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An Alternative Approach to the Analysis of Two-Point Boundary Value Problems for Linear Evolution PDEs and Applications

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Abstract

A new transform method recently developed by Fokas, is used for the study of two-point boundary value problems of the form $q_t(x,t) + Tq(x,t) = 0$ for linear evolution partial differential equations of arbitrary order, posed on the finite space domain $[0, L]$. Here T is an appropriately defined x-differential operator and suitable initial and boundary data is assumed.

The solution representation is expressible as an integral in the complex plane. For problems of odd order such representations are new, while for even orders it is shown that they are equivalent to classical series representations.

Spectral codes are developed for the numerical solution of a variety of illustrative examples, with many different types of boundary conditions. Finally, these codes are generalised and developed for linear third order problems for the solution of two-point boundary value problems for the important nonlinear equation, the Korteweg-deVries equation.

Acknowledgements

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Finally, special thanks goes to my family - my Mum, Dad and sisters, Maria and Amanda - and also the special person in my life David, for the unconditional love and support they all continue to give me. I thank you all from the bottom of my heart.

Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

Contents

Chapter 1

Introduction

In this work we consider the analytical and numerical solution of boundary value problems for evolution PDEs with constant coefficients, posed on a finite space domain. The equations that we consider are of the form

$$q_t(x,t) + Tq(x,t) = 0\,, \quad t > 0\,, \quad x \in [0, L]\,, \tag{1.1a}$$

$$q(x,0) = q_0(x), \quad x \in [0, L]\,, \tag{1.1b}$$

where T is an x-differential operator, and $q_0(x)$ is a given smooth function. T will mostly be linear, but we will also consider the nonlinear KdV equation, given by

$$q_t(x,t) + q_x(x,t) + q(x,t)\, q_x(x,t) + q_{xxx}(x,t) = 0\,. \tag{1.2}$$

These equations are equipped with the appropriate number and type of boundary conditions.

This work is divided into two main parts. In the first, we derive appropriate representations of the solution of various boundary value problems posed for such PDEs, and develop a general theory for linear evolution PDEs of arbitrary order. These are classical problems and many second order examples are extensively treated in the literature. However, the classical methods are not naturally or easily generalisable to higher order, and in particular to odd order problems. Here, we were able to give a systematic and unified treatment by using a new transform method, introduced by Fokas [12], which is based on the fact that linear and integrable nonlinear equations can be written as the compatibility condition of two linear ODEs. This is called a Lax pair formulation [32]. This method yields a complex integral representation of the solution, using classical tools of complex analysis such as the Cauchy integral formula and the Riemann-Hilbert problem. It also yields a relation between the initial and all the boundary values of the solution, which is known as the *global relation* and plays a crucial role in the methodology of Fokas.

The integral representation has explicit x and t exponential dependence (we say that it is *spectrally decomposed*) and, after exploiting appropriately the global relation, it can be given in terms of only the prescribed initial and boundary data of the problem.

The integral representation can be deformed to the classical series representation, without any appeal to classical theory. The series representation can also be derived directly from the global relation, but in this case the classical theory regarding the eigenvalues and eigenfunctions of an associated linear differential operator must be called into play. The method of Fokas for solving linear evolution two-point boundary value problems, and his relation to classical methods, is the general theme of Chapter 2.

In the second part, we study a variety of linear and nonlinear boundary value problems numerically. The final aim and motivation is the study of the nonlinear KdV equation, given by (1.2), posed on a bounded domain. All our numerical schemes use a spectral method for the discretisation of the space variable x. We develop such schemes for the imposition of a variety of boundary conditions for linear problems of third and fourth order, before turning to the nonlinear case.

We begin by a brief review of the work that has led to the development of the Fokas transform method, and the motivation for studying two-point boundary value problems for evolution PDEs, both analytically and numerically. We then give a brief account of the history of the KdV equation and the discovery of the famous soliton solution, and a review of spectral methods. Finally, we list some results in complex analysis and the theory of linear differential operators with constant coefficients, that will be referred to in the sequel. We conclude with a brief overview of the work presented in this thesis.

1.1 Background and Motivation

The initial value (Cauchy) problem for linear evolution equations in one space dimension, posed on the real line, is solved by the Fourier transform [26]. The analogous problem for integrable nonlinear evolution equations is solved by the inverse scattering transform method, which is conceptually similar to the classical Fourier transform approach and amounts in essence to a nonlinear version of it [27]. The inversion formula for this nonlinear version of the Fourier transform is obtained through the solution of a Riemann-Hilbert problem posed on the real line [1].

Here, we consider initial and boundary value problems. In this case, it is well known that the classical Fourier, sine and cosine transforms can be used to solve certain linear initial boundary value problems of even order. The choice of which transform to use, is governed by the PDE and the imposed boundary conditions, and for some boundary value problems, there exists a procedure, based on the so-called Green's function, for deriving the appropriate transform [44]. However, these methods to solve linear boundary value problems have some disadvantages. To illustrate this point, consider as a simple example the linear initial value problem

$$iq_t(x,t) + q_{xx}(x,t) = 0\,, \quad t > 0\,, \quad x \in (-\infty, \infty)\,,$$

$$q(x,0) = q_0(x)\,, \quad x \in (-\infty, \infty)\,,$$

and the corresponding boundary value problem posed on $[0, \infty)$ with $q_x(0,t) = f_1(t)$, where $q_0(x)$ and $f_1(t)$ are given functions satisfying $q_0(0) = f_1(0)$.

The solution of the initial value problem is given by

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - ik^2 t} \hat{q}(0,k)\, \mathrm{d}k\,. \tag{1.3}$$

The traditional approach that is used to solve the initial boundary value problem is the cosine transform in x, given by the pair of equations

$$\hat{q}(t,k) = \int_0^{\infty} \cos(kx)\, q(x,t)\, \mathrm{d}x\,, \quad q(x,t) = \frac{2}{\pi} \int_0^{\infty} \cos(kx)\, \hat{q}(t,k)\, \mathrm{d}k\,.$$

The solution of the boundary value problem is given by

$$q(x,t) = \frac{2}{\pi} \int_0^{\infty} \cos(kx) \left(e^{-ik^2 t} \hat{q}(0,k) - i \int_0^t e^{-ik^2(t-t')} f_1(t')\, \mathrm{d}t' \right) \mathrm{d}k\,, \tag{1.4}$$

where $\hat{q}(0,k)$ is the Fourier transform of the initial data.

