\partial n} = g$$



We already know that the Fundamental solution $G(\underline{x}, \underline{y}) = \frac{1}{2\pi} \ln \frac{1}{|\underline{x} - \underline{y}|}$ and that the form of the general integral equation is:

$$u(x) + \int_{\Gamma} \mathrm{K}(x, y) u(y) dy = f(x) .$$

Thus, we set up the interior problem as an integral equation of the above form and we will solve it using a code in Matlab.

From (2.7) of the bounded problem we have:

$$\int_{\partial D} (T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n}) d\underset{-}{x} = \frac{1}{2}T(y) \qquad y \in \partial D$$

$$\Rightarrow -\frac{1}{2}T(y) + \int_{\partial D} T\frac{\partial G}{\partial n} d\underset{-}{x} = \int_{\partial D} G\frac{\partial T}{\partial n} d\underset{-}{x} \qquad y \in \partial D$$

Where in this case K(x,y)= $\frac{\partial G}{\partial n}$

$$f(x) = \int_{\partial D} (G\frac{\partial T}{\partial n}) d\underset{-}{x} = \int_{\partial D} (Gg) d\underset{-}{x}$$

So we have to solve :

$$-T(y) + 2\int_{\partial D} T\frac{\partial G}{\partial n} d\underset{-}{x} = 2\int_{\partial D} G\frac{\partial T}{\partial n} d\underset{-}{x}$$

Firstly, we have to find $\frac{\partial G}{\partial n}$ with respect to x.

We know G$(\underset{-}{x},\underset{-}{y}) = \frac{1}{2\pi}\ln\frac{1}{|\underset{-}{x}-\underset{-}{y}|}$ where $\underset{-}{x} = (x_1, x_2)$

$\underset{-}{y} = (y_1, y_2)$

We substitute $\underset{-}{x}$ and $\underset{-}{y}$ in G and then we have the following expression for G:

$$G(\underset{-}{x},\underset{-}{y}) = \frac{1}{2\pi}\ln\frac{1}{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}}$$

$$\frac{\partial G}{\partial n(x)} = \underset{-}{n}(\underset{-}{x}).\nabla_x G = \begin{pmatrix} n_1(x) \\ n_2(x) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial G}{\partial x_1} \\ \frac{\partial G}{\partial x_2} \end{pmatrix} = n_1(x)\frac{\partial G}{\partial x_1} + n_2(x)\frac{\partial G}{\partial x_2}$$

(4.1)

$$\frac{\partial G}{\partial x_1}=\frac{1}{2\pi}\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\left(-\frac{1}{2}\right)\left[(x_1-y_1)^2+(x_2-y_2)^2\right]^{-\frac{3}{2}}2(x_1-y_1)$$

$$\frac{\partial G}{\partial x_1}=-\frac{1}{2\pi}\frac{[(x_1-y_1)^2+(x_2-y_2)^2]^{\frac{1}{2}}}{[(x_1-y_1)^2+(x_2-y_2)^2]^{\frac{3}{2}}}\cdot(x_1-y_1)$$

$$\frac{\partial G}{\partial x_1}=-\frac{1}{2\pi}\frac{(x_1-y_1)}{[(x_1-y_1)^2+(x_2-y_2)^2]}$$ and in the same way

$$\frac{\partial G}{\partial x_2}=-\frac{1}{2\pi}\frac{(x_2-y_2)}{[(x_1-y_1)^2+(x_2-y_2)^2]}$$

We set $x=(\cos(\varsigma),\sin(\varsigma))\Rightarrow n_1=\cos\varsigma, n_2=\sin\varsigma$

$y=(\cos(t),\sin(t))$

After that we replace $\frac{\partial G}{\partial x_1}$, $\frac{\partial G}{\partial x_2}$, n_1 and n_2 into equation (4.1)

And finally we have:

$$\frac{\partial G}{\partial n}=-\frac{1}{2\pi}\frac{(x_1-y_1)}{[(x_1-y_1)^2+(x_2-y_2)^2]}\cos(\varsigma)-\frac{1}{2\pi}\frac{(x_2-y_2)}{[(x_1-y_1)^2+(x_2-y_2)^2]}\sin(\varsigma)$$

$$\Rightarrow\frac{\partial G}{\partial n}=-\frac{1}{2\pi[(x_1-y_1)^2+(x_2-y_2)^2]}\left[\cos(\varsigma)(x_1-y_1)+\sin(\varsigma)(x_2-y_2)\right]$$

We also replace $x_1=\cos(\varsigma)$ $\qquad y_1=\cos(t)$

$x_2=\sin(\varsigma)$ $\qquad y_2=\sin(t)$

$$\Rightarrow \frac{\partial G}{\partial n} = -\frac{[\cos\varsigma(\cos\varsigma - \cos t) + \sin\varsigma(\sin\varsigma - \sin t)]}{2\pi[(\cos(\varsigma) - \cos(t))^2 + (\sin(\varsigma) - \sin(t))^2]}$$

(Note: $[(\cos\varsigma - \cos t)^2 + (\sin\varsigma - \sin t)^2] = 2 - 2(\cos\varsigma\cos t + \sin\varsigma\sin t)$

$$= 2 - 2\cos(\varsigma - t)$$

$$= 2 - 2\cos 2\left(\frac{\varsigma - t}{2}\right)$$

$$= 2 - 2[1 - 2\sin^2\left(\frac{\varsigma - t}{2}\right)]$$

$$= \cancel{2} - \cancel{2} + 4\sin^2\left(\frac{\varsigma - t}{2}\right)$$

$$= 4\sin^2\left(\frac{\varsigma - t}{2}\right)$$).

$$\Rightarrow \frac{\partial G}{\partial n} = \frac{-\cos^2\varsigma + \cos\varsigma\cos t - \sin^2\varsigma + \sin\varsigma\sin t}{2\pi \cdot 4\sin^2(\frac{\varsigma - t}{2})}$$

$$\Rightarrow \frac{\partial G}{\partial n} = \frac{-(\cos^2\varsigma + \sin^2\varsigma) + \cos\varsigma\cos t + \sin\varsigma\sin t}{8\pi\sin^2(\frac{\varsigma - t}{2})}$$

$$\Rightarrow \frac{\partial G}{\partial n} = \frac{-(\cos^2\varsigma + \sin^2\varsigma) + \cos\varsigma\cos t + \sin\varsigma\sin t}{8\pi\sin^2(\frac{\varsigma - t}{2})}$$

