

UNIVERSITY OF READING

Department of Mathematics

**NUMERICAL  
APPROXIMATION  
OF SIMILARITY IN  
NONLINEAR DIFFUSION  
EQUATIONS**

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# Abstract

This dissertation investigates the numerical solution of similarity solution of the Porous Medium and the Thin Film equations. Scaling transformations are introduced to reduce the original equations to ordinary differential equations. Self-similar solutions are found for all  $n > 0$  in the case of the Porous Medium equation, but only for  $n = 1$  in the Thin Film equation. The ordinary differential equations are solved numerically and the numerical results are compared with the self-similar solutions to verify the accuracy of the numerical schemes used. The main idea is to find a numerical self-similar solution for  $n > 1$  in the Thin Film equation.

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# Declaration

I confirm that this is my own work, and the use of all material from other sources has been properly and fully acknowledged.

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Siti Mazulianawati Haji Majid

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# Chapter 1

## Introduction

### 1.1 Idea of Similarity

The study of time-dependent partial differential equations (PDEs) has had a sporadic history up to the present time. Such equations arise in many branches of applied mathematics. For example, in physical sciences, PDEs are used to describe phenomena such as the propagation of sound or heat, fluid flow, and laws such as the conservation of mass, energy and momentum. Their solutions give an insight into the physical processes they are modelling, with some initial information provided on a given domain. Although there are methods for solving the underlying model equations, sometimes the solution (either general or particular) is difficult, if not impossible, to find.

When dealing with a linear PDE, various techniques including integral transforms and eigenfunction expansions help to reduce the equation into an ordinary differential equation (ODE), which can be easily solved. Such techniques are much less prevalent when they come to non-linear PDEs. However, there is an approach which identifies equations for which the solution depends on appropriate groupings of the independent variables.

One way of grouping these independent variables to obtain analytic solutions is by similarity. This method works under (group) transformations of the original independent variables into new independent variables, called similarity variables, leaving the equation in question invariant. This process was systematised in the early 20th century by Lie who observed that groupings of variables in many partial differential problems made effective transformed variables [5]. The solutions are termed similarity solutions, which often satisfy much simpler

equations than the original PDE. The importance of similarity solutions lies in their ease of calculation, the fact that they often act as attractors for the more general solutions of the PDE.

The examples below show different types of PDEs for which they are invariant under certain groups of transformations [1].

Example 1: Fisher's equation, given by

$$u_t = u_{xx} + u(1 - u), \quad u \in C^2(\infty, -\infty)$$

plays an important role in the study of mathematical biology and in probability.

It is invariant under:

- i. translations in time,  $t \rightarrow t + \lambda$
- ii. translations in space,  $x \rightarrow x + \lambda$
- iii. reflexions in  $x$ ,  $x \rightarrow -x$

Example 2: The universal heat equation

$$u_t = u_{xx}, \quad u \in C^2(\infty, -\infty)$$

is invariant under the same transformations as in Example 1, and also the stretching groups given by

$$\begin{aligned} t &\rightarrow \lambda t, \\ x &\rightarrow \lambda^{\frac{1}{2}} x, \end{aligned}$$

where  $\lambda > 0$  is arbitrary.

Example 3: The blow-up equation, of the form

$$u_t = u_{xx} + u^2, \quad u \in C^2(\infty, -\infty)$$

is used in modelling combustion processes in which materials become hot very quickly. It is invariant under the action of the same groups as in Example 2, with an additional stretching group given by

$$u \rightarrow \frac{u}{\lambda}.$$

## 1.2 Scale Invariance

Scale invariance is a basic idea which originates from the analysis of the consequences of changes of units of measurement on the mathematical form of the laws of physics. It is viewed as a particular aspect of study of differential equations under groups of transformations [7].

Basically, this type of transformation maps all the variables in the original differential equation (DE) to newly transformed variables by different scaling parameters for each of the original variables. The DE is then said to be scale-invariant if the system remains unchanged by the transformations.

### 1.2.1 Scaling Invariance on an ODE

Suppose a ODE is given by

$$\frac{dy}{dx} = F(x, y). \quad (1.1)$$

Introduce a mapping from the original system  $(x, y)$  to a new system  $(\bar{x}, \bar{y})$  by the transformations

$$\begin{aligned} \bar{x} &= \alpha x \\ \text{and } \bar{y} &= \beta y \end{aligned} \quad (1.2)$$

where  $\alpha$  and  $\beta$  are the scaling parameters.

Substituting (1.2) into equation (1.1), the left-hand side becomes

$$\begin{aligned} \frac{dy}{dx} &= \frac{d\left(\frac{\bar{y}}{\beta}\right)}{d\left(\frac{\bar{x}}{\alpha}\right)} \\ &= \frac{\alpha d\bar{y}}{\beta d\bar{x}} \end{aligned}$$

and the right-hand side gives

$$F(x, y) = F\left(\frac{\bar{x}}{\alpha}, \frac{\bar{y}}{\beta}\right).$$

The new system is therefore

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\beta}{\alpha} F\left(\frac{\bar{x}}{\alpha}, \frac{\bar{y}}{\beta}\right).$$

The definition of invariance is then

$$\frac{\beta}{\alpha} F\left(\frac{\bar{x}}{\alpha}, \frac{\bar{y}}{\beta}\right) = F(\bar{x}, \bar{y}).$$

### 1.2.2 Scaling Invariance on a PDE

Consider a PDE of general form

$$u_t = f(x, u, u_x, u_{xx}, \dots). \quad (1.3)$$

A scaling transformation is described by a mapping of  $(u, x, t)$  to  $(\bar{u}, \bar{x}, \bar{t})$  such that

$$\begin{aligned} t &= \lambda \bar{t}, \\ x &= \lambda^\beta \bar{x}, \\ u &= \lambda^\gamma \bar{u}, \end{aligned} \quad (1.4)$$

for an arbitrary parameter  $\lambda > 0$  and scaling powers  $\beta, \gamma$ .

It is an observed fact that scaling relationships have wide applications in science and in engineering. Such scalings give evidence of deep properties of the phenomena they represent. They can be found in fluid mechanics, turbulence, mathematical biology and structural geology, to name a few. Scaling invariance is also closely related to the theories of fractals and dimensional analysis [6].

## 1.3 Self-Similar Solutions

A time-dependent phenomenon is called self-similar if the spatial distributions at different moments in time can be derived from one another by a similarity transform [6]. The self-similarity of the solutions of time-dependent PDEs has allowed their reduction to ODEs, which simplifies matters, and hence they had attracted attention, due to the simplicity of obtaining and analyzing them.

### 1.3.1 Self-Similarity for an ODE

Rearranging (1.2) results in

$$\alpha = \frac{\bar{x}}{x}$$

and

$$\beta = \frac{\bar{y}}{y}.$$

Assuming there exists a functional relationship between  $\alpha$  and  $\beta$  such that  $\beta = \beta(\alpha)$ , self-similar solutions can be found to satisfy the ODE in (1.1), which is invariant under the transformations in (1.2).

### 1.3.2 Self-Similarity for a PDE

Making  $\lambda$  the subject from (1.4), we have

$$\lambda = \frac{u^{\frac{1}{\gamma}}}{\bar{u}^{\frac{1}{\gamma}}} = \frac{t}{\bar{t}} = \frac{x^{\frac{1}{\beta}}}{\bar{x}^{\frac{1}{\beta}}}$$

Define similarity variables to be

$$\eta = \frac{u}{t^\gamma} = \frac{\bar{u}}{\bar{t}^\gamma}, \tag{1.5}$$

$$\xi = \frac{x}{t^\beta} = \frac{\bar{x}}{\bar{t}^\beta}, \tag{1.6}$$

which are independent of  $\lambda$  and hence scale-invariant under (1.4).

A functional relationship between the similarity variables is assumed to take the form  $\eta = \eta(\xi)$ . We can then find self-similar solutions satisfying the ODE, that is derived by the transformation of the original PDE into the variables  $\eta$  and  $\xi$  with scaling exponents  $\beta$  and  $\gamma$ .

## 1.4 Outline of Dissertation

In Chapter Two, we look at non-linear diffusion equation and its applications. An illustration of the method of similarity under scaling transformation to this equation is presented. A detailed account of the construction of similarity variables to obtain self-similar solutions are shown for both the Porous Medium equation, for all  $n$ , and the Thin Film equation, in the case of  $n = 1$ , with given initial conditions. We then go on to consider the applications of both diffusion equations. Furthermore, we develop another approach to solving both equations by including the time variable in the relationship between the new transformed variables. This results in scale-invariant but not self-similarity solutions. It was hoped that when method is run to convergence, within some tolerance, the

solutions would converge to give the numerical self-similar solutions for both equations. In practice, we found another way of achieving our aims so this was left to further work.

Chapter Three provides various numerical methods in solving the simplified ODE, in terms of the transformed variables, for the Porous Medium equation. The numerical results are then verified with the self-similar solutions for all values of  $n$ . We also implement numerical scheme to the second approach using time-stepping. The program is run to finite time to obtain non-self-similar solutions. Then we include some tolerance so that the solutions converge to give self-similar solutions. Graphs are produced to show these different solutions.

In Chapter Four, we apply the same methods to solving the Thin Film equation. In this case, we introduce a new variable so that the ODE looks similar to that of transformed Porous Medium equation. We compare the result from the numerical scheme with that from self-similar, for  $n = 1$ . As for the other values of  $n$ , we find the solutions numerically.

Finally, we end with conclusions and possible areas of further work.

# Chapter 2

## Non-Linear Diffusion

We focus mainly on non-linear diffusion equations of general form

$$\frac{\partial u}{\partial t} = (-1)^m \frac{\partial}{\partial x} \left( u^n \frac{\partial^{2m+1} u}{\partial x^{2m+1}} \right), \quad (2.1)$$

where  $u^n$  represents the diffusion coefficient and  $n > 0$  is a diffusion growth exponent. It describes many physical processes such as heat conduction in a solid body, insect population dispersion, radiation hydrodynamics, and many others.