We now compare these two representations. The solution given by (1.3) is *spectrally decomposed*, i.e., x and t appear explicitly and the spectral data $\hat{q}(0,k)$ depends only on k. For this reason, it is easy to study the long time behaviour of the solution. In contrast, the expression (1.4) is not separable, hence not spectrally decomposed, and it is not straightforward to determine the long time behaviour of the solution of the initial boundary value problem. In addition, expression (1.4) is not uniformly convergent at $x = 0$.

In spite of these disadvantages, for a variety of linear boundary value problems, the transform methods are successful. However, the appropriate transform does not always exist. For example, if the boundary conditions are sufficiently complicated, or if the associated x-differential operator is not symmetric, then the method above cannot be employed and one needs to resort to the Laplace transform in t, which is more complicated and involves further restrictions on the data.

Recently, Fokas has proposed a method for solving boundary value problems for linear PDEs in two variables by giving an algorithmic way to construct a transform in both x and t [12]. This transform yields an integral representation of the solution which depends only on the given data of the problem. This method has proved very important for linear problems, and in addition, it can be generalised to the solution of boundary value problems for integrable nonlinear evolution PDEs. We remark that the latter problem was open for almost forty years after the discovery of the inverse scattering transform, as it is not possible to follow the same procedure to derive a nonlinear analogue of these 'boundary-value' transforms, and of course there was no starting point when such a boundary linear transform does not exist in the first place. The crucial idea of Fokas was that of constructing a transform simultaneously in x and t.

This new approach has proved very successful, and it has been used to study initial boundary value problems for linear dispersive evolution equations [23, 17, 15, 19], several integrable nonlinear evolution equations [16, 14, 49], and two-point boundary value problems for linear evolution partial differential equations [22, 38, 39, 21]. The KdV equation is an important example of an integrable nonlinear evolution equation for which boundary value problems have been studied by this method, and we focus here on this particular example [19]. In the next section we give a brief account of the history of this equation.

1.1.1 The Korteweg-deVries Equation

In August 1834, John Scott-Russell (1808-1882) witnessed a phenomenon on the Edinburgh-Glasgow canal that was the first ever recorded observation of what is now termed a solitary wave. What he observed was a large mass of water, some thirty feet in length and a foot and a half in height, which was rounded, smooth, well-defined and in the form of a large solitary elevation. The wave travelled down the channel without changing in form or decreasing in speed, for several miles before the wave was finally lost. Russell observed that the volume of the water in the wave was exactly equal to the volume of the water that was displaced. He concluded that the wave speed was proportional to the amplitude, demonstrating that the larger the amplitude and narrower the wave, the faster it travels.

In the 1870's Boussinesq (1871, 1872, 1877) and Rayleigh (1876) considered the two dimensional Euler equation and independently derived expressions for the shape and speed of the type of wave that Russell had observed.

In 1895 Korteweg and deVries, both of whom were seemingly unaware of the earlier work of Boussinesq, obtained a partial differential equation, which is now commonly referred to as the Korteweg-deVries, or KdV equation [31]. The equation is nonlinear and based on the assumption that the depth of the water is small compared to the width of the wave, and relates the amplitude of the wave and its changes in space, with the change of the amplitude in time.

Korteweg and deVries demonstrated that the equation possesses a particular solution with the type of behaviour witnessed by Russell. This solution, called a soliton, takes the form

$$q(x,t) = 3A^2 \mathrm{sech}^2 \frac{1}{2} \left(Ax - A^3 t \right) ,$$

where the parameter A determines both the amplitude and speed. The term soliton was introduced by Kruskal and Zabusky in 1965 to emphasise the analogy with particles [52].

We define the soliton as a particular solution of a nonlinear equation, which is localised in space and retains its shape upon interaction with any other localised disturbance. The dispersive effects cause the wave to spread and decrease in amplitude, and the nonlinear effects cause the wave to steepen and become narrower. Under certain circumstances these opposing effects complement each other and the balance between them is such that the soliton propagates without changing in form.

Moreover, solitons are important because they characterise the long time behaviour of evolution equations in one space dimension. The pure initial value problem for the KdV equation posed on the real line, and the periodic initial value problem posed on a bounded domain, are the focus of [3], and existence, uniqueness, regularity and continuous dependence results have been established. For a general description, regarding the derivation of such model equations for long waves in nonlinear dispersive systems, together with references to their derivation in specific physical situations, see [2]. More recently, the analysis of the KdV equation posed on a bounded domain has received much attention. The exact boundary controllability of linear and nonlinear equations with various boundary conditions has been studied by Rosier [41], and initial boundary value problems for the KdV equation are the focus of the works by Colin *et al.* [8] and Bona *et al.* [4].

The problem of finding a solution describing the interaction of two solitons, was not addressed by Korteweg and deVries, but instead, in 1967, by Gardner *et al.*, who derived an explicit solution describing the interaction of an arbitrary number of solitons [26]. If we consider a two soliton initial solution, then the soliton with the larger amplitude will travel faster than the other soliton and will overtake it. The effect of the interaction will simply be a phase shift, i.e., the two solitons will have reached the positions they would have otherwise reached had they not have interacted.

Numerically, the solution of nonlinear dispersive wave equations has been of interest since the 1960s. Explicit methods, used to approximate the KdV equation, include schemes by Zabusky and Kruskal [52] and Vliegenhart [50]. Implicit methods include a hopscotch method [29] and a scheme due to Goda [28]. Finite Fourier transform or pseudospectral methods include the split-step Fourier method introduced by Tappert [46] and the pseudospectral method introduced by Fornberg and Whitham [25], both of which will be the focus of the numerical schemes of Chapter 6. All of the schemes are reviewed in detail, and their efficiency compared, in the paper of Taha and Ablowitz [45]. Fourier spectral methods have also been applied to the KdV equation by Schamel and Elsasser [43] and Chan and Kerkhoven [6], and more recently a comparison of the Fourier pseudospectral methods for the solution of the KdV equation, has appeared in the literature [37].

Such methods are also of great interest in the solution of higher order problems [34], and indeed will be the focus of Section 5.2 for our numerical approximations. In the next section we give a brief review of spectral methods.

1.2 An Overview of Spectral Methods

Before the introduction of spectral methods in the 1970s, finite difference methods (1950s) and finite element methods (1960s) were the main two classes of methods that were used for the numerical solution of partial differential equations. Afterwards, theoretical and numerical studies have shown that when the functions involved are smooth, such methods converge must faster than finite difference or finite element methods [5], and nonlinearities do not pose any special difficulties. By now there is a very large literature on spectral methods. In this thesis, we are interested in particular in the use of such methods for solving third order differential equations [30].