(Note: Trigonometric identity: $\cos^2\varsigma + \sin^2\varsigma = 1$)

$$\Rightarrow \frac{\partial G}{\partial n} = \frac{-1 + \cos\varsigma\cos t + \sin\varsigma\sin t}{8\pi\sin^2(\frac{\varsigma - t}{2})}$$

$$\Rightarrow \frac{\partial G}{\partial n} = -\frac{1}{8\pi}\frac{(1-(\cos\varsigma\cos t+\sin\varsigma\sin t))}{\sin^2(\frac{\varsigma - t}{2})}$$

Now we are going to simplify the numerator of this fraction.

$$\begin{aligned}1-(\cos\varsigma\cos t+\sin\varsigma\sin t) &= 1-\cos(\varsigma - t)\\ &= 1-(1-2\sin^2(\frac{\varsigma - t}{2}))\\ &= 2\sin^2(\frac{\varsigma - t}{2})\end{aligned}$$

(Note: Trigonometric identities: $\cos(\varsigma - t) = \cos\varsigma\cos t + \sin\varsigma\sin t$

$$\cos 2\varsigma = \cos^2\varsigma - \sin^2\varsigma$$
$$\Rightarrow \cos 2\varsigma = 1 - \sin^2\varsigma - \sin^2\varsigma$$
$$\Rightarrow \cos 2\varsigma = 1 - 2\sin^2\varsigma$$).

Finally,

$$\Rightarrow \frac{\partial G}{\partial n} = -\frac{1}{8\pi}\frac{2\cancel{\sin^2(\frac{\varsigma - t}{2})}}{\cancel{\sin^2(\frac{\varsigma - t}{2})}}$$

$$\Rightarrow \frac{\partial G}{\partial n} = -\frac{1}{4\pi} \tag{4.2}$$

Hence our integral equation is:

$$-T(y) - 2\int_{\partial D} T\frac{1}{4\pi} d\underline{x} = \underbrace{2\int_{\partial D} Ggd\underline{x}}_{f(y)}$$

We know G($\underline{x},\underline{y}) = \frac{1}{2\pi}\ln\frac{1}{|\underline{x}-\underline{y}|}$ therefore

$$f(y)=\int_0^{2\pi}2Ggdx=\frac{1}{\cancel{2}\pi}\int_0^{2\pi}\cancel{2}\ln\frac{1}{\left|\underline{x}-\underline{y}\right|}g(x)dx=\frac{1}{\pi}\int_0^{2\pi}\ln\frac{1}{\left|\underline{x}-\underline{y}\right|}g(x)dx$$

Lets substitute $\hat{x}=x-y\Rightarrow x=\hat{x}+y$

$$dx=d\hat{x}$$

$$g(x)=g(\hat{x}+y)$$

$$f(y)=\frac{1}{\pi}\int_{-y}^{2\pi-y}\ln\frac{1}{|\hat{x}|}g(\hat{x}+y)d\hat{x}=\int_0^{2\pi}\ln\frac{1}{|x|}g(x+y)dx$$

because is periodic

To find $f(y)$ which is equal to $\int_{\partial D}Ggd\,\underline{x}$ we have to apply some quadrature rule. The composite midpoint rule is appropriate for this equation, since the integrand is singular at x=0.

<u>Composite midpoint rule:</u>

For any function F and for any N≥ 1 we have:

$$\int_0^{2\pi}f(x)dx\approx h\sum_{J=0}^{N-1}f\left(\frac{(2_j+1)h}{2}\right)\qquad\text{where }\ h=\frac{2\pi}{N}$$

Therefore, our $f(y)$ is:

$$f(y)=\int_0^{2\pi}\ln\frac{1}{|x|}g(x+y)dx\approx h\sum_{j=0}^{N-1}\ln\left(\frac{2}{(2_j+1)h}\right)g\left(\frac{(2_j+1)h}{2}+y\right)$$

where $x=\dfrac{(2_j+1)h}{2}$.

Finally, we know the kernel (K(x,y))and the right hand side of the integral equation f(y). Thus we will solve it numerically in Matlab to find T(y).

Below are the graphs of the numerical solution of the integral equation:

$$-T(y)+2\int_{\partial D} T\frac{\partial G}{\partial n}d\underline{x} = 2\int_{\partial D} Ggd\underline{x}$$ for different values of n.

Where

$$\frac{\partial G}{\partial n} = -\frac{1}{4\pi}$$

$$f(y) = 2\int_{\partial D} 2Ggd\underline{x} = \int_0^{2\pi} \ln\frac{1}{|x|}g(x+y)dx$$

$$g(\theta) = \cos(10\theta) + \sin(12\theta)$$

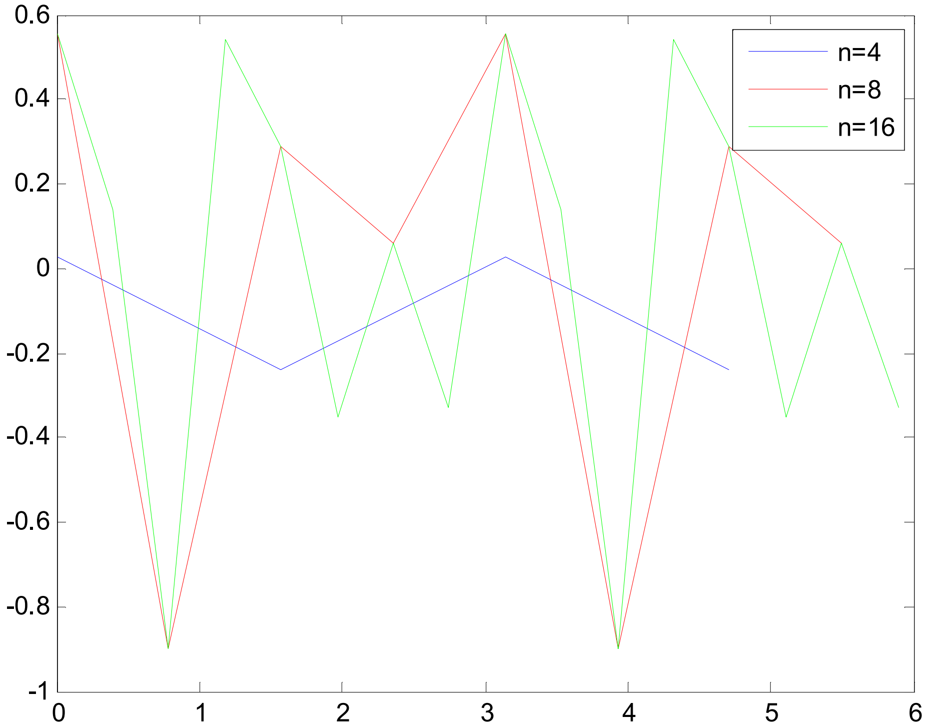


Figure 4.1: Numerical solution of the interior problem for Laplace's equation for n=4,8,16.