Transforming the left-hand side of equation (2.1) under the mapping described in (1.4) into the variables  $(\bar{u}, \bar{x}, \bar{t})$  gives

$$\frac{\partial u}{\partial t} = \frac{\partial (\lambda^\gamma \bar{u})}{\partial (\lambda \bar{t})} = \lambda^{\gamma-1} \frac{\partial \bar{u}}{\partial \bar{t}} \quad (2.2)$$

and the right-hand side becomes

$$\begin{aligned} (-1)^m \frac{\partial}{\partial x} \left( u^n \frac{\partial^{2m+1} u}{\partial x^{2m+1}} \right) &= (-1)^m \frac{\partial}{\partial (\lambda^\beta \bar{x})} \left( \lambda^{n\gamma} \bar{u}^n \frac{\partial^{2m+1} (\lambda^\gamma \bar{u})}{\partial (\lambda^\beta \bar{x})^{2m+1}} \right) \\ &= (-1)^m \lambda^{\gamma(n+1)-(2m+2)\beta} \frac{\partial}{\partial \bar{x}} \left( \bar{u}^n \frac{\partial^{2m+1} \bar{u}}{\partial \bar{x}^{2m+1}} \right). \end{aligned} \quad (2.3)$$

Hence, the transformed general PDE is

$$\lambda^{\gamma-1} \frac{\partial \bar{u}}{\partial \bar{t}} = (-1)^m \lambda^{\gamma(n+1)-(2m+2)\beta} \frac{\partial}{\partial \bar{x}} \left( \bar{u}^n \frac{\partial^{2m+1} \bar{u}}{\partial \bar{x}^{2m+1}} \right). \quad (2.4)$$

For the original equation (2.1) to be invariant under the transformation (1.4),

we require

$$\gamma - 1 = (n + 1) \gamma - (2m + 2) \beta. \quad (2.5)$$

To determine  $\gamma$  and  $\beta$  we need another equation. Integrating equation (2.1) over the domain gives

$$\int_{a(t)}^{b(t)} \frac{\partial u}{\partial t} dx = \int_{a(t)}^{b(t)} (-1)^m \frac{\partial}{\partial x} \left( u^n \frac{\partial^{2m+1} u}{\partial x^{2m+1}} \right) dx,$$

which simplifies to

$$\frac{d}{dt} \int_{a(t)}^{b(t)} u dx = (-1)^m \left[ u^n \frac{\partial^{2m+1} u}{\partial x^{2m+1}} \right]_{a(t)}^{b(t)}.$$

Taking the boundary conditions at  $u(a(t)) = u(b(t)) = 0$ , we have

$$\frac{d}{dt} \int_{a(t)}^{b(t)} u dx = 0$$

and therefore

$$\int_{a(t)}^{b(t)} u dx = k, \quad (2.6)$$

where  $k$  is a constant. This shows that mass is conserved over the whole domain.

Transforming the integral in (2.6) to the variables  $(\bar{u}, \bar{x}, \bar{t})$ , by (1.4), we obtain

$$\begin{aligned} \int_{a(t)}^{b(t)} \lambda^\gamma \bar{u} d(\lambda^\beta \bar{x}) &= k \\ \lambda^{\gamma+\beta} \int_{a(t)}^{b(t)} \bar{u} d\bar{x} &= \lambda^0 k. \end{aligned}$$

For (2.6) to be invariant under the transformation (1.4), we require

$$\gamma + \beta = 0. \quad (2.7)$$

Solving (2.5) and (2.7) simultaneously, we find that

$$\begin{aligned} \gamma &= \frac{-1}{n + (2m + 2)} \\ \text{and } \beta &= \frac{1}{n + (2m + 2)} \end{aligned} \quad (2.8)$$

By (1.5) and (1.6), the self-similar solutions for non-linear diffusion equations are of the form

$$\begin{aligned} \eta &= \eta(\xi) \\ \text{i.e. } u(x, t) &= t^\gamma \eta\left(\frac{x}{t^\beta}\right) \end{aligned} \tag{2.9}$$

under the scalings  $\gamma$  and  $\beta$  defined in (2.8).

## 2.1 The Porous Medium Equation

In the case when  $m = 0$ , equation (2.1) becomes the Porous Medium equation (PME), a second-order non-linear diffusion equation. It has the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( u^n \frac{\partial u}{\partial x} \right), \tag{2.10}$$

where  $n$  is as described in equation (2.1), with  $u = u^n \frac{\partial u}{\partial x} = 0$  at the boundaries. Such problems with zero boundary conditions are degenerate in the sense that  $u = 0$  is a sufficient boundary condition.

Equation (2.10) has been widely used to model many different applications. In physical problems, it is used to model the flow in thin saturated region in a porous medium, the percolation of gas through porous media, the spreading of thin viscous spreading under gravity over a horizontal plane, and many other processes [2],[10].

Further applications of the second-order case arise in the modelling of bacterial growth on agar plates and in medicine. An example for the latter is the development of a tumour inside a human body. The tumour gains nutrients and oxygen for growth by diffusion from already existing vasculature surrounding them. Thus the size of the tumour is limited by diffusion through a porous medium [3].

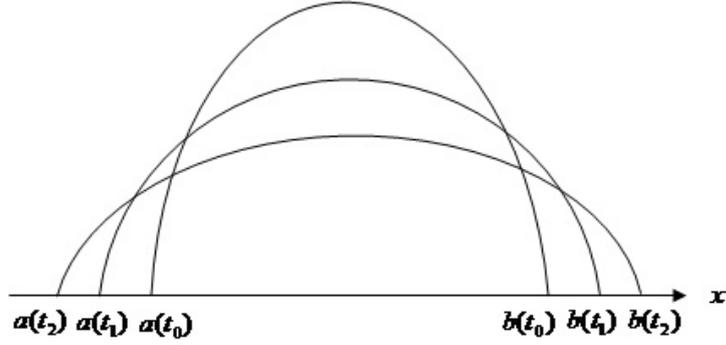


Figure 2.1: Evolution of PME at Different Times for a fixed  $n$

The above figure represents a self-similar solution at three points in time. The solution at time  $t_0$  is transformed onto the solution at a different time, say at time  $t_1$  by the scaling transformation in equation (1.4).

### 2.1.1 Self-Similar Solutions

From (2.8),  $\gamma$  and  $\beta$  for the second-order equation (2.10) are

$$\begin{aligned} \gamma &= \frac{-1}{(n+2)} \quad \text{and} \\ \beta &= \frac{1}{(n+2)}. \end{aligned} \tag{2.11}$$

To construct self-similar solutions in the form given in (2.9), we first derive the ODE in terms of the similarity variables  $\eta = \eta(\xi)$  and  $\xi$ , which are defined in (1.5) and (1.6).

The left-hand side of equation (2.10) gives

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial(\eta t^\gamma)}{\partial t} \\
&= \eta\gamma t^{\gamma-1} + t^\gamma \frac{\partial \eta}{\partial t} \\
&= \eta\gamma t^{\gamma-1} + t^\gamma \left( \frac{d\eta}{d\xi} \times \frac{d\xi}{dt} \right) \\
&= \eta\gamma t^{\gamma-1} + t^\gamma \left( \frac{d\eta}{d\xi} \times \frac{-\beta x}{t^{\beta+1}} \right) \\
&= \eta\gamma t^{\gamma-1} + t^\gamma \left( \frac{d\eta}{d\xi} \times \frac{-\beta \xi}{t} \right) \\
&= \eta\gamma t^{\gamma-1} - \beta t^{\gamma-1} \xi \frac{d\eta}{d\xi}
\end{aligned} \tag{2.12}$$

and the right-hand side becomes

$$\begin{aligned}
\frac{\partial}{\partial x} \left( u^n \frac{\partial u}{\partial x} \right) &= \frac{\partial \xi}{\partial x} \frac{d}{d\xi} \left( \eta^n t^{\gamma n} \frac{\partial u}{\partial \eta} \frac{d\eta}{d\xi} \frac{\partial \xi}{\partial x} \right) \\
&= \frac{1}{t^\beta} \frac{d}{d\xi} \left( \eta^n t^{\gamma n} t^\gamma \frac{d\eta}{d\xi} \frac{1}{t^\beta} \right) \\
&= \frac{t^{\gamma n + \gamma}}{t^{2\beta}} \frac{d}{d\xi} \left( \eta^n \frac{d\eta}{d\xi} \right) \\
&= t^{\gamma n + \gamma - 2\beta} \frac{d}{d\xi} \left( \eta^n \frac{d\eta}{d\xi} \right).
\end{aligned} \tag{2.13}$$

Substituting (2.12) and (2.13) into equation (2.10), we have

$$\begin{aligned}
\eta\gamma t^{\gamma-1} - \beta t^{\gamma-1} \xi \frac{d\eta}{d\xi} &= t^{\gamma n + \gamma - 2\beta} \frac{d}{d\xi} \left( \eta^n \frac{d\eta}{d\xi} \right) \\
t^{\gamma-1} \left( \eta\gamma - \beta \xi \frac{d\eta}{d\xi} \right) &= t^{\gamma n + \gamma - 2\beta} \frac{d}{d\xi} \left( \eta^n \frac{d\eta}{d\xi} \right) \\
\eta\gamma - \beta \xi \frac{d\eta}{d\xi} &= t^{\gamma n + 1 - 2\beta} \frac{d}{d\xi} \left( \eta^n \frac{d\eta}{d\xi} \right).
\end{aligned} \tag{2.14}$$

From (2.11), we see that  $\beta = -\gamma$ , therefore the exponent of  $t$  can be simplified,

$$\begin{aligned}
\gamma n + 1 - 2\beta &= \gamma n + 1 + 2\gamma \\
&= \left( \frac{-n}{(n+2)} + 1 \right) - \frac{2}{(n+2)} \\
&= 0
\end{aligned} \tag{2.15}$$

and hence

$$\begin{aligned}
-\eta\beta - \beta\xi\frac{d\eta}{d\xi} &= \frac{d}{d\xi}\left(\eta^n\frac{d\eta}{d\xi}\right) \\
-\beta\left(\eta + \xi\frac{d\eta}{d\xi}\right) &= \frac{d}{d\xi}\left(\eta^n\frac{d\eta}{d\xi}\right) \\
-\beta\frac{d}{d\xi}(\eta\xi) &= \frac{d}{d\xi}\left(\eta^n\frac{d\eta}{d\xi}\right), \tag{2.16}
\end{aligned}$$

which is a second-order ODE in the function  $\eta(\xi)$  with boundary conditions  $\eta = 0$  at  $\xi = \pm 1$ .