Both finite difference and finite element methods are based on approximations that are local in nature. In contrast, spectral methods use the information given over the whole domain to approximate the solution at a certain point. In this sense, the nature of the approximation is global. Spectral methods are usually described as expansions based on infinitely differentiable global functions:

$$u(x) \approxeq \sum_{k=0}^{N} a_k \, \phi_k(x) \, ,$$

where the functions $\phi_k(x)$ are for example Chebyshev polynomials, or trigonometric functions, and the coefficients a_k are called the expansion coefficients. These functions are then differentiated exactly. To define a spectral numerical scheme to solve a specific PDE problem, one needs to choose the trial functions $\phi_k(x)$ and to determine the expansion coefficients a_k, $k = 0, 1, \ldots, N$.

In the next section, we give a detailed review of the spectral differentiation process for the cases for which the prescribed function is both periodic and nonperiodic. In the former case we consider the approximation of the derivative of a given function using the discrete Fourier transform (DFT), which we then contrast with the Toeplitz differentiation matrix approach. This is to be compared to the latter case of approximating the derivative of a function when the domain is bounded but not periodic using Chebyshev polynomial interpolants.

The introductory material given below, follows closely the exposition in Trefethen [48]. For a more historical background on these methods, see [5].

1.2.1 Periodic Problems

Suppose we are given a set of N grid points $\{x_1, ..., x_N\}$ and corresponding function values $\{u_1, ..., u_N\}$ and we wish to approximate the derivative of $u(x)$ spectrally. The first approach that we consider is the approximation of the derivative of a given function using the discrete Fourier transform (DFT), which we then contrast with the Toeplitz differentiation matrix approach.

Discrete Fourier Transforms

Let us begin by considering the continuous case. The Fourier transform pair is given by

$$\hat{u}(k) \;=\; Fu(x) = \int_{-\infty}^{\infty} e^{-ikx} u(x)\,\mathrm{d}x\,, \qquad x\,,k \in \mathbb{R}\,, \tag{1.5a}$$

$$u(x) \;=\; F^{-1}\hat{u}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{u}(k)\,\mathrm{d}k\,, \tag{1.5b}$$

where $u(x)$ is a sufficiently smooth function and $\hat{u}(k)$ is its Fourier transform.

The Fourier transform converts differentiation into multiplication by ik: $\widehat{u'}(k) = ik\hat{u}(k)$. This is probably its most important property, as it means that PDEs in physical space can be transformed to ODEs in Fourier space.

Consider now the discrete spatial domain with grid points $x_j = jh$ for $j \in \mathbb{Z}$. Because of the phenomenon known as 'aliasing', the wavenumbers k will no longer range over $\mathbb{R}$, but will instead range over a bounded domain of length $\frac{2\pi}{h}$. Indeed, if one considers the points $x_j = jh$ for $j \in \mathbb{Z}$, then $e^{ik_1 x_j} = e^{ik_2 x_j}$ for each j, provided $k_1 - k_2$ is an integer multiple of $\frac{2\pi}{h}$.

We now consider spectral differentiation on a bounded periodic grid. Our basic periodic grid is $[0, 2\pi]$ discretised by N equispaced points. Because the spacing $h = \frac{2\pi}{N}$ of the grid points implies that wavenumbers differing by an integer multiple of $\frac{2\pi}{h}$ are indistinguishable on the grid, the spectral variable is $k \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$.

The formula for the discrete Fourier transform (DFT) which transforms our function $u(x)$ to the discrete Fourier space is given by

$$Fu_j = \hat{u}_k = h \sum_{j=1}^{N} e^{-ikx_j} u_j \,, \qquad k = -N/2+1, \dots , N/2 \,, \tag{1.6}$$

and the inversion formula is

$$F^{-1}\hat{u}_k = u_j = \frac{1}{2\pi} \sum_{k=-N/2+1}^{N/2} e^{ikx_j} \hat{u}_k \,, \qquad j = 1, \dots , N \,. \tag{1.7}$$

Via the formula for differentiation of the continuous function $u(x)$, we differentiate our discrete function analogously and approximate the first derivative of $u(x)$ as $u'_j \approxeq F^{-1}(ikFu_j)$, and similarly approximate the third derivative as $u'''_j \approxeq -F^{-1}(ik^3Fu_j)$, where Fu_j and $F^{-1}\hat{u}_k$ are given by (1.6) and (1.7) respectively.

Matlab has built in functions $X = \mathit{fft}(x)$ and $x = \mathit{ifft}(X)$ that implement the Fourier transform and its inverse. For a vector of length N the pair are given explicitly by

$$\begin{aligned} X(k) &= \sum_{j=1}^{N} x(j)\, e^{-\frac{2\pi i}{N}(j-1)(k-1)}, \\ x(j) &= \frac{1}{N}\sum_{k=1}^{N} X(k)\, e^{\frac{2\pi i}{N}(j-1)(k-1)}. \end{aligned} \tag{1.8}$$

Matlab stores the wavenumbers in the order $0, 1, \dots , \frac{N}{2}, -\frac{N}{2}+1, -\frac{N}{2}+2, \dots , -1$.

For convenience, these equations are often written as matrix $\times$ vector products. For example expression (1.8) for the inverse discrete Fourier transform, can be written as

$$\begin{pmatrix} x(1) \\ x(2) \\ x(3) \\ \vdots \\ x(N) \end{pmatrix} = \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^3 & \dots & \omega^{2N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2N-2} & \dots & \omega^{(N-1)^2} \end{pmatrix} \begin{pmatrix} X(1) \\ X(2) \\ X(3) \\ \vdots \\ X(N) \end{pmatrix}, \tag{1.9}$$

where $\omega = e^{\frac{2\pi i}{N}}$. The discrete Fourier transform (DFT) on the N-point grid can be computed via the fast Fourier transform (FFT), discovered by J.W.Cooley and J.W.Tukey in 1965, and used to approximate the derivative of $u(x)$. The execution time for the discrete Fourier transform depends on the length N of the transform, and the matrix-vector multiplication, given by (1.9), appears to require $O(N^2)$ operations. This is because the process requires N^2 complex multiplications (along with a small number of operations to generate the required powers of ω). However, if N is chosen to be the product of small prime factors, then the discrete Fourier transform can be computed using the fast Fourier transform (FFT) in $O(N \log N)$ operations. The FFT algorithm writes the DFT matrix as the product of sparse matrices, and this factorisation becomes particularly efficient when N is chosen to be a power of 2. For this reason, all of the numerical schemes that are presented in Chapter 6 will choose N to be a power of 2. For a detailed explanation of the algorithm, see [40].