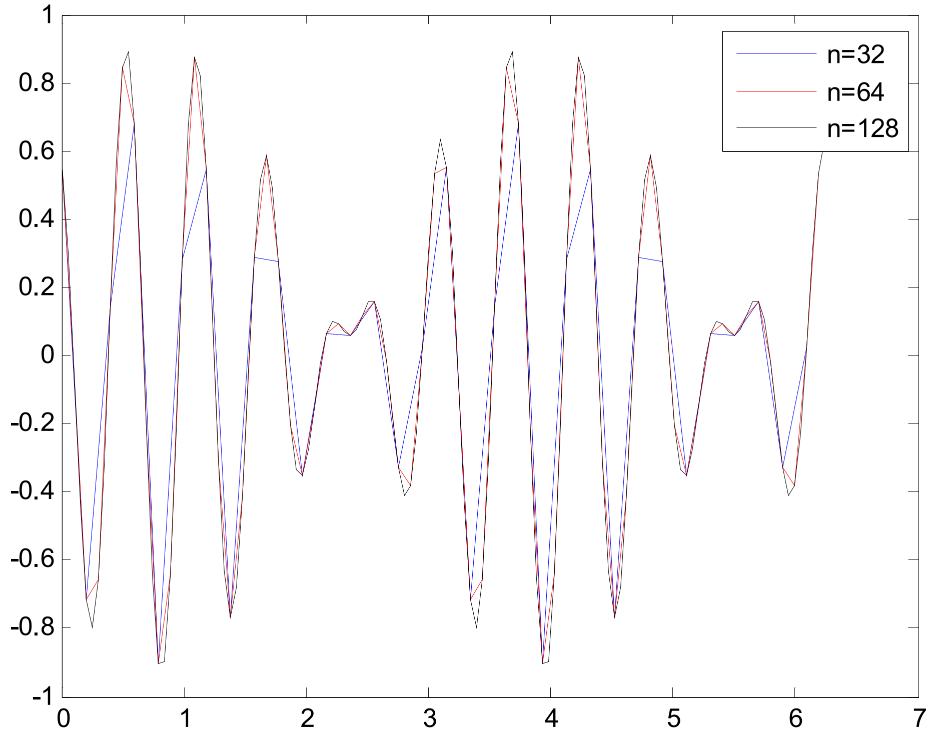


Figure 4.2: Numerical solution of the interior problem for Laplace's equation for n=32,64,128.

In the above diagrams the solution looks like converging.
By increasing the value of n the solution becomes more accurate.

4.1 Separation of variables in polar coordinates

We will use polar coordinates and separation of variables to solve analytically the interior problem:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{4.3}$$

We seek a solution of the form:

$$u = R(r).\ \Theta(\theta)$$

Therefore,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(R(r)\Theta(\theta)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\big(R(r)\Theta(\theta)\big) = 0$$

$$\Rightarrow \Theta(\theta)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + R(r)\frac{1}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

Multiplying by $r^2/R\Theta$ gives us:

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} = 0$$

Let us now separate the variables: i.e. let us collect all of the r -dependent terms on one side of the equation, and all of the θ-dependent terms on the other side. Thus,

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2}$$

The above equation has the form:

f(r)=f(θ)

where f(r) is a function of r and f(θ) is a function of θ. The only way in which the above equation can be satisfied, for general r and θ, is if both sides are equal to the same constant. Thus,

$$\Rightarrow \frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\frac{1}{\Theta}\frac{\partial^2 \Theta}{\partial \theta^2} = c \text{ (constant)}$$

The ordinary differential equations we get are then:

(a) $r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - cR = 0$

(b) $\frac{\partial^2 \Theta}{\partial \theta^2} + c\Theta = 0$

We take (b) $\frac{\partial^2 \Theta}{\partial \theta^2} + c\Theta = 0$

Try Θ=$e^{\lambda\theta}$ so $\lambda^2 e^{\lambda\theta} + ce^{\lambda\theta} = 0 \Rightarrow \lambda^2 + c = 0 \Rightarrow \lambda = \pm\sqrt{-c}$

$$\Rightarrow \Theta = Ae^{\theta\sqrt{-c}} + Be^{-\theta\sqrt{-c}}$$

We know that Θ(θ) must be 2π periodic because is around a periodic boundary.

If c<0 then is not periodic

If c=0 then is not periodic

Thus when c>0 $\Rightarrow \Theta = Ae^{i\theta\sqrt{c}} + Be^{-i\theta\sqrt{c}}$

$$= \breve{A}\cos(\sqrt{c}\,\theta) + \breve{B}\sin(\sqrt{c}\,\theta)$$

So write c=$\nu^2 \Rightarrow \boxed{\Theta = A\cos(\nu\theta) + B\sin(\nu\theta)}$ (4.4)

We take (a) $r\dfrac{\partial}{\partial r}\left(r\dfrac{\partial R}{\partial r}\right) - \nu^2 R = 0$ (where c=ν^2)

$$\Rightarrow r\left(r\frac{\partial^2 R}{\partial r^2} + \frac{\partial R}{\partial r}\right) - \nu^2 R = 0$$

$$\Rightarrow r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} - \nu^2 R = 0$$

This is an Euler differential equation. The general solution to this simple case of Euler's ordinary differential equation is given as:

$$\boxed{R(r) = C_1 r^{\nu} + C_2 r^{-\nu}} \tag{4.5}$$

Combining equations (4.4) and (4.5) we have:

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\Rightarrow u(r,\theta) = (C_1 r^{\nu} + C_2 r^{-\nu}).\ (A\cos(\nu\theta) + B\sin(\nu\theta)) \tag{4.6}$$

As $r \to 0$ the term involving $r^{-\nu}$ is unbounded. The only way to fix this is to take $C_2 = 0$.