In order to solve the ODE, we integrate (2.16) to get

$$\eta^n\frac{d\eta}{d\xi} = -\beta(\eta\xi) + C,$$

where  $C$  is an integration constant. Since  $\eta = 0$  at the boundary,  $C = 0$  and thus

$$\eta^n\frac{d\eta}{d\xi} = -\beta(\eta\xi).$$

Dividing through by  $\eta$  knowing that  $\eta$  is non-zero, gives

$$\eta^{n-1}\frac{d\eta}{d\xi} = -\beta\xi.$$

Separating the variables,

$$\begin{aligned}
\int \eta^{n-1} d\eta &= -\beta \int \xi d\xi \\
\frac{\eta^n}{n} &= \frac{-\beta\xi^2}{2} + K,
\end{aligned}$$

where  $K$  is a constant of integration. Applying the zero boundary conditions again,

$$\frac{\eta^n}{n} = \frac{-\beta\xi^2}{2} + \frac{\beta}{2}.$$

Bearing in mind that  $u = t^\gamma\eta$  in (1.5), we make  $\eta$  the subject and setting  $A_n = \frac{\beta n}{2}$ ,

$$\eta(\xi) = \left(A_n - \frac{\beta n \xi^2}{2}\right)^{\frac{1}{n}}.$$

Hence, there exists the following self-similar solution in terms of  $\eta$  and  $\xi$  with

zero boundary conditions,

$$\eta(\xi) = \begin{cases} \left(A_n - \frac{\beta n \xi^2}{2}\right)^{\frac{1}{n}} & \frac{\beta n \xi^2}{2} \leq A_n, \\ 0 & \frac{\beta n \xi^2}{2} > A_n. \end{cases}$$

Mapping this back to the original variables  $u, x$  and  $t$  using the definitions in (1.5) and (1.6) with  $\beta$  and  $\gamma$  given by (2.11), we obtain

$$u(x, t) = \frac{1}{t^{\frac{1}{n+2}}} \left( A_n - \frac{nx^2}{2(n+2)t^{\frac{2}{n+2}}} \right)^{\frac{1}{n}}_+, \quad (2.17)$$

where the notation  $(\cdot)_+^{\frac{1}{n}}$  indicates that we take the positive solution of the argument. Equation (2.17) is a self-similar solution of the original PME with  $u = 0$  at the boundaries, for all values of  $n > 0$ . The original derivation is due to Barenblatt[6].

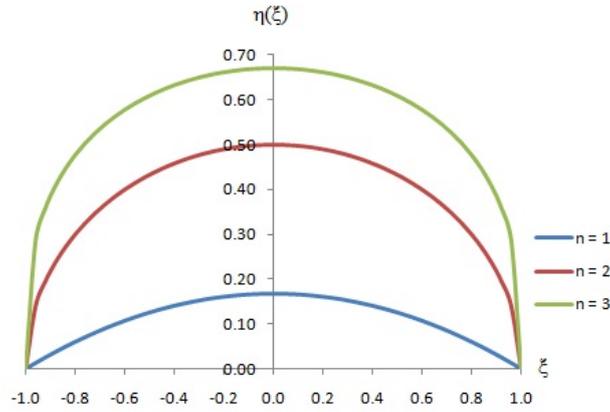


Figure 2.2: Self-Similar Solutions of the transformed PME for  $n = 1, 2, 3$  when  $t = 1$

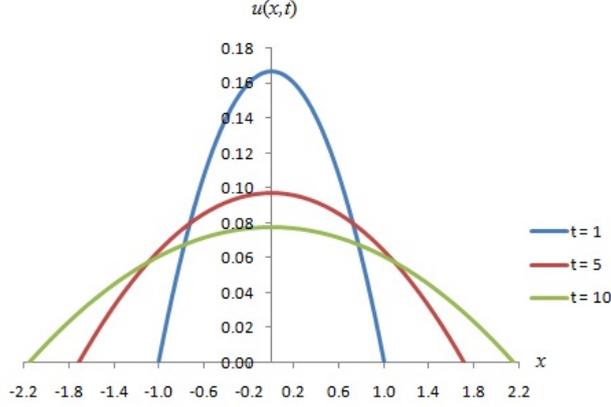


Figure 2.3: Self-Similar Solutions of the original PME when  $n = 1$  at different times

### 2.1.2 Non-Self-Similar Solutions

More generally, the equation (2.10) can be solved with  $\eta$  as a function of  $\xi$  and the time  $t$ , giving scale-invariant solutions but not self-similarity. This results in an additional term to the transformed left-hand side expressed in (2.12),

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial (\eta t^\gamma)}{\partial t} \\
&= \eta \gamma t^{\gamma-1} + t^\gamma \frac{\partial \eta}{\partial t} \\
&= \eta \gamma t^{\gamma-1} + t^\gamma \left( \frac{\partial \eta}{\partial \xi} \times \frac{d\xi}{dt} + \frac{\partial \eta}{\partial t} \right) \\
&= \eta \gamma t^{\gamma-1} + t^\gamma \left( \frac{\partial \eta}{\partial \xi} \times \frac{-\beta x}{t^{\beta+1}} + \frac{\partial \eta}{\partial t} \right) \\
&= \eta \gamma t^{\gamma-1} + t^\gamma \left( \frac{\partial \eta}{\partial \xi} \times \frac{-\beta \xi}{t} + \frac{\partial \eta}{\partial t} \right) \\
&= \eta \gamma t^{\gamma-1} - \beta t^{\gamma-1} \xi \frac{\partial \eta}{\partial \xi} + t^\gamma \frac{\partial \eta}{\partial t}. \tag{2.18}
\end{aligned}$$

The right-hand side has the same transformed expression as in (2.13),

$$\frac{\partial}{\partial x} \left( u^n \frac{\partial u}{\partial x} \right) = t^{\gamma n + \gamma - 2\beta} \frac{\partial}{\partial \xi} \left( \eta^n \frac{\partial \eta}{\partial \xi} \right). \tag{2.19}$$

Equating (2.18) and (2.19), we have

$$\begin{aligned}
\eta\gamma t^{\gamma-1} - \beta t^{\gamma-1}\xi \frac{\partial\eta}{\partial\xi} + t^\gamma \frac{\partial\eta}{\partial t} &= t^{\gamma n + \gamma - 2\beta} \frac{\partial}{\partial\xi} \left( \eta^n \frac{\partial\eta}{\partial\xi} \right) \\
t^{\gamma-1} \left( \eta\gamma - \beta\xi \frac{\partial\eta}{\partial\xi} + t \frac{\partial\eta}{\partial t} \right) &= t^{\gamma n + \gamma - 2\beta} \frac{\partial}{\partial\xi} \left( \eta^n \frac{\partial\eta}{\partial\xi} \right) \\
\eta\gamma - \beta\xi \frac{\partial\eta}{\partial\xi} + t \frac{\partial\eta}{\partial t} &= t^{\gamma n + 1 - 2\beta} \frac{\partial}{\partial\xi} \left( \eta^n \frac{\partial\eta}{\partial\xi} \right). \tag{2.20}
\end{aligned}$$

Using (2.11) and that  $\beta = -\gamma$ , and because of (2.15), gives

$$\begin{aligned}
-\eta\beta - \beta\xi \frac{\partial\eta}{\partial\xi} + t \frac{\partial\eta}{\partial t} &= \frac{\partial}{\partial\xi} \left( \eta^n \frac{\partial\eta}{\partial\xi} \right) \\
-\beta \left( \eta + \xi \frac{\partial\eta}{\partial\xi} \right) + t \frac{\partial\eta}{\partial t} &= \frac{\partial}{\partial\xi} \left( \eta^n \frac{\partial\eta}{\partial\xi} \right) \\
-\beta \frac{\partial}{\partial\xi} (\eta\xi) + t \frac{\partial\eta}{\partial t} &= \frac{\partial}{\partial\xi} \left( \eta^n \frac{\partial\eta}{\partial\xi} \right) \\
t \frac{\partial\eta}{\partial t} &= \beta \frac{\partial}{\partial\xi} (\eta\xi) + \frac{\partial}{\partial\xi} \left( \eta^n \frac{\partial\eta}{\partial\xi} \right), \tag{2.21}
\end{aligned}$$

which is a PDE for the function  $\eta(\xi, t)$ , with chosen domain  $|\xi| \leq 1$  and boundary conditions  $\eta = 0$ . This PDE can be solved numerically.

## 2.2 The Thin Film Equation

The Thin Film equation (TFE) is a fourth-order non-linear diffusion equation, given by

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x} \left( u^n \frac{\partial^3 u}{\partial x^3} \right), \tag{2.22}$$

where  $n$  is as stated in equation (2.1), and with zero boundary conditions. This is when  $m = 1$  in equation (2.1). When  $n = 1$ , the equation is used to describe flow in a Hele-Shaw cell. The fluid placed between two parallel plates moves in response to pressure gradients due to surface tension and other externally imposed forces.

With  $n = 3$ , it models the lubrication of a surface tension driven thin viscous liquid on a horizontal surface with a no-slip condition at the interface. However, the no-slip condition implies that an infinite force occurs at the interface; but this can be avoided by having more realistic models allowing slip, which are of Navier-type slip condition type [8]. Other applications involve the spreading of

fluid on textiles and the overflow of rainwater over soils.