As a simple example to demonstrate the ease with which the FFT and IFFT can be implemented, we consider the following where we approximate the derivative of a given function $u(x)$, defined on the N point equispaced periodic grid $[0, 2\pi]$. The numerical derivative $w(x)$ is computed as follows:

- Begin by computing $\hat{u}_k$.
- Define $\hat{w}_k = ik\hat{u}_k$.
- Compute $w(x)$ from $\hat{w}_k$.

To approximate the n^{th} order derivative, we simply multiply by the n^{th} power of ik. However, for odd order derivatives, we must set the term $\hat{w}_{\frac{N}{2}} = 0$, due to a loss of symmetry (which will be explained in detail in the next section when we discuss Toeplitz differentiation matrices). Hence the numerical n^{th} derivative is computed as follows:

- Begin by computing $\hat{u}_k$.
- Define $\hat{w}_k = (ik)^n\hat{u}_k$ and if n is odd set $\hat{w}_{\frac{N}{2}} = 0$.
- Compute $w(x)$ from $\hat{w}_k$.

Toeplitz Differentiation Matrices

Toeplitz differentiation matrices provide an alternative method for approximating the derivatives of a given function, provided the grid upon which the function is defined is uniform and the problem is periodic i.e., $u_0 = u_N$ and $u_1 = u_{N+1}$. Therefore, under the assumption of periodicity, we can use trigonometric interpolants and represent the discrete differentiation process as a matrix $\times$ vector multiplication. Our basic periodic grid will be $[0, 2\pi]$ with spacing $h = 2\pi/N$:

- Let $p(x)$ be a single function such that $p(x_j) = u_j$ for all j.

- Set $w_j = p^{'}(x_j)$.

For spectral differentiation, we use the inverse Fourier transform, given by (1.7), to derive an interpolant. We begin by determining $\hat{u}_k$ and then define the interpolant $p(x)$ according to the formula given by (1.7), evaluated for $x \in [0, 2\pi]$. This yields

$$p(x) = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} e^{ikx} \hat{u}_k \,, \quad x \in [0, 2\pi] \,.$$

Evaluating this expression achieves a term of the form $e^{iNx/2}$ with derivative $(iN/2)e^{iNx/2}$. At the points $x_j = jh$, $j = 1, \ldots, N$, the function $e^{iNx/2}$ represents a real, sawtooth wave, and therefore the derivative should vanish and not be equal to a complex exponential. The reason for this problem is that (1.7) treats the highest wavenumber asymmetrically, but this problem can be fixed by defining the band-limited interpolant as follows

$$p(x) = \frac{1}{2\pi} {\sum_{k=-N/2}^{N/2}}' e^{ikx} \hat{u}_k \,, \quad x \in [0, 2\pi] \,, \tag{1.10}$$

where the prime indicates that the terms $k = \pm N/2$ are multiplied by $\frac{1}{2}$.

Now, the periodic Delta function is given by

$$\delta_j = \begin{cases} 1 \,, & j \equiv 0 \pmod{N} \,, \\ 0 \,, & j \not\equiv 0 \pmod{N} \,, \end{cases}$$

and the Fourier transform of δ_j is a constant, $\hat{\delta}_k = h$ for each k. Therefore, via (1.10), we have

$$\begin{aligned} p(x) = \frac{h}{2\pi} {\sum_{k=-N/2}^{N/2}}' e^{ikx} &= \frac{h}{2\pi} \left(\frac{1}{2} \sum_{k=-N/2}^{N/2-1} e^{ikx} + \frac{1}{2} \sum_{k=-N/2+1}^{N/2} e^{ikx} \right) \\ &= \frac{h}{2\pi} \cos\left(\frac{x}{2}\right) \sum_{k=-N/2+1/2}^{N/2-1/2} e^{ikx} \\ &= \frac{h}{2\pi} \cos\left(\frac{x}{2}\right) \left(\frac{e^{-i(N/2)x} - e^{i(N/2)x}}{e^{-ix/2} - e^{ix/2}} \right) \end{aligned}$$

$$= \frac{h}{2\pi}\left(\frac{\cos\left(\frac{x}{2}\right)\sin\left(\frac{Nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)$$

$$= \frac{\sin\left(\frac{\pi x}{h}\right)}{\left(\frac{2\pi}{h}\right)\tan\left(\frac{x}{2}\right)}.$$

Hence for a periodic domain, $p(x)$ is a trigonometric polynomial on the equispaced grid and is constructed using the *periodic sinc function* $S_N(x)$, given by

$$S_N(x) = \frac{\sin\left(\frac{\pi x}{h}\right)}{\left(\frac{2\pi}{h}\right)\tan\left(\frac{x}{2}\right)}.$$

To approximate the first derivative of $u(x)$, we evaluate the derivative of $S_N(x)$:

$$S_N'(x_j) = \begin{cases} 0\,, & j \equiv 0 \ (\mathrm{mod} N)\,, \\ \\ \frac{1}{2}(-1)^j \cot\left(\frac{jh}{2}\right)\,, & j \not\equiv 0 \ (\mathrm{mod} N)\,, \end{cases}$$

arrange the entries in the N^{th} column of the following $N \times N$ matrix, and perform the following matrix $\times$ vector multiplication:

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ \vdots \\ \vdots \\ w_N \end{pmatrix} = \begin{pmatrix} 0 & & & & -\frac{1}{2}\cot\frac{1h}{2} \\ -\frac{1}{2}\cot\frac{1h}{2} & \ddots & & \ddots & \frac{1}{2}\cot\frac{2h}{2} \\ \frac{1}{2}\cot\frac{2h}{2} & & \ddots & & -\frac{1}{2}\cot\frac{3h}{2} \\ -\frac{1}{2}\cot\frac{3h}{2} & & & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1}{2}\cot\frac{1h}{2} \\ \frac{1}{2}\cot\frac{1h}{2} & & & & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ \vdots \\ u_N \end{pmatrix}, \tag{1.11}$$

which we write as

$$w = D_N\, u\,.$$

The matrix D_N is Toeplitz, meaning that the entries along the diagonals are constant (a_{ij} depends only on $i-j$), and circulant, meaning that a_{ij} depends only on $(i-j)(\mathrm{mod} N)$. D_N will be the differentiation matrix that we will use in Chapter 6 to approximate the derivative of a periodic function defined on $[0, 2\pi]$.