Therefore,

$$u(r,\theta) = r^{\nu} \ (A\cos(\nu\theta) + B\sin(\nu\theta)) \quad \text{for any } \nu.$$

We know that equation (b) $\dfrac{\partial^2 \Theta}{\partial \theta^2} + c\Theta = 0$ which its general solution is :

$$\boxed{\Theta = A\cos(\nu\theta) + B\sin(\nu\theta)}$$

We use the boundary conditions Θ (0) =Θ (2π) (i.e. periodic)

$$\left.\begin{array}{l} \Theta\ (0) = A \\ \Theta\ (2\pi) = A\cos(2\pi\nu) \end{array}\right\} \Leftrightarrow \Theta\ (0) = \Theta\ (2\pi)$$

Thus,

$$\cos\ (2\pi\nu)\ ) = 1 \Rightarrow \nu = 0, \pm 1, \pm 2, \ldots.$$

Therefore the general solution of the problem is:

$$u(r,\theta) = \sum_{\nu=0}^{\infty} r^{\nu} \left(A_{\nu} \cos(\nu\theta) + B_{\nu} \sin(\nu\theta)\right) \tag{4.7}$$

In our problem $\dfrac{\partial u}{\partial n} = g$ for a circle then $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial n} = g$ on r=a

$$\frac{\partial u}{\partial r} = \sum_{m=0}^{\infty} m a^{m-1} \left(A_m \cos(m\theta) + B_m \sin(m\theta)\right) = g(\theta) \tag{4.8}$$

The interior Neumann problem is solvable if and only if:

$$\int_0^{2\pi} g(\theta)d\theta = 0$$

but there is no existence of a unique solution.

(Theorem 6.26, Kress 'Linear Integral Equations') .

The coefficients A_m and B_m may be determined by a Fourier expansion on $0 \le \theta \le 2\pi$.

The important observation is that sine and cosine functions of different frequency are orthogonal. This means that, when multiplied and integrated, give zero result:

$$\int_0^{2\pi} \cos(m\theta)\cos(n\theta)d\theta = \begin{cases} \pi & \text{if} \quad m = n \neq 0 \\ 2\pi & \text{if} \quad m = n = 0 \\ 0 & \text{if} \quad m \neq n \end{cases}$$

$$\int_0^{2\pi} \sin(m\theta)\sin(n\theta)d\theta = \begin{cases} \pi & \text{if} \quad m = n \neq 0 \\ 0 & \text{if} \quad m = n = 0 \\ 0 & \text{if} \quad m \neq n \end{cases}$$

$$\int_0^{2\pi} \cos(m\theta)\sin(n\theta)d\theta = 0 \quad , \qquad \text{all } m, n$$

(Fourier Series and Applications, Beatrice Pelloni,2006)

Fix an integer value $n \neq 0$ and multiply (4.8) by cos($n\theta$) and sin($n\theta$), respectively. Then perform the integration term by term:

$$\int_0^{2\pi} g(\theta)\cos(n\theta)d\theta = \sum_{m=0}^{\infty}[ma^{m-1}A_m \int_0^{2\pi}\cos(m\theta)\cos(n\theta)d\theta + B_m \underbrace{\int_0^{2\pi}\cos(n\theta)\sin(m\theta)d\theta}_{=0}]$$

$$= na^{n-1}\pi A_n$$

Hence,

$$\boxed{A_n = \frac{1}{\pi n a^{n-1}}\int_0^{2\pi} g(\theta)\cos(n\theta)d\theta} \qquad n \neq 0$$

While

$$\int_0^{2\pi} g(\theta)\sin(n\theta)d\theta = ma^{m-1}\sum_{m=0}^{\infty}[A_m \underbrace{\int_0^{2\pi}\cos(m\theta)\sin(n\theta)d\theta}_{=0} + B_m \int_0^{2\pi}\sin(n\theta)\sin(m\theta)d\theta]$$

$$= na^{n-1}\pi B_n$$

Hence,

$$\boxed{B_n = \frac{1}{\pi n a^{n-1}}\int_0^{2\pi} g(\theta)\sin(n\theta)d\theta} \qquad n \neq 0$$

The solution of the interior problem is not unique because

If $\left.\begin{array}{l} \Delta u = 0 \\ \dfrac{\partial u}{\partial n} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \Delta(u+c) = 0 \\ \dfrac{\partial(u+c)}{\partial n} = 0 \end{array}$ for all constants c.

Chapter 5

Exterior Problem for Laplace's equation

For the exterior problem (unbounded problem) as we have seen in chapter 2 (2.8) the general integral is:

$$\int_{\partial D} (T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})ds(x) = T(y) - \frac{1}{2\pi R}\int_{\partial\Omega_R} T(x)ds(x) \qquad (5.1)$$

What happens as $R \to \infty$?

For the half plane problem we have the asymptotic condition as y→∞, x →±∞

$$T \to T_{gg}(y) = gy + T_g \qquad \text{so } \frac{\partial T}{\partial y} = g$$

We solve for u=T-T_{gg} and we consider u=T-$gy - T_g$

As $R \to \infty$ then $u \to 0$.

But because we have exterior problem and as $R \to \infty$ we need T(R)=o(1)

i.e. $T(R) \to 0$

Therefore, the part $\frac{1}{2\pi R}\int_{\partial\Omega_R} T(x)ds(x)$ of equation (5.1) disappears and then we only have :

$$\int_{\partial D} (T\frac{\partial G}{\partial n} - G\frac{\partial T}{\partial n})ds(x) = T(y) \qquad (5.2)$$

The above equation (5.2) is the same equation as (2.6) .
Thus, we are going to solve it numerically with the same way as the interior problem.