### 2.2.1 Self-Similar Solutions

Similarly, we follow the same procedure in finding self-similar solutions for TFE.  $\gamma$  and  $\beta$  defined in (2.8) become

$$\begin{aligned} \gamma &= \frac{-1}{(n+4)} \\ \text{and } \beta &= \frac{1}{(n+4)}. \end{aligned} \quad (2.23)$$

A similarity solution of the form  $\eta = \eta(\xi)$  is sought, by obtaining an ODE for  $\eta$  in terms of  $\xi$ . The transformed left-hand side of (2.22) is the same as in (2.12),

$$\frac{\partial u}{\partial t} = \eta \gamma t^{\gamma-1} - \beta t^{\gamma-1} \xi \frac{d\eta}{d\xi} \quad (2.24)$$

To transform the right-hand side, first consider

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \eta} \frac{d\eta}{d\xi} \frac{\partial \xi}{\partial x} \\ &= t^\gamma \frac{d\eta}{d\xi} \frac{1}{t^\beta} \\ &= t^{\gamma-\beta} \frac{d\eta}{d\xi} \end{aligned}$$

from which,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= t^{\gamma-\beta} \frac{d^2 \eta}{d\xi^2} \frac{\partial \xi}{\partial x} \\ &= t^{\gamma-\beta} \frac{d^2 \eta}{d\xi^2} \frac{1}{t^\beta} \\ &= t^{\gamma-2\beta} \frac{d^2 \eta}{d\xi^2} \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^3 u}{\partial x^3} &= t^{\gamma-2\beta} \frac{d^3 \eta}{d\xi^3} \frac{\partial \xi}{\partial x} \\
&= t^{\gamma-2\beta} \frac{d^3 \eta}{d\xi^3} \frac{1}{t^\beta} \\
&= t^{\gamma-3\beta} \frac{d^3 \eta}{d\xi^3}.
\end{aligned}$$

Then

$$\begin{aligned}
-\frac{\partial}{\partial x} \left( u^n \frac{\partial^3 u}{\partial x^3} \right) &= -\frac{\partial \xi}{\partial x} \frac{d}{d\xi} \left( \eta^n t^{\gamma n} t^{\gamma-3\beta} \frac{d^3 \eta}{d\xi^3} \right) \\
&= -\frac{1}{t^\beta} \frac{d}{d\xi} \left( \eta^n t^{\gamma n} t^{\gamma-3\beta} \frac{d^3 \eta}{d\xi^3} \right) \\
&= -t^{\gamma n + \gamma - 4\beta} \frac{d}{d\xi} \left( \eta^n \frac{d^3 \eta}{d\xi^3} \right). \tag{2.25}
\end{aligned}$$

Equating (2.24) with (2.25) gives

$$\begin{aligned}
\eta \gamma t^{\gamma-1} - \beta t^{\gamma-1} \xi \frac{d\eta}{d\xi} &= -t^{\gamma n + \gamma - 4\beta} \frac{d}{d\xi} \left( \eta^n \frac{d^3 \eta}{d\xi^3} \right) \\
t^{\gamma-1} \left( \eta \gamma - \beta \xi \frac{d\eta}{d\xi} \right) &= -t^{\gamma n + \gamma - 4\beta} \frac{d}{d\xi} \left( \eta^n \frac{d^3 \eta}{d\xi^3} \right) \\
\eta \gamma - \beta \xi \frac{d\eta}{d\xi} &= -t^{\gamma n + 1 - 4\beta} \frac{d}{d\xi} \left( \eta^n \frac{d^3 \eta}{d\xi^3} \right). \tag{2.26}
\end{aligned}$$

From (2.23), since  $\beta = -\gamma$ , the power of  $t$  in (2.27) becomes

$$\begin{aligned}
\gamma n + 1 - 4\beta &= \gamma n + 1 + 4\gamma \\
&= \left( \frac{-n}{(n+4)} + 1 \right) - \frac{4}{(n+4)} \\
&= 0 \tag{2.27}
\end{aligned}$$

and hence

$$\begin{aligned}
-\eta \beta - \beta \xi \frac{d\eta}{d\xi} &= -\frac{d}{d\xi} \left( \eta^n \frac{d^3 \eta}{d\xi^3} \right) \\
-\beta \left( \eta + \xi \frac{d\eta}{d\xi} \right) &= -\frac{d}{d\xi} \left( \eta^n \frac{d^3 \eta}{d\xi^3} \right) \\
-\beta \frac{d}{d\xi} (\eta \xi) &= -\frac{d}{d\xi} \left( \eta^n \frac{d^3 \eta}{d\xi^3} \right), \tag{2.28}
\end{aligned}$$

a fourth-order ODE for  $\eta$  in terms of  $\xi$ . We impose boundary conditions  $\eta = 0$  and  $\frac{d\eta}{d\xi} = 0$  over our chosen domain  $|\xi| \leq 1$ . Integrating (2.28) once, we get

$$-\beta(\eta\xi) = -\eta^n \frac{d^3\eta}{d\xi^3} + D,$$

where  $D$  is a constant of integration. Substituting the boundary conditions,  $D = 0$  and thus

$$-\beta(\eta\xi) = -\eta^n \frac{d^3\eta}{d\xi^3}.$$

We divide through by  $\eta \neq 0$ , gives

$$\eta^{n-1} \frac{d^3\eta}{d\xi^3} = \beta\xi. \quad (2.29)$$

It is stated in [9] that TFE admits similarity solution only for  $n = 1$ . Substituting this value of  $n$  into (2.23) for the value of  $\beta$  and into (2.29) results in the following ODE,

$$\frac{d^3\eta}{d\xi^3} = \frac{\xi}{5}. \quad (2.30)$$

We integrate (2.30) thrice,

$$\eta = \frac{\xi^4}{120} + \frac{K_1\xi^2}{2} + K_2\xi + K_3,$$

where  $K_1, K_2$ , and  $K_3$  are integration constants. Using the boundary conditions at  $\eta = 0$  and  $\frac{d\eta}{d\xi} = 0$ , results in

$$K_2 = 0$$

and therefore

$$\eta(\xi) = \frac{\xi^4}{120} - \frac{\xi^2}{60} + K_3.$$

Mapping this back to the original variables  $u, x$  and  $t$  using the definitions in (1.5) and (1.6) with  $\beta$  and  $\gamma$  given by (2.23), we have

$$\begin{aligned} u(x, t) &= \frac{1}{t^{\frac{1}{5}}} \left( \frac{x^4}{120t^{\frac{4}{5}}} - \frac{x^2}{60t^{\frac{2}{5}}} + K_3 \right) \\ &= \frac{1}{t^{\frac{1}{5}}} \left( \frac{1}{120} \left( \frac{x^2}{t^{\frac{2}{5}}} \right)^2 - \frac{1}{60} \left( \frac{x^2}{t^{\frac{2}{5}}} \right) + K_3 \right). \end{aligned} \quad (2.31)$$

To evaluate  $K_3$ , we note that

$$\begin{aligned} \eta = 0 & \quad \text{at} \quad \xi = \pm 1 \\ \Rightarrow \frac{u}{t^{\frac{-1}{5}}} = 0 & \quad \text{at} \quad \frac{x}{t^{\frac{1}{5}}} = \pm 1. \end{aligned} \quad (2.32)$$

Substituting (2.32) into (2.31) gives  $K_3 = \frac{1}{120}$ . Hence the self-similar solution of the original TFE for  $n = 1$  is

$$u(x, t) = \frac{1}{120t^{\frac{1}{5}}} \left(1 - \frac{x^2}{t^{\frac{2}{5}}}\right)^2. \quad (2.33)$$

For values of  $n > 1$ , we find their approximate solutions from the numerical schemes presented in the next chapter.

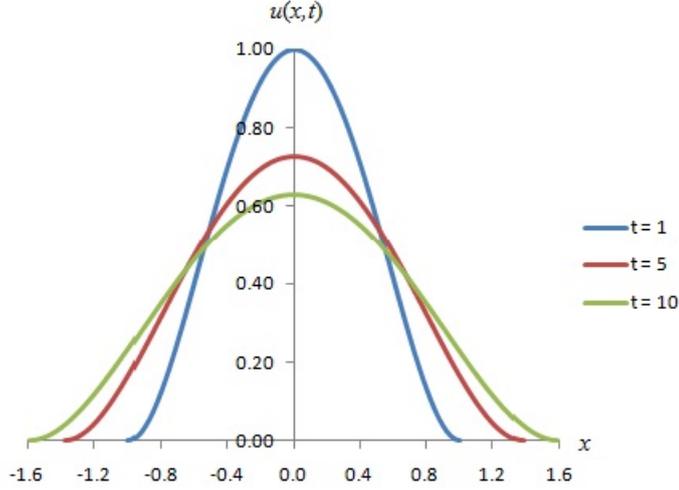


Figure 2.4: Self-Similar Solutions of original TFE when  $n = 1$  at different times

## 2.2.2 Non-Self-Similar Solutions

We can also find the solutions of equation (2.22) with  $\eta$  as a function of  $\xi$  and  $t$ , again giving scale-invariant solutions but not self-similarity. The transformed expression of the left-hand side in (2.24) now has an extra term,

$$\frac{\partial u}{\partial t} = \eta \gamma t^{\gamma-1} - \beta t^{\gamma-1} \xi \frac{\partial \eta}{\partial \xi} + t^\gamma \frac{\partial \eta}{\partial t}. \quad (2.34)$$

The right-hand side is transformed into the same expression as in (2.25),

$$-\frac{\partial}{\partial x} \left( u^n \frac{\partial^3 u}{\partial x^3} \right) = -t^{\gamma n + \gamma - 4\beta} \frac{\partial}{\partial \xi} \left( \eta^n \frac{\partial^3 \eta}{\partial \xi^3} \right). \quad (2.35)$$

Equating (2.34) and (2.35), gives

$$\begin{aligned} \eta \gamma t^{\gamma-1} - \beta t^{\gamma-1} \xi \frac{\partial \eta}{\partial \xi} + t^\gamma \frac{\partial \eta}{\partial t} &= -t^{\gamma n + \gamma - 4\beta} \frac{\partial}{\partial \xi} \left( \eta^n \frac{\partial^3 \eta}{\partial \xi^3} \right) \\ t^{\gamma-1} \left( \eta \gamma - \beta \xi \frac{\partial \eta}{\partial \xi} + t \frac{\partial \eta}{\partial t} \right) &= -t^{\gamma n + \gamma - 4\beta} \frac{\partial}{\partial \xi} \left( \eta^n \frac{\partial^3 \eta}{\partial \xi^3} \right) \\ \eta \gamma - \beta \xi \frac{\partial \eta}{\partial \xi} + t \frac{\partial \eta}{\partial t} &= -t^{\gamma n + 1 - 4\beta} \frac{\partial}{\partial \xi} \left( \eta^n \frac{\partial^3 \eta}{\partial \xi^3} \right). \end{aligned} \quad (2.36)$$

Deducing from (2.23) that  $\beta = -\gamma$ , and because of (2.27),

$$\begin{aligned} -\eta \beta - \beta \xi \frac{\partial \eta}{\partial \xi} + t \frac{\partial \eta}{\partial t} &= -\frac{\partial}{\partial \xi} \left( \eta^n \frac{\partial^3 \eta}{\partial \xi^3} \right) \\ -\beta \left( \eta + \xi \frac{\partial \eta}{\partial \xi} \right) + t \frac{\partial \eta}{\partial t} &= -\frac{\partial}{\partial \xi} \left( \eta^n \frac{\partial^3 \eta}{\partial \xi^3} \right) \\ -\beta \frac{\partial}{\partial \xi} (\eta \xi) + t \frac{\partial \eta}{\partial t} &= -\frac{\partial}{\partial \xi} \left( \eta^n \frac{\partial^3 \eta}{\partial \xi^3} \right) \\ -\beta \frac{\partial}{\partial \xi} (\eta \xi) + \frac{\partial}{\partial \xi} \left( \eta^n \frac{\partial^3 \eta}{\partial \xi^3} \right) &= -t \frac{\partial \eta}{\partial t}, \end{aligned} \quad (2.37)$$

which is a PDE in the function  $\eta(\xi, t)$ . We choose the domain to be  $|\xi| \leq 1$  with zero boundary conditions. The numerical solutions are presented in chapter Four.