To calculate higher spectral derivatives we differentiate $S_N(x)$ more times. i.e., we construct the matrix for differentiation of any order by differentiation of $S_N(x)$. For a detailed explanation, see [48].

We conclude this section with a simple example illustrating the use of the Toeplitz matrix D_N for differentiating the smooth function $u(x) = e^{\cos(x)}$. The program is then repeated using the FFT instead of matrices and the results are given in Figure 1.1. The program discretises the interval $[0, 2\pi]$ using 24 grid points.

To calculate the errors, the infinity norms for the two functions given by the difference between the spectral derivatives and the exact derivatives were numerically calculated. The maximum error for the derivative calculated using the Toeplitz matrix is 9.6811×10^{-13}, compared to 9.5468×10^{-13} from the FFT approach.

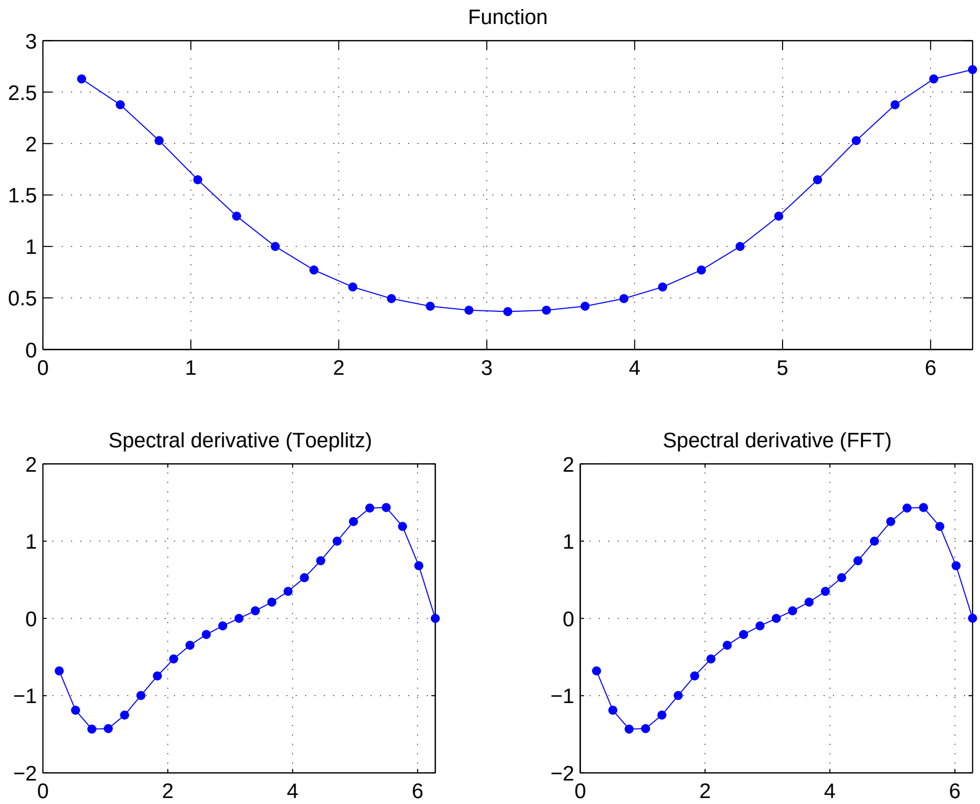


Figure 1.1: Spectral differentiation of the function $u(x) = e^{\cos(x)}$.

1.2.2 Non-Periodic Problems

We now construct interpolants that can be used for approximating the derivative of a given function when the domain is bounded but not periodic.

Polynomial Interpolation

Consider a non-periodic smooth function defined on $[-1, 1]$. We wish to choose $\{\phi_k\}$ so that for smooth functions the series approximation, given by $\sum_{k=0}^{N} a_k \, \phi_k(x)$, converges rapidly. In general, a smooth function becomes non-smooth when periodically extended, and if we simply pretend the function is periodic and use trigonometric (Fourier) interpolation in equispaced points, then the required condition will not be satisfied. As a result, the spectral accuracy will be lost, and the coefficients a_k will decrease like $O(1/N)$ as $N \to \infty$.

This is called Gibb's phenomenon, and it was first observed by Wilbraham (1848) and Gibbs (1899). It basically describes how the Fourier series of a piecewise continuously differentiable function behaves at a jump discontinuity. It reflects the difficulties encountered by approximating a discontinuous function by a series of continuous functions, and is the most significant example of how an irregularity of a piecewise smooth function can effect the convergence of interpolants and truncated series expansions.

To overcome this we must replace trigonometric polynomials by algebraic polynomials, $p(x) = a_0 + a_1(x) + \cdots + a_N(x)$, and use polynomial interpolation on unevenly spaced points. The set of points we shall use are the Chebyshev points:

$$x_j = \cos\left(\frac{j\pi}{N}\right), \qquad j = 0, 1, \ldots, N, \tag{1.12}$$

which are the projections onto the interval $[-1, 1]$, of equispaced points along the unit circle in the complex plane.

We now give an example to demonstrate the accuracy that is lost from interpolating a non-periodic function using equispaced points. Figure 1.2 shows the results from interpolating the function $u(x) = \frac{2}{3+40x^2}$ on a 17 point grid.

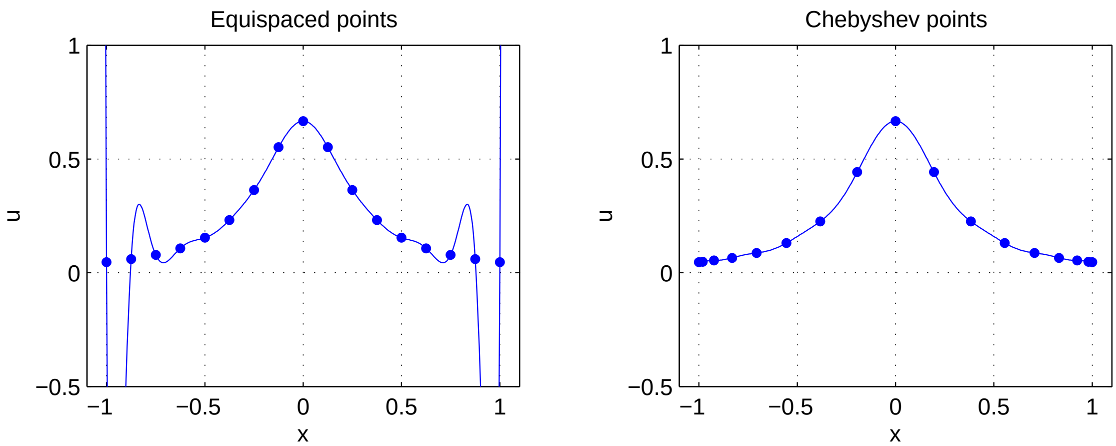


Figure 1.2: Interpolation of $u(x) = \frac{2}{3+40x^2}$ for equispaced points and Chebyshev points.