For the exterior problem the analytical solution is the same as (4.6) :

$$u(r,\theta) = (C_1 r^{\nu} + C_2 r^{-\nu}).\ \ (A\cos(\nu\theta) + B\sin(\nu\theta))$$

except

as r $\rightarrow \infty$ the solution tends to zero. Thus, we take $C_1=0$ and $C_2 \neq 0$.

$$u(r,\theta) = r^{-\nu}.\ \ (A\cos(\nu\theta) + B\sin(\nu\theta))$$

The exterior Neumann problem is uniquely solvable if and only if:

$$\int_0^{2\pi} g(\theta)d\theta = 0$$

(Theorem 6.28, Kress 'Linear Integral Equations') .

Below are the graphs of the numerical solution integral equation:

$$-T(y)+2\int_{\partial D} T\frac{\partial G}{\partial n}d\underset{-}{x}=2\int_{\partial D} Ggd\underset{-}{x}$$ for different values of n.

Where

$\frac{\partial G}{\partial n}=\frac{1}{4\pi}$ which is the same with the $\frac{\partial G}{\partial n}$ of the interior problem but with different sign.

$$f(y)=2\int_{\partial D} 2Ggd\underset{-}{x}=\int_0^{2\pi}\ln\frac{1}{|x|}g(x+y)dx$$

$$g(\theta)=\cos\theta$$

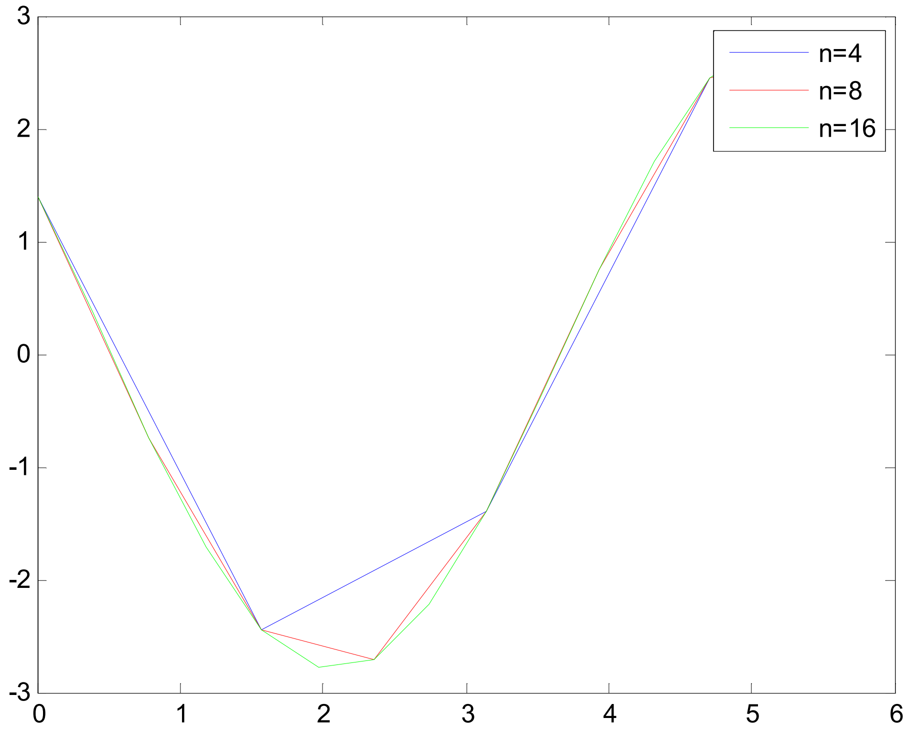


Figure 5.1: The numerical solution of the exterior problem for Laplace's equation for n=4,8,16.