# Chapter 3

## Numerical Results for the Porous Medium Equation

In this chapter, we compare numerical results obtained for PME with the similarity solutions found earlier, in order to check the accuracy of the numerical methods, in approximating nonlinear diffusion.

### 3.1 Self-Similar Solutions

Discretising equation (2.16) using a finite difference method, we have

$$\begin{aligned} & -\beta \left( \frac{(i+1)\eta_{i+1} - (i-1)\eta_{i-1}}{2h} \right) \\ = & \left( \frac{\eta_{i+1} + \eta_i}{2} \right)^n \left( \frac{\eta_{i+1} - \eta_i}{h^2} \right) - \left( \frac{\eta_i + \eta_{i-1}}{2} \right)^n \left( \frac{\eta_i - \eta_{i-1}}{h^2} \right) \\ \Rightarrow & 0 = \beta \left( \frac{(i+1)\eta_{i+1} - (i-1)\eta_{i-1}}{2h} \right) + \\ & \left( \frac{\eta_{i+1} + \eta_i}{2} \right)^n \left( \frac{\eta_{i+1} - \eta_i}{h^2} \right) - \\ & \left( \frac{\eta_i + \eta_{i-1}}{2} \right)^n \left( \frac{\eta_i - \eta_{i-1}}{h^2} \right) \end{aligned}$$

where  $h$  is the distance between adjacent points in  $\xi$  space.

The system can be represented in a matrix form

$$A(\eta)\underline{\eta} = \underline{0}, \tag{3.1}$$

where  $\underline{\eta} = [\eta_1, \eta_2, \dots, \eta_{N-1}]^T$  with known boundary conditions  $\eta_0 = 0$  and  $\eta_N = 0$ , and  $N$  represents the number of uniformly spaced intervals between  $-1 < \xi < 1$ .

From the matrix system in (3.1),  $\eta$  is non-unique and one of the solutions for  $\eta$  is zero. To remedy this, we use symmetry and solve on  $0 < \xi < 1$  instead with boundary conditions  $\eta = 0$  at  $\xi = 1$  and  $\eta_\xi = 0$  at  $\xi = 0$ . Using the fact that  $\eta_{-1} = \eta_1$  (by symmetry) and  $\eta_0 \neq 0$ , we obtain a non-singular system of non-linear equations for all the  $\eta_i$  in terms of  $\eta_0$ . Knowing that  $\eta = 0$  at the last point  $i = N$ , we then have an equation for  $\eta_0$ , which is non-linear. A good way to go is by the method of bisection.

The matrix yields an infinite number of solutions owing to its singularity. The problem can be solved by removing any one row in the matrix and replacing it by

$$\frac{1}{2}\eta_0 + \eta_2 + \dots + \frac{1}{2}\eta_N = 1, \quad (3.2)$$

which is equivalent to stating that the total mass  $= \int \eta d\xi = 1$ , using the Trapezoidal rule. This is similar to what we obtained in (2.6) in terms of the original variables.

We now consider another way of solving equation (2.16) numerically, by first integrating it once to give

$$\begin{aligned} \eta^n \frac{d\eta}{d\xi} &= -\beta(\eta\xi) \\ \frac{d\eta}{d\xi} &= \frac{-\beta\xi}{\eta^{n-1}} \end{aligned}$$

and applying the 2nd-order Runge-Kutta method, also known as the Improved Euler method. The explicit form is

$$\eta_{i+1} = \eta_i + \frac{h}{2}(k_1 + k_2)$$

where

$$k_1 = \frac{-\beta\xi}{\eta^{n-1}}$$

and

$$k_2 = \frac{-\beta(\xi + h)}{(\eta + hk_1)^{n-1}}$$

in which  $k_1$  and  $k_2$  are evaluated at the previous  $\xi_i$ .

By symmetry, we solve on  $0 < \xi < 1$  starting from  $\xi = 0$  and the bisection method is employed to get the zero value of  $\eta$  at  $\xi = 0$ . This method starts by

choosing two initial values at  $\xi = 0$  such that the solutions of  $\eta$  at the boundary  $\xi = 1$  will result in a positive  $\eta$  and a negative  $\eta$  at the last point.

For  $n = 1$ , the value of  $\eta$  at  $\xi = 0$  is the same regardless of the number of intervals between  $\xi = 1$  and  $\xi = 0$  for a fixed value of tolerance. This is because the solution is a quadratic. The numerical method used, Runge-Kutta of order 2, is exact for approximating a quadratic curve.

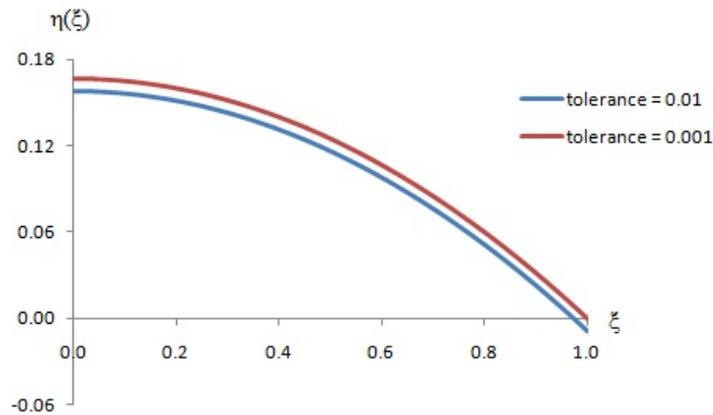


Figure 3.1: Numerical Solution of PME for  $n = 1$  at different tolerances

Comparing two different tolerance values of 0.01 and 0.001, we see that  $\eta = 0.1580$  and  $\eta = 0.1664$  respectively.

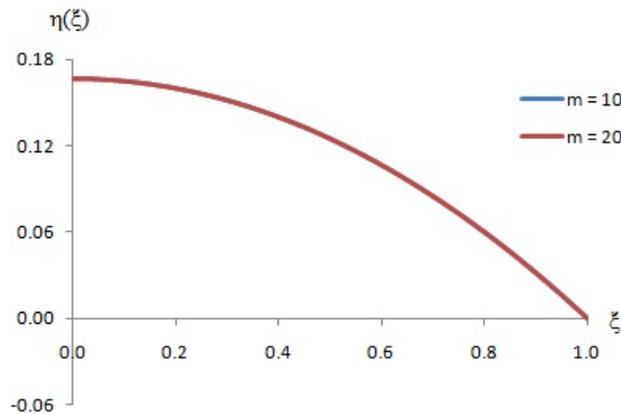


Figure 3.2: Numerical Solution of PME for  $n = 1$  for different number of intervals

Comparing two different numbers of intervals, 100 and 200, the values of  $\eta$  is the same.

For  $n = 2$ , the values of  $\eta$  at  $\xi = 0$  are similar for different tolerances but fixed number of intervals. As the number of intervals increases,  $\eta$  converges to a value of 0.50. Comparing two different tolerance values of 0.01 and 0.001,

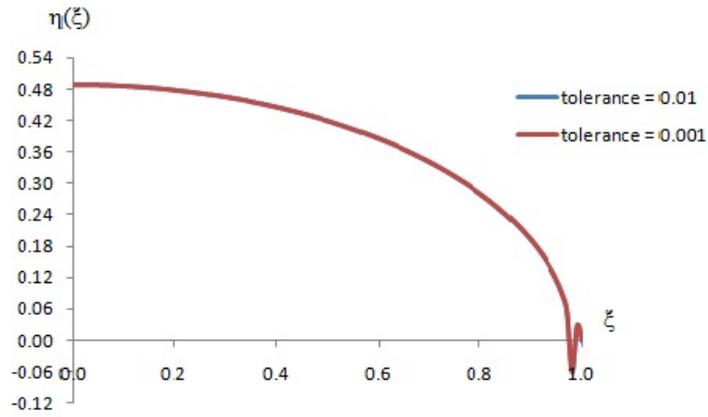


Figure 3.3: Numerical Solution of PME for  $n = 2$  at different tolerances

$\eta = 0.4881916$  and  $\eta = 0.4881922$  respectively.

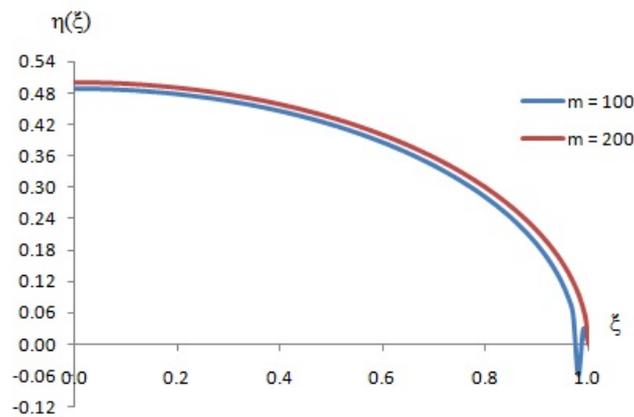


Figure 3.4: Numerical Solution of PME for  $n = 2$  for different number of intervals

Comparing two different numbers of intervals of 100 and 200, the values of  $\eta$  are converging.