As the number of grid points is increased, the errors from equispaced grid point interpolation increase exponentially, whereas if Chebyshev interpolation is used the errors decrease exponentially.

Chebyshev Differentiation Matrices

Given a grid function u_j defined on the Chebyshev points x_j, $j = 0, 1, \dots, N$, we obtain a discrete derivative w_j in two steps:

- Let $p(x)$ be the unique polynomial of degree $\leqslant N$ with $p(x_j) = u_j$, $0 \leqslant j \leqslant N$.
- Set $w_j = p'(x_j)$.

This operation is linear, because the interpolating polynomial through the grid points is linear, and is represented by multiplication by an $(N+1) \times (N+1)$ matrix, which we shall denote D_N:

$$w = D_N \, u \, .$$

The differentiation matrix D_N is constructed according to Theorem 1.2.1:

Theorem 1.2.1. *For each $N \geqslant 1$, the first-order spectral differentiation matrix D_N has entries*

$$\begin{aligned}
(D_N)_{00} &= \frac{2N^2+1}{6}\,, & (D_N)_{NN} &= -\frac{2N^2+1}{6}\,, \\
(D_N)_{jj} &= \frac{-x_j}{2\left(1-x_j^2\right)}\,, & & 1 \leqslant j \leqslant N-1\,, \\
(D_N)_{ij} &= \frac{c_i\,(-1)^{i+j}}{c_j\,(x_i - x_j)}\,, & & i \neq j, \quad i, j = 0, \dots, N\,,
\end{aligned}$$

where the rows and columns of the $(N+1) \times (N+1)$ matrix are indexed from 0 to N, and

$$c_i = \begin{cases} 2\,, & i = 0 \text{ or } N\,, \\ 1\,, & \text{otherwise}\,. \end{cases}$$

The Chebyshev differentiation matrix D_N can be computed via an eight-line *Matlab* program called *cheb* [48], which returns a vector x and a matrix D_N, and we will use this program for all non-periodic numerical schemes in Chapter 5 and Chapter 6. To obtain higher order differentiation matrices, we will compute the relevant powers of D_N. i.e., $D_N^j u$ approximates the j^{th} derivative of $u(x)$.

Remark 1.2.2. The distinction between the use of D_N to represent both Toeplitz and Chebyshev differentiation matrices, will be made explicitly clear in Chapter 5 and Chapter 6.

Remark 1.2.3. All of the numerical schemes presented in Chapter 5 and Chapter 6 are performed using *Matlab*, and our goal is on achieving simplistic codes using as many built-in functions as possible, rather than achieving high performance abstract codes.

1.3 Preliminary Results

We begin this section by summarising some basic results on the theory of functions of one complex variable, which will be used throughout this work. The presentation follows the exposition given in the book by Ablowitz and Fokas [1].

We also use some of these complex techniques to give an alternative derivation of the Fourier transform inversion formula for smooth decaying functions. This derivation is a model for many of the results that follow.

We then include some general results in the theory of linear differential operators with constant coefficients, which follows closely the book of Naimark [35], and conclude the section with a theorem, due to Levin, about the distribution of the zeros of a certain class of entire functions [33].

1.3.1 Cauchy's Theorem

In this section we give the statement of Cauchy's Theorem, one of the most significant results in complex analysis, and list some of its consequences.

Theorem 1.3.1. ***(Cauchy)*** *If a function $f(k)$ is analytic and bounded in a simply connected domain D, then along any simple closed contour C in D*

$$\oint_C f(k)\ \mathrm{d}k = 0\,.$$

The proof of Cauchy's Theorem, which is omitted, uses a well-known result from vector analysis, known as Green's Theorem in the plane.

Theorem 1.3.2. ***(Green)*** *Let the functions $P(x,t)$ and $Q(x,t)$, along with their partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial t}$, $\frac{\partial Q}{\partial x}$ and $\frac{\partial Q}{\partial t}$ be continuous throughout a simply connected region D*

consisting of points interior to and on a simple closed contour ∂D in the x-t plane. Let ∂D be described in the positive (counter-clockwise) direction, then

$$\oint_{\partial D} (Q\,\mathrm{d}t + P\,\mathrm{d}x) = \iint_D \left(\frac{\partial P}{\partial t} - \frac{\partial Q}{\partial x} \right) \mathrm{d}t\,\mathrm{d}x\,.$$

The following lemma is of importance for evaluating exponential integrals on open unbounded domains, such as the real line.

Lemma 1.3.3. ***(Jordan)*** *Let C be the circular arc given in Figure 1.3, obtained by considering the intersection of the circle of radius R with the upper half complex plane $\mathbb{C}^+$. Suppose that on C we have $f(k) \to 0$ uniformly as $R \to \infty$. Then*

$$\lim_{R\to\infty} \int_C e^{i\lambda k} f(k)\,\mathrm{d}k = 0\,, \quad (\lambda > 0)\,.$$

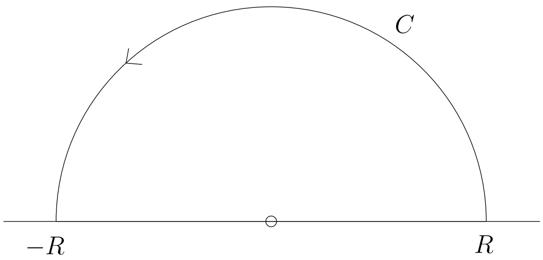


Figure 1.3: The contour C corresponding to Jordan's Lemma (Lemma 1.3.3).

We conclude this section with the most important consequence of Cauchy's Theorem (Theorem 1.3.1). This formula can be used to determine the values of an analytic function $f(k)$ on the interior of a closed contour C using the known values of the function on the boundary of C.