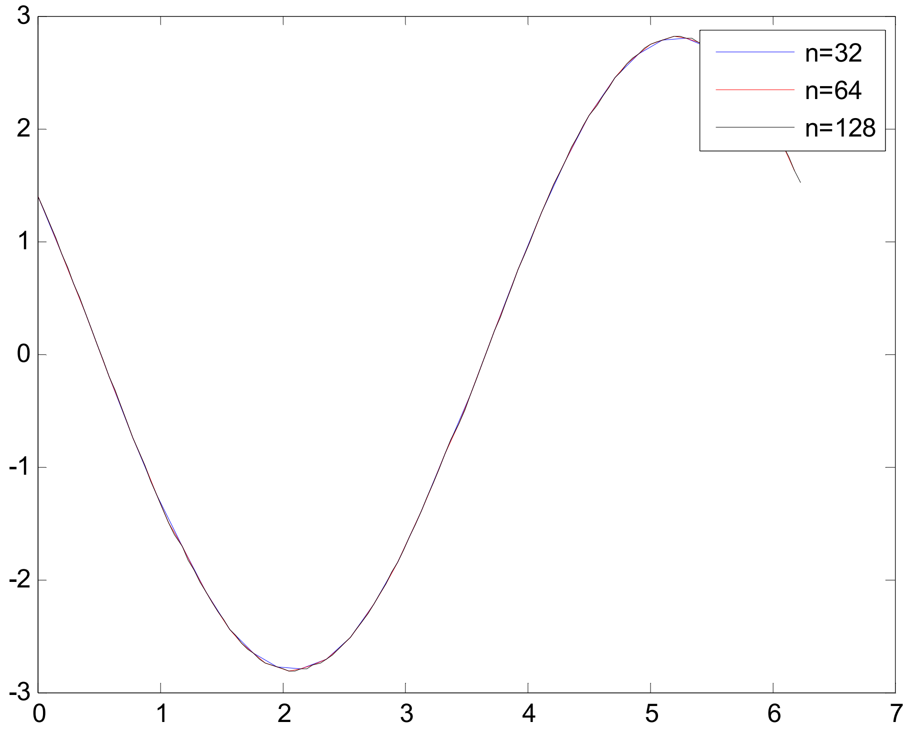


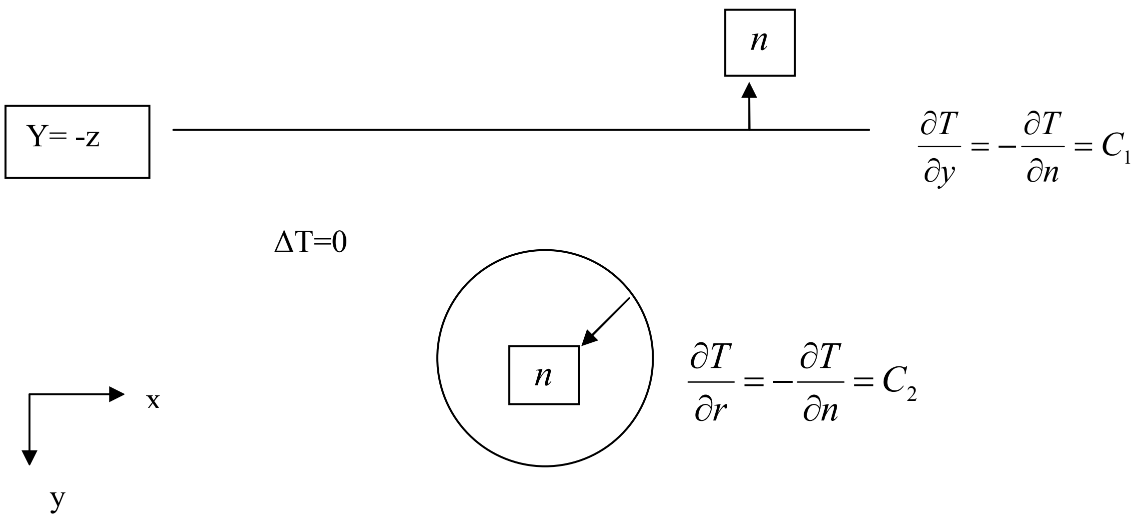
Figure 5.2: The numerical solution of the exterior problem for Laplace's equation for n=32,64,128.

Again we can see a convergence in both diagrams and as the value of n increases the solution becomes more accurate.

Chapter 6

Full pipe flow Problem

To find the general integral equation for the full pipe flow problem we assume the following plot.



where

C_1, C_2 are constants.

We consider the asymptotic condition as $y \to \infty, x \pm \infty$

$T \to gy + T_g$. where T_g is a constant.

We solve for $u = T - gy - T_g$ as $y \to \infty, x \pm \infty$ and we end up :
$u \to 0$ as $y \to \infty, x \pm \infty$

$$\frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} - g = C_1 - g$$

$$\Rightarrow \frac{\partial u}{\partial n} = -\frac{\partial u}{\partial y} = g - C_1 = C_3$$

The following is an integral equation approach:

$$\int (u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n})ds(x) + \int_{-\infty}^{\infty}(-GC_3)ds(x) + \frac{1}{2}u(x) = 0$$ if x is on the boundary of the pipe.

We assume that C_3 is zero .Therefore, we have the following integral equation:

$$\frac{1}{2}u(x) = \int_{\Gamma}(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n})ds(x) + 0$$

$$\Rightarrow \frac{1}{2}u(x) + \int_{\Gamma}(u\frac{\partial G}{\partial n})ds(x) = \int_{\Gamma}(G\frac{\partial u}{\partial n})ds(x)$$

This integral equation is the same as the integral equation of interior and exterior problem except that here the Green's function is:

$$\hat{G}(\underline{x},\underline{y}) = \frac{1}{2\pi}\ln\frac{1}{|\underline{x}-\underline{y}|} + \frac{1}{2\pi}\ln\frac{1}{|\underline{x}-\underline{y}'|} \tag{6.1}$$

We want to find $\frac{\partial \hat{G}}{dn}$.

We separate $\hat{G}(\underline{x},\underline{y})$ into two parts.

The 1st part is G$(\underline{x},\underline{y}) = \frac{1}{2\pi}\ln\frac{1}{|\underline{x}-\underline{y}|}$ and the

2nd part we call it P$(\underline{x},\underline{y}') = \frac{1}{2\pi}\ln\frac{1}{|\underline{x}-\underline{y}'|}$

We already know from (4.2) $\frac{\partial G}{\partial n} = -\frac{1}{4\pi}$

Thus, now we want to find $\frac{\partial P}{\partial n}$.

We know $P(\underline{x}, \underline{y}') = \frac{1}{2\pi} \ln \frac{1}{|\underline{x} - \underline{y}'|}$ where $\underline{x} = (x_1, x_2)$

$\underline{y}' = (y_1, -2z - y_2)$

We substitute $\underline{x}$ and $\underline{y}'$ in P and then we have the following expression for P:

$$P(\underline{x}, \underline{y}') = \frac{1}{2\pi} \ln \frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 + 2z + y_2)^2}}$$

$$\frac{\partial P}{\partial n} = \underline{n}(\underline{x}).\nabla_x P = \begin{pmatrix} n_1(x) \\ n_2(x) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial P}{\partial x_1} \\ \frac{\partial P}{\partial x_2} \end{pmatrix} = n_1(x)\frac{\partial P}{\partial x_1} + n_2(x)\frac{\partial P}{\partial x_2} \tag{6.2}$$

$$\frac{\partial P}{\partial x_1} = \frac{1}{2\pi}\sqrt{(x_1 - y_1)^2 + (x_2 + 2z + y_2)^2}\left(-\frac{1}{2}\right)\left[(x_1 - y_1)^2 + (x_2 + 2z + y_2)^2\right]^{-\frac{3}{2}} 2(x_1 - y_1)$$

$$\frac{\partial P}{\partial x_1} = -\frac{1}{2\pi}\frac{[(x_1 - y_1)^2 + (x_2 + 2z + y_2)^2]^{\frac{1}{2}}}{[(x_1 - y_1)^2 + (x_2 + 2z + y_2)^2]^{\frac{3}{2}}} \cdot (x_1 - y_1)$$

$$\frac{\partial P}{\partial x_1} = -\frac{1}{2\pi}\frac{(x_1 - y_1)}{[(x_1 - y_1)^2 + (x_2 + 2z + y_2)^2]}$$ and in the same way

$$\frac{\partial P}{\partial x_2} = -\frac{1}{2\pi}\frac{(x_2 + 2z + y_2)}{[(x_1 - y_1)^2 + (x_2 + 2z + y_2)^2]}$$

We set $x = (\cos(\varsigma), \sin(\varsigma)) \Rightarrow n_1 = \cos\varsigma, n_2 = \sin\varsigma$

$y = (\cos(t), \sin(t))$

After that we replace $\dfrac{\partial P}{\partial x_1}$, $\dfrac{\partial P}{\partial x_2}$, n_1 and n_2 into equation (6.2)