Similarly, for  $n = 3$ , the values of  $\eta$  at  $\xi = 0$  are almost the same for different tolerances but fixed number of intervals.  $\eta$  converges to a value of 0.67 as we increase the number of intervals. Comparing two different tolerance values of

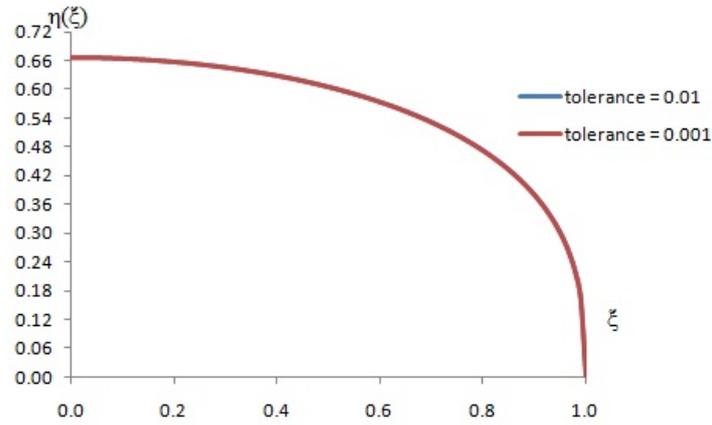


Figure 3.5: Numerical Solution of PME for  $n = 3$  at different tolerances

0.01 and 0.001,  $\eta = 0.66814$  and  $\eta = 0.66811$  respectively.

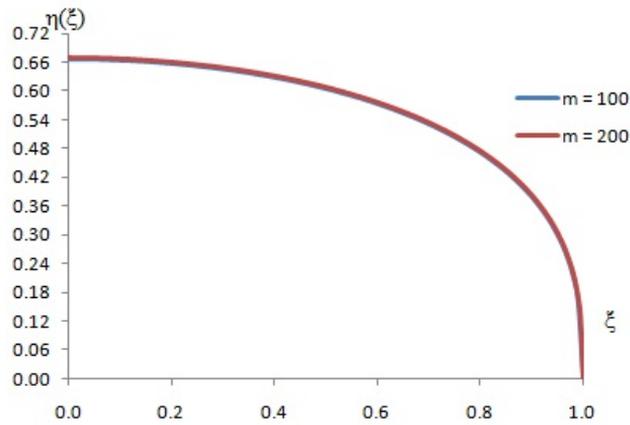


Figure 3.6: Numerical Solution of PME for  $n = 3$  for different number of intervals

Comparing two different numbers of intervals of 100 and 200, the values of  $\eta$  are converging.

Another approach of obtaining the solution is by solving (2.21) and run it to convergence, which we look at next.

## 3.2 Another Approach via Non-Self-Similar Equation

We discretise the equation (2.21) to give

$$\eta^{k+1} = \eta^k + \frac{\Delta t}{t} \left[ \beta \left( \frac{(i+1)\eta_{i+1} - (i-1)\eta_{i-1}}{2h} \right) \right] + \frac{\Delta t}{t} \left[ \left( \frac{\eta_{i+1} + \eta_i}{2} \right)^n \left( \frac{\eta_{i+1} - \eta_i}{h^2} \right) - \left( \frac{\eta_i + \eta_{i-1}}{2} \right)^n \left( \frac{\eta_i - \eta_{i-1}}{h^2} \right) \right]^k, \quad (3.3)$$

where  $k$  represents the time level and  $\Delta t$  is the local distance between time steps. We regard (3.3) as an iteration and perform two iterative methods, Jacobi and Gauss-Seidel. We run the program for a few time steps for each of  $n = 1, 2, 3$ . The evolutions are shown below:

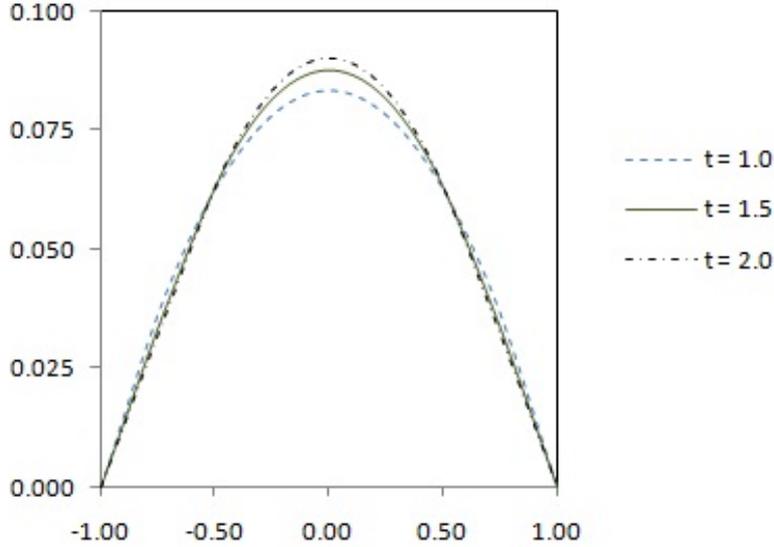


Figure 3.7: Evolution of Self-Similar Solution of PME for  $n = 1$

If we run this program to convergence, that is when  $\eta^{k+1} = \eta^k$ , within some tolerance, we can then find solutions for equation (2.16). The Gauss-Seidel

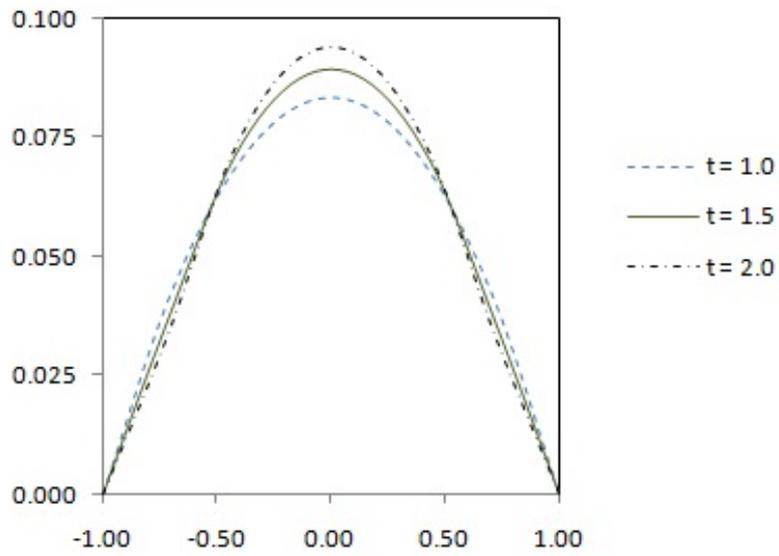


Figure 3.8: Evolution of Self-Similar Solution of PME for  $n = 2$

method converges about twice as fast as Jacobi does because we use updated values of  $\eta_i$  and  $\eta_{i-1}$ .

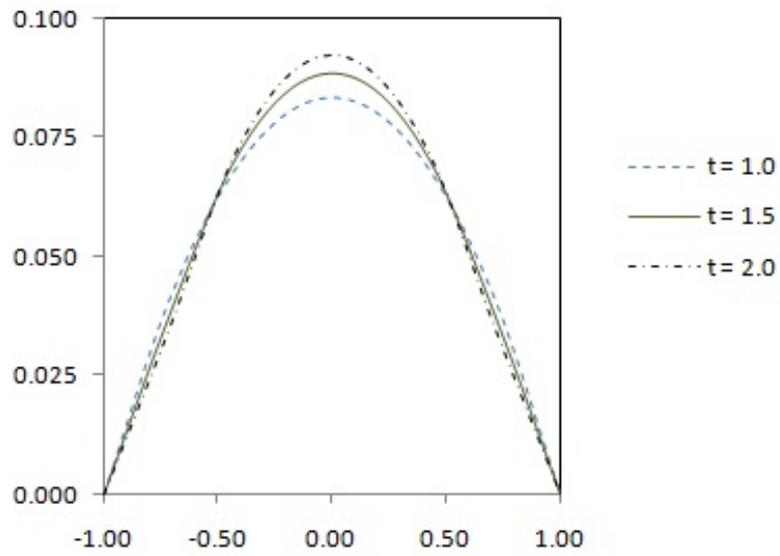


Figure 3.9: Evolution of Self-Similar Solution of PME for  $n = 3$

We check the results against the exact solutions and they both have the give same graphs.

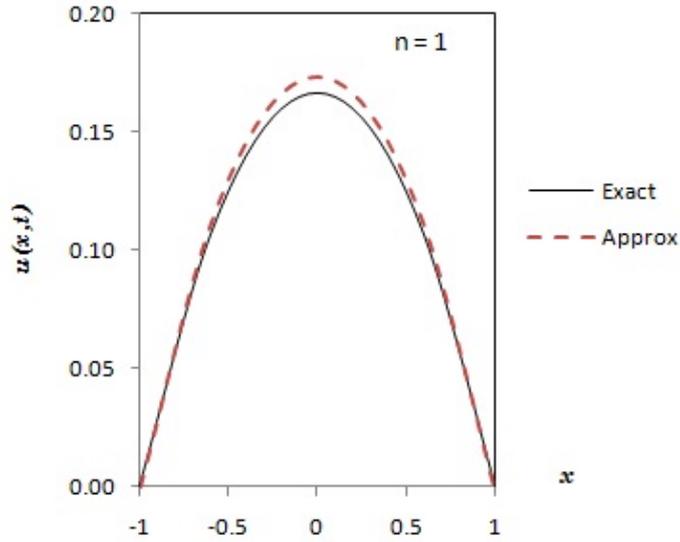


Figure 3.10: Exact and Approximate Solutions for PME for  $n = 1$

### 3.3 Accuracy

We then investigated the accuracy of the method used in finding self-similar solutions by evaluating the sum of the global errors taken at all points for different number of intervals with value of  $n$  fixed. The global error is the difference between the exact and the approximate (self-similar) solutions at  $\xi_i$ . Table 1 below shows the total error at intervals of 200, 400, and 800. The error can be

No. of Intervals	Total Exact	Total Approx	Total Global Errors
200	112.5393	112.0696	0.4725
400	225.2001	224.7100	0.4913
800	0.0000	0.0000	0.0000

represented graphically as given below:

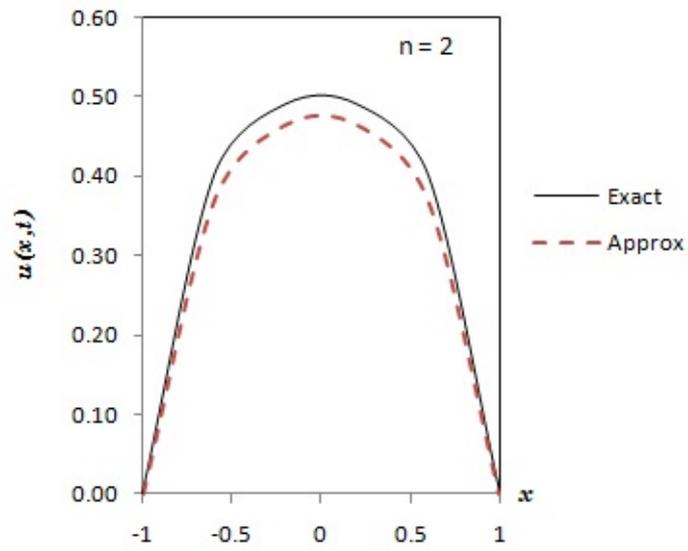


Figure 3.11: Exact and Approximate Solutions for PME for  $n = 2$

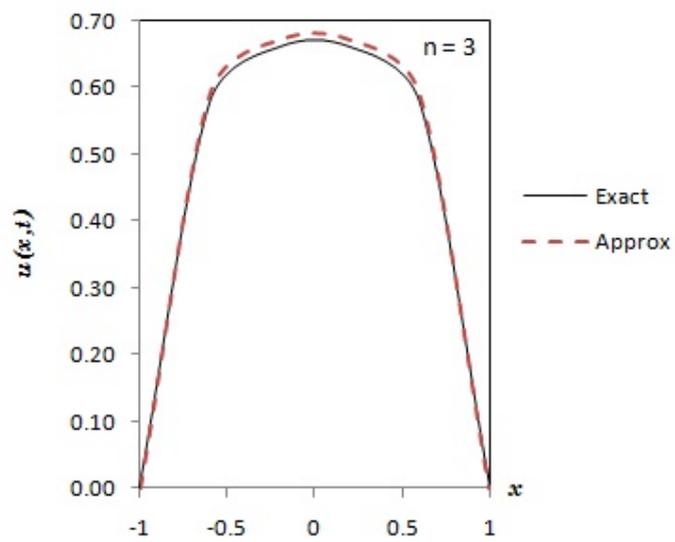


Figure 3.12: Exact and Approximate Solutions for PME for  $n = 3$

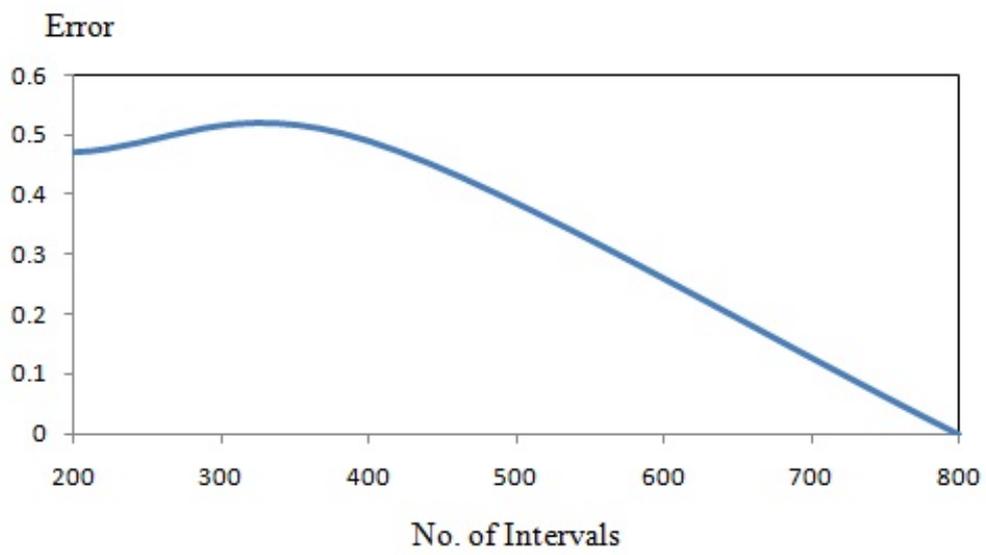


Figure 3.13: Error Analysis for PME for  $n = 3$

# Chapter 4

## Numerical Results for the Thin Film Equation

Here, we present numerical results for the TFE and check against the solution for  $n = 1$  in chapter Two. This is to see whether the numerical schemes are accurate enough to produce the same solutions.

### 4.1 Self-Similar Solutions

The method to find the numerical solutions to equation (2.28) works in a similar way as that for solving equation (2.16). This time we introduce a new variable,

$$\chi = -\frac{d^2\eta}{d\xi^2}, \quad (4.1)$$

and hence the new expression is

$$-\beta \frac{d}{d\xi}(\eta\xi) = \frac{d}{d\xi} \left( \eta^n \frac{d\chi}{d\xi} \right). \quad (4.2)$$

Discretising (4.1) and (4.2) gives

$$\chi_i = \frac{(\eta_{i+1} - 2\eta_i + \eta_{i-1}))}{h^2}$$

and

$$\left( \frac{\eta_{i+1} + \eta_i}{2} \right)^n \left( \frac{\chi_{i+1} - \chi_i}{h^2} \right) - \left( \frac{\eta_i + \eta_{i-1}}{2} \right)^n \left( \frac{\chi_i - \chi_{i-1}}{h^2} \right) +$$

$$\beta \left( \frac{(i+1)\eta_{i+1} - (i-1)\eta_{i-1}}{2h} \right) = 0.$$

From equation (4.2), integrating it once gives

$$-\beta\eta\xi = -\eta^n \frac{d\chi}{d\xi}.$$

Substituting (4.1), we have

$$\begin{aligned} \beta\eta\xi &= \eta^n \frac{d^3\eta}{d\xi^3} \\ \frac{\beta\xi}{\eta^{n-1}} &= \frac{d^3\eta}{d\xi^3}. \end{aligned}$$

We introduce

$$\frac{d\eta}{d\xi} = \mu \tag{4.3}$$

and

$$\frac{d\mu}{d\xi} = \nu. \tag{4.4}$$

Therefore,

$$\frac{d\nu}{d\xi} = \frac{\beta\xi}{\eta^{n-1}}. \tag{4.5}$$

Applying the 2nd-order Runge-Kutta method to each of (4.3), (4.4), and (4.5) with the explicit form

$$\nu_{i+1} = \nu_i + \frac{h}{2} (k_1 + k_2)$$

where

$$k_1 = \frac{\beta\xi}{\nu^{n-1}}$$

and

$$k_2 = \frac{\beta(\xi+h)}{(\nu+hk_1)^{n-1}}$$

in which  $k_1$  and  $k_2$  are evaluated at the previous  $\xi_i$ .

By symmetry, we solve on  $-1 < \xi < 0$  starting from  $\xi = -1$  and the bisection method is employed to get the value of  $\eta$  at  $\xi = 0$ . This method starts by choosing two initial values at  $\xi = -1$  such that the solutions of  $\mu$  at the boundary  $\xi = 0$  will result in a positive  $\mu$  and a negative  $\mu$  at the last point (remembering that we need  $\mu = 0$ ).

We can also find the solution by iterating equation (2.37) numerically and

running it to convergence.

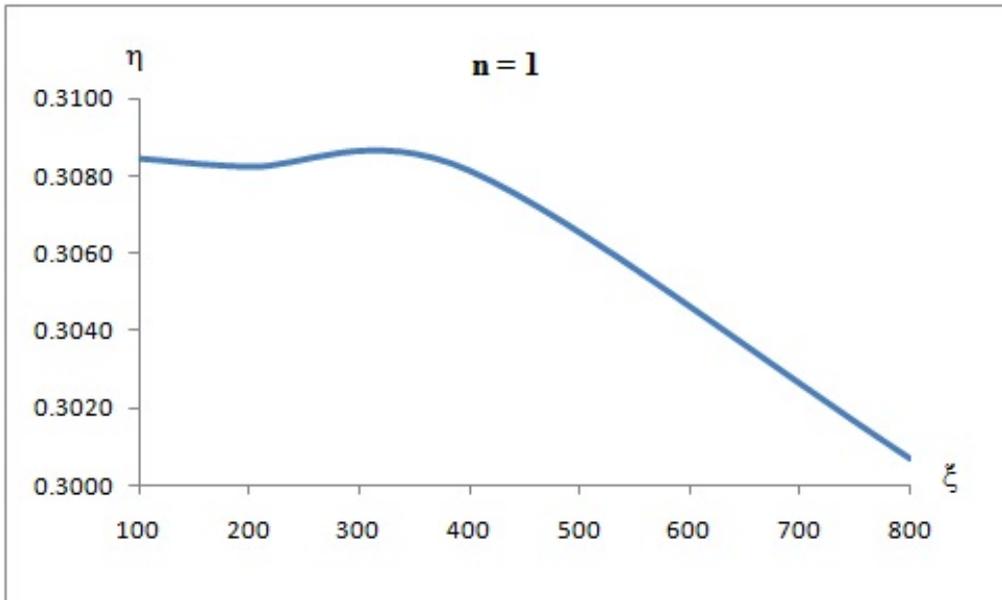


Figure 4.1: Numerical Solution of TFE at for  $n = 1$

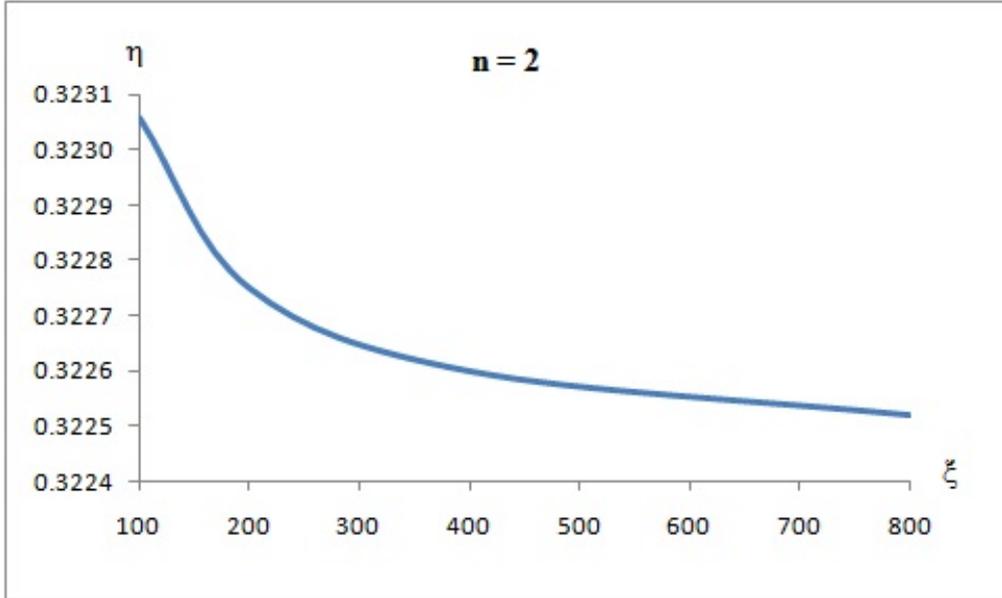


Figure 4.2: Numerical Solution of TFE at for  $n = 2$

## 4.2 Another Approach via Non-Self-Similar Equation

Substituting (4.1) into equation (2.37), we have

$$-\beta \frac{\partial}{\partial \xi} (\eta \xi) + \frac{\partial}{\partial \xi} \left( \eta^n \frac{\partial \chi}{\partial \xi} \right) = -t \frac{\partial \eta}{\partial t}. \quad (4.6)$$