Theorem 1.3.4. ***(Cauchy's Integral Formula)*** *Let $f(k)$ be analytic interior to and on a simple closed contour C. Then at any interior point k*

$$f(k) = \frac{1}{2\pi i} \oint_C \frac{f(k')}{k' - k}\,\mathrm{d}k'\,. \tag{1.13}$$

Equation (1.13) is referred to as Cauchy's integral formula.

1.3.2 Residue Calculus

If the function $f(k)$ is analytic everywhere on and inside a simple closed contour C, then it follows by Cauchy's Theorem (Theorem 1.3.1) that

$$\oint_C f(k)\,\mathrm{d}k = 0\,.$$

It may be however that $f(k)$ has a singularity at a point $k = k_n$, and this leads us to the following definition:

Definition 1.3.5. *A singularity of $f(k)$ at $k = k_n$ is called a simple pole if $f(k)$ has the following representation:*

$$f(k) = \frac{p(k)}{k - k_n}\,,$$

where $p(k)$ is analytic in a neighbourhood of k_n and $p(k_n) \neq 0$. The coefficient $p(k)$ is called the residue of $f(k)$ at $k = k_n$.

The following definition allows us to compute residues, and is of significant importance in the evaluation of complex integrals.

Definition 1.3.6. *The residue at a simple pole is calculated using the formula*

$$f(k) = \frac{p(k)}{q(k)}\,,$$

where $p(k_n) \neq 0$ and $q(k)$ has a simple zero at k_n (implying that $f(k)$ has s simple pole at k_n by Definition 1.3.5). The formula for the residue at a simple pole is then given by

$$\operatorname*{Res}_{k=k_n} f(k) = \operatorname*{Res}_{k=k_n} \frac{p(k)}{q(k)} = \frac{p(k_n)}{q'(k_n)}\,.$$

Note that if $k = k_n$ is a simple zero of $q(k)$, then $q'(k_n)$ cannot vanish. Now that we can compute residues, we turn our attention to residue integration and begin with the example of an analytic function $f(k)$ with several singularities inside a simple closed contour C.

Theorem 1.3.7. ***(Residue)*** *Let $f(k)$ be analytic inside and on a simple closed contour C, except for finitely many singular points $k_1, k_2, \ldots k_n$ inside C. Then the integral of $f(k)$, taken counter-clockwise around C, equals $2\pi i$ times the sum of the residues of $f(k)$ at $k_1, k_2, \ldots k_n$:*

$$\oint_C f(k)\,\mathrm{d}k = 2\pi i \sum_{j=1}^{n} \operatorname*{Res}_{k=k_j} f(k)\,.$$

If the function $f(k)$ is such that there is just one simple pole on the boundary of the contour C, then the integral of $f(k)$ is equal to exactly πi times the sum of the residue at the pole.

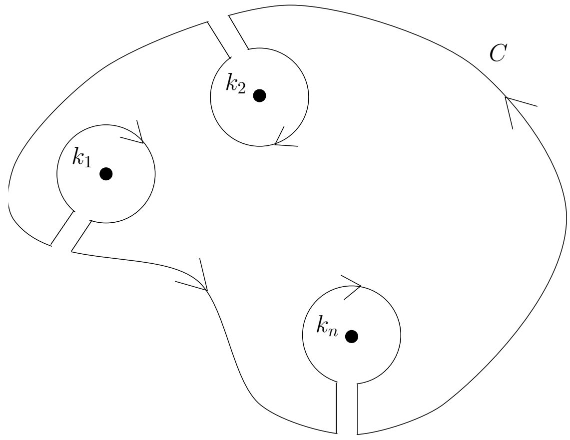


Figure 1.4: The graphical representation of the Residue Theorem (Theorem 1.3.7).

Theorem 1.3.8. ***(Simple Poles on the Real Axis)*** *Let $f(k)$ be analytic inside and on a simple closed contour C, except for a simple pole at $k = k_n$ on the real axis. Then*

$$\lim_{r\to 0} \oint_C f(k)\,\mathrm{d}k = \pi i \operatorname*{Res}_{k=k_n} f(k)\,.$$

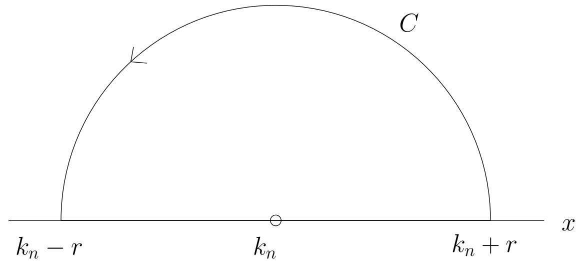


Figure 1.5: The graphical representation of Theorem 1.3.8.

Remark 1.3.9. The integral around the semicircle is exactly one-half the value obtained by integration over the full circle. A general theorem regarding this result is given as follows: Let $f(k)$ be analytic at $k = k'$. Consider the integral

$$F_\theta = \int_{k_1}^{k_2} \frac{f(k)}{k - k'}\,\mathrm{d}k\,,$$

taken from $k_1 = k' + re^{i\theta_1}$ to $k_2 = k' + re^{i\theta_2}$ along the circle $|k - k'| = r$, (Figure 1.6). Then

$$\lim_{r\to 0} F_\theta = \theta i f(k')\,,$$

where $\theta = \theta_2 - \theta_1 + 2n\pi$ and n is chosen such that $|\theta| \leqslant 2\pi$.

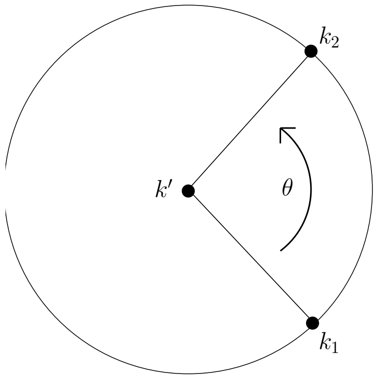


Figure 1.6: The graphical representation of Remark 1.3.9.