And finally we have:

$$\frac{\partial P}{\partial n} = -\frac{1}{2\pi} \frac{(x_1 - y_1)}{[(x_1 - y_1)^2 + (x_2 + 2z + y_2)^2]} \cos(\varsigma) - \frac{1}{2\pi} \frac{(x_2 + 2z + y_2)}{[(x_1 - y_1)^2 + (x_2 + 2z + y_2)^2]} \sin(\varsigma)$$

$$\Rightarrow \frac{\partial P}{\partial n} = -\frac{1}{2\pi[(x_1 - y_1)^2 + (x_2 + 2z + y_2)^2]} \left[\cos(\varsigma)(x_1 - y_1) + \sin(\varsigma)(x_2 + 2z + y_2)\right]$$

We also replace $x_1 = \cos(\varsigma)$ $\qquad y_1 = \cos(t)$

$x_2 = \sin(\varsigma)$ $\qquad y_2 = \sin(t)$

$$\Rightarrow \frac{\partial P}{\partial n} = -\frac{\left[\cos\varsigma(\cos\varsigma - \cos t) + \sin\varsigma(\sin\varsigma + 2z + \sin t)\right]}{2\pi[(\cos(\varsigma) - \cos(t))^2 + (\sin(\varsigma) + 2z + \sin(t))^2]}$$

(Note: $[(\cos\varsigma - \cos t)^2 + (\sin\varsigma + 2z + \sin t)^2] = 2 - 2(\cos\varsigma\cos t - \sin\varsigma\sin t)$

$$+ 4z^2 + 4z\sin\varsigma + 4z\sin t$$

$$= 2 - 2\cos(\varsigma + t) + 4z^2$$
$$+ 4z\sin\varsigma + 4z\sin t$$

$$= 2 - 2\cos 2\left(\frac{\varsigma + t}{2}\right) + 4z^2$$
$$+ 4z\sin\varsigma + 4z\sin t$$

$$= 2 - 2[1 - 2\sin^2\left(\frac{\varsigma + t}{2}\right)]$$
$$+ 4z^2 + 4z\sin\varsigma + 4z\sin t$$

$$= 4\sin^2\left(\frac{\varsigma + t}{2}\right) + 4z^2 +$$

$$4z\sin\varsigma + 4z\sin t$$

).

$$\Rightarrow \frac{\partial P}{\partial n} = \frac{-\cos^2\varsigma + \cos\varsigma\cos t - \sin^2\varsigma - \sin\varsigma\sin t - 2z\sin\varsigma}{2\pi\cdot[4\sin^2(\frac{\varsigma + t}{2}) + 4z^2 + 4z\sin\varsigma + 4z\sin t]}$$

$$\Rightarrow \frac{\partial P}{\partial n} = \frac{-1 + \cos\varsigma\cos t - \sin\varsigma\sin t - 2z\sin\varsigma}{2\pi\cdot[4\sin^2(\frac{\varsigma + t}{2}) + 4z^2 + 4z\sin\varsigma + 4z\sin t]}$$

$$\Rightarrow \frac{\partial P}{\partial n} = -\frac{1}{2\pi}\frac{(1 - (\cos\varsigma\cos t - \sin\varsigma\sin t) + 2z\sin\varsigma}{4\sin^2(\frac{\varsigma + t}{2}) + 4z^2 + 4z\sin\varsigma + 4z\sin t}$$

(Note: Trigonometric identity: $\cos^2\varsigma + \sin^2\varsigma = 1$)

Now we are going to simplify the numerator of this fraction.

$$1 - (\cos\varsigma\cos t - \sin\varsigma\sin t) = 1 - \cos(\varsigma + t)$$

$$= 1 - (1 - 2\sin^2(\frac{\varsigma + t}{2}))$$

$$= 2\sin^2(\frac{\varsigma + t}{2})$$

(Note: Trigonometric identities: $\cos(\varsigma + t) = \cos\varsigma\cos t - \sin\varsigma\sin t$

$$\cos 2\varsigma = \cos^2\varsigma - \sin^2\varsigma$$

$$\Rightarrow \cos 2\varsigma = 1 - \sin^2\varsigma - \sin^2\varsigma$$

$\Rightarrow \cos 2\varsigma = 1 - 2\sin^2\varsigma$).

$$\Rightarrow \frac{\partial P}{\partial n} = -\frac{1}{2\pi}\frac{2\sin^2\left(\frac{\varsigma + t}{2}\right) + 2z\sin\varsigma}{4\sin^2(\frac{\varsigma + t}{2}) + 4z^2 + 4z\sin\varsigma + 4z\sin t}$$

Finally,

$$\frac{\partial \hat{G}}{dn} = \frac{\partial G}{\partial n} + \frac{\partial P}{\partial n} = -\frac{1}{4\pi} + -\frac{1}{2\pi}\frac{2\sin^2\left(\frac{\varsigma+t}{2}\right)+2z\sin\varsigma}{4\sin^2(\frac{\varsigma+t}{2})+4z^2+4z\sin\varsigma+4z\sin t}$$

Therefore, the integral equation of the full pipe flow problem is solved with the same way as the exterior and interior problem. The only difference is the Kernel.

Below are the graphs of the numerical solution of the integral equation:

$$-T(y) + 2\int_{\partial D} T\frac{\partial G}{\partial n} d\underset{-}{x} = 2\int_{\partial D} Ggd\underset{-}{x}$$ for different values of n.