Then we discretise (4.6) and this gives a similar expression with that for equation (2.21) but with  $\chi$ . Before running the program, the boundary values of  $\chi$  are determined by linear extrapolation, that is

$$\begin{aligned} \chi_0 &= 2\chi_1 - \chi_2, \\ 2\chi_{n-1} &= \chi_{n-2}. \end{aligned}$$

We run the program for a few time steps for each  $n = 1, 2, 3$  using self-similar solution as initial conditions. The evolutions are shown below:

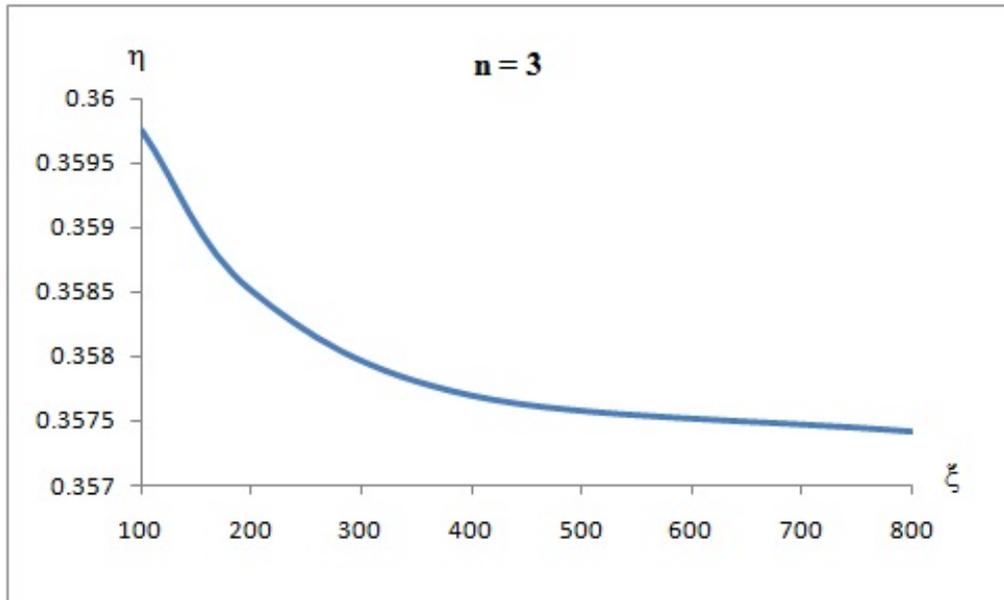


Figure 4.3: Numerical Solution of TFE for  $n = 3$

However, when running this program to convergence, that is when  $\eta^{k+1} = \eta^k$ , within some tolerance, we find that the numerical solutions were unstable. If we had more time, we could refine the method by using smaller  $\Delta t$ .

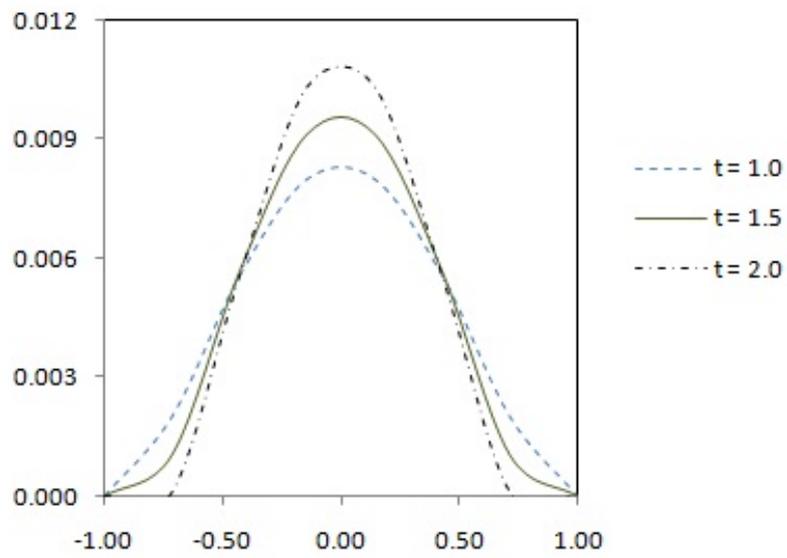


Figure 4.4: Evolution of Self-Similar Solution of TFE at  $n = 1$

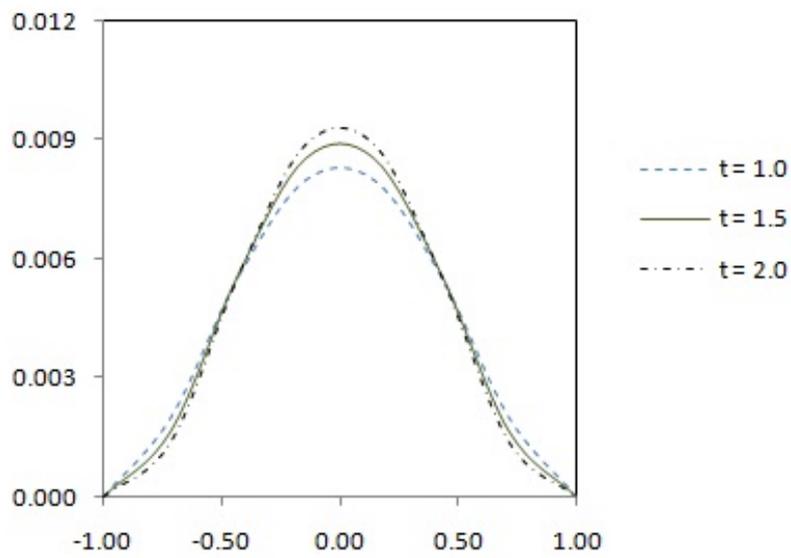


Figure 4.5: Evolution of Self-Similar Solution of TFE at  $n = 2$

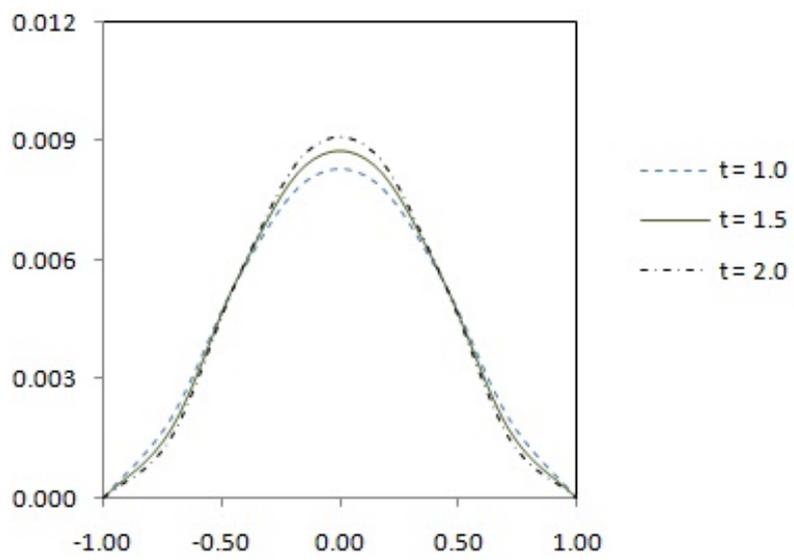


Figure 4.6: Evolution of Self-Similar Solution of TFE at  $n = 3$

# Chapter 5

## Conclusions and Further Work

### 5.1 Summary

This chapter summarises the work carried out in this dissertation. We then discuss some of the results and suggest possible improvements to this work.

In this dissertation, we implemented similarity method under the action of scaling transformations to non-linear diffusion equations, paying particular attention to PME and TFE. These equations were reduced to ODEs, which are then used to find self-similar solutions.

In Chapter Two, some applications of both diffusion equations were considered. The derivation of self-similarity variables which led to the transformed ODEs were shown. The ODEs were solved analytically to give self-similar solutions for all  $n > 0$  for PME and  $n = 1$  for TFE. In principle, the diffusion equations could be solved by iterating a time-stepping scheme. In this scheme, we had time as an extra variable apart from the two similarity variables.

In Chapter Three, a different numerical method for solving the transformed ODE for PME was presented. The ODE was then solved numerically and the results were checked with the self-similar solutions. As for the time-stepping method, the evolutions of the solutions at different times were shown on graphs. By running this method to convergence, we obtained the same self-similar solutions for all  $n$ .

In Chapter Four, the same numerical schemes were applied to the ODE for TFE to get self-similar solution for  $n = 1$ . In addition, in order to obtain the numerical self-similar solutions for  $n > 1$ , the time-stepping scheme was also run to finite time to see the evolutions of the solutions and then to convergence.

Unfortunately, the solutions blew up.

## 5.2 Further Work

We could implement the Crank-Nicolson method to solve equations (2.37) and (3.3) by passing to a limit as  $t \rightarrow \infty$  to obtain self-similar solutions. This could produce results in a shorter time as when compared to the iterative methods used. Another area of improvement includes solving the matrix system (3.1) by Crank-Nicolson to be solved for  $\eta$ .

One other approach is by attempting a sequence of increasingly accurate guesses to solving the transformed ODE in (2.16) of PME, known as the shooting method. This method could be applied to the transformed ODE of TFE, which is given in (2.28), by solving a system of three equations.

The total global errors can be used to find the ratio of  $\frac{e_{i+1}}{e_i}$  for the different values of space step  $h$  in order to check the order of convergence of the errors. We use the formula

$$error \propto h^p$$

where  $p$ , the order of convergence, is to be found.

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