Where

$$\frac{\partial G}{\partial n} = -\frac{1}{4\pi} + -\frac{1}{2\pi}\frac{2\sin^2\left(\frac{\varsigma+t}{2}\right)+2z\sin\varsigma}{4\sin^2(\frac{\varsigma+t}{2})+4z^2+4z\sin\varsigma+4z\sin t}$$

$$f(y) = 2\int_{\partial D} 2Ggd\underset{-}{x} = \int_0^{2\pi}\ln\frac{1}{|x|}g(x+y)dx$$

$$g(\theta) = \cos(10\theta)$$

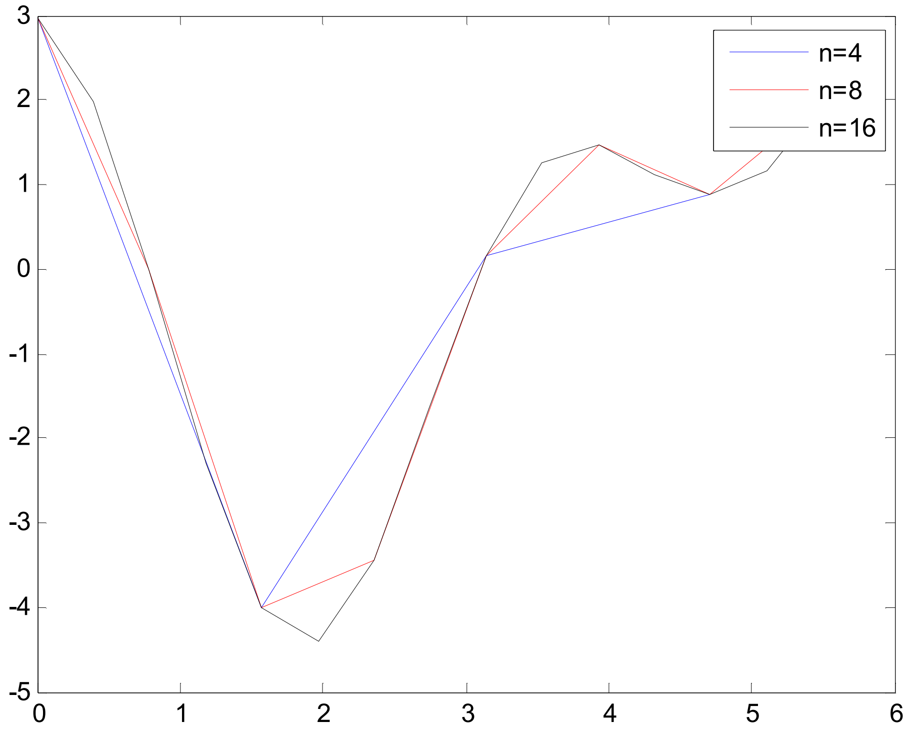


Figure 6.1: The numerical solution of the full pipe flow problem for n=4,8,16.

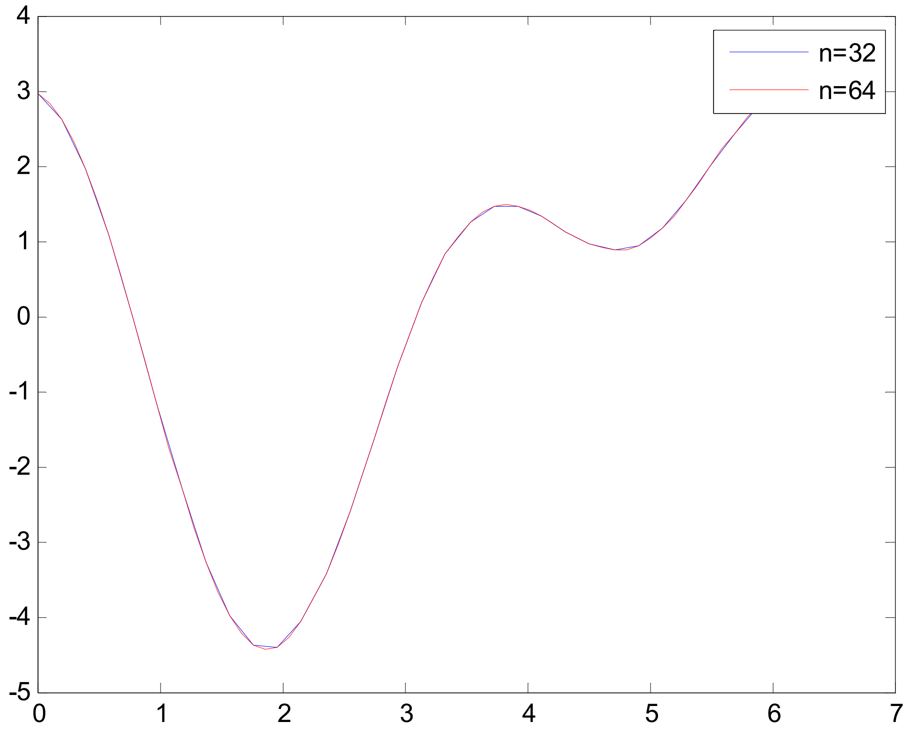


Figure 6.2: The numerical solution of the full pipe flow problem for n=32,64.

As we can see from the diagrams the solution converges and by increasing the value of n , the solution becomes more accurate.

Chapter 7

Summary and Conclusions

This dissertation has used the Boundary Element Method to approximate the heat transfer in a buried pipe. In chapter 1 we started by becoming familiar with the pipe flow problem. We explained what the heat transfer is and simplified our problem within the pipe to one-dimension and made some assumptions for the problem.
In chapter 2 we introduce the Boundary Element Method. The Boundary Element Method is a numerical method for solving Partial Differential Equations which have been formulated as integral equations. The advantages in the BEM arise from the fact that only the boundary of the domain of the PDE requires sub-division. So, the dimension of the problem is effectively reduced by one. We also reformulate the PDE as a Boundary Integral Equation. We looked at the bounded problem when y is on the domain and y is on the boundary. We also looked at the unbounded problem where y is not on the domain. Moreover, we looked at the full pipe problem.
In chapter 3 we talked about the methods that we have used to solve a Fredholm integral equation of the 2nd kind. We had two examples of a single integral equation. One example of a periodic function and one of the non-periodic function. We end up that the periodic function is faster than the non-periodic function.
In chapter 4 we saw the numerical solution of the interior boundary integral equation arising from the Laplace's equation. We also looked at the analytical solution of the problem using polar coordinates and separation of variables. We came to the conclusion that the numerical solution of the interior problem of Laplace's equation is converging and that by increasing the value of n the solution becomes more accurate. Furthermore, the solution of the interior problem is not unique.
In chapter 5 we saw the numerical solution of the exterior boundary integral equation arising from the Laplace's equation. We end up that the numerical solution of the exterior problem of Laplace's equation is converging. Moreover, the solution of the exterior problem is unique.
Finally, in chapter 6 we have approached an integral equation for the full pipe flow problem and we saw that the numerical solution of the full pipe flow problem is converging.

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