1.3.3 Cauchy Type Integrals

Consider the integral

$$\mu(k) = \frac{1}{2\pi i} \int_C \frac{f(k')}{k' - k} \, \mathrm{d}k' \,, \tag{1.14}$$

where C is a smooth contour and $f(k')$ is a function satisfying the *Hölder condition* on C. i.e., for any two points k_1' and k_2' on C, $\exists\, \alpha > 0$ and $0 \leqslant \lambda \leqslant 1$ such that

$$|\, f(k_1') - f(k_2')| \leqslant \alpha |\, k_1' - k_2'|^{\lambda} \,.$$

If $\lambda = 1$ then the Hölder condition is the so-called Lipschitz condition. Provided that k is not on C then the integral, given by (1.14), is well defined and $\mu(k)$ is analytic. If k is on C then it is necessary to know how k approaches C, so as to give the integral meaning.

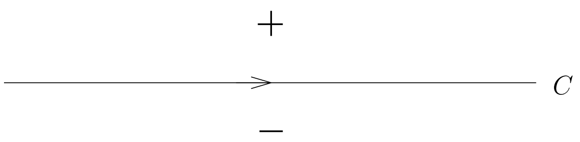


Figure 1.7: The regions $+$ and $-$ on either side of the contour C.

The two regions to the left and right of the positive direction of C, are denoted by $+$ and $-$ respectively, and are given in Figure 1.7. When k approaches C along a curve that is entirely in the $+$ region, the function $\mu(k)$ has a limit $\mu^+(\tau)$, where τ on C. Similarly, $\mu(k)$ has a limit $\mu^-(\tau)$ when k approaches C along a curve entirely in the $-$ region. These limits are given by the so-called Plemelj Formulae and are called the boundary values of the function $\mu(k)$.

Lemma 1.3.10. ***(Plemelj Formulae)*** *Let C be a smooth contour (open or closed) and let $f(k')$ satisfy a Hölder condition on C. Then the Cauchy type integral $\mu(k)$, defined by (1.14), has the limiting values $\mu^+(\tau)$ and $\mu^-(\tau)$ as k approaches C from the left and the right, respectively, and τ is not an endpoint of C. These limits are given by*

$$\mu^{\pm}(\tau) = \pm\frac{1}{2}f(\tau) + \frac{1}{2\pi i}\fint_C \frac{f(k')}{k'-\tau}\,\mathrm{d}k'\,.$$

In these equations, $\fint$ denotes the principal value integral defined by

$$\fint_C \frac{f(k')}{k'-\tau}\,\mathrm{d}k' = \lim_{\epsilon\to 0}\int_{C-C_\epsilon}\frac{f(k')}{k'-\tau}\,\mathrm{d}k'\,,$$

where C_ϵ is the part of C that has length 2ϵ and is centred around τ, (Figure 1.8).

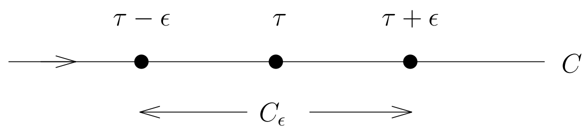


Figure 1.8: The graphical representation of C_ϵ.

1.3.4 Scalar Riemann-Hilbert Problems

The formulae introduced above, indicates the behaviour of a Cauchy integral as k approaches any point on the contour C, and can be used to solve scalar Riemann-Hilbert problems. We now introduce some definitions.

Definition 1.3.11. *Let C be a simple, smooth, closed contour dividing the complex k-plane into two regions $D^{\pm}$, where the orientation of C is such that D^+ is always on the left of the positive direction.*

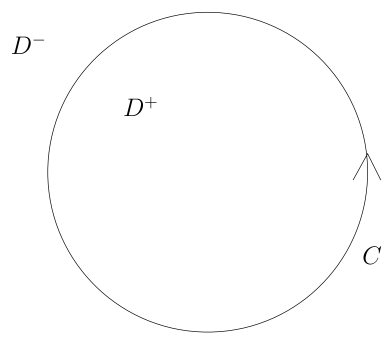


Figure 1.9: The simple closed contour C and the regions D^+ and D^- of Definition 1.3.11.

A scalar function $\mu(k)$ defined in the entire plane, except for points on C, is called sectionally analytic if

i.) the function $\mu(k)$ is analytic in each of the regions $D^{\pm}$ except perhaps at $k = \infty$, and

ii.) the function $\mu(k)$ is sectionally continuous with respect to C. i.e., as k approaches any point τ on C along any path which lies completely in either D^{+} or D^{-}, the function $\mu(k)$ approaches a definite limiting value $\mu^{+}(\tau)$ or $\mu^{-}(\tau)$ respectively.

It follows that $\mu(k)$ is continuous in the closed region $D^{+} \cup C$ if it is assigned the value $\mu^{+}(\tau)$ on C, and is continuous in the closed region $D^{-} \cup C$ if it is assigned the value $\mu^{-}(\tau)$ on C. The values $\mu^{\pm}(\tau)$ are called the boundary values of $\mu(k)$.

A simple example of a scalar Riemann-Hilbert problem, involves finding a sectionally analytic function $\mu(k)$, with $\mu(k) = O(1/k)$ as $k \to \infty$, such that the two limiting functions $\mu^{+}(\tau)$ and $\mu^{-}(\tau)$, defined inside and outside the closed contour C given by Figure 1.9, of the complex k-plane, satisfy

$$\mu^{+}(\tau) - \mu^{-}(\tau) = f(\tau)\,, \quad \tau \text{ on } C\,,$$

for a given sufficiently smooth function $f(\tau)$. This problem is closely related to the Cauchy type integral, given by (1.14), and the unique solution satisfying the boundary condition at $k \to \infty$, is given by

$$\mu(k) = \frac{1}{2\pi i}\int_C \frac{f(k')}{k'-k}\,\mathrm{d}k'\,. \tag{1.15}$$

Remark 1.3.12. Consider the integral

$$\int_{-\infty}^{\infty} f(k)\,\mathrm{d}k = \lim_{R\to\infty}\int_{-R}^{R} f(k)\,\mathrm{d}k\,,$$

where $f(k)$ is analytic and bounded in $\mathbb{C}^{+}$. This complex integral over the open real axis, may be regarded as closed at infinity. i.e., it can be treated as a portion of the complex integral $\oint_C f(k)\,\mathrm{d}k$, evaluated over the closed contour C (Figure 1.10):

$$\oint_C f(k)\,\mathrm{d}k = \int_{-R}^{R} f(k)\,\mathrm{d}k + \int_{C_R} f(k)\,\mathrm{d}k\,.$$

This important observation allows for the application, to open contours, of Cauchy's Theorem (Theorem 1.3.1), Jordan's Lemma (Lemma 1.3.3) and the results of Residue Calculus. An analogous observation, allows for the application of these results to functions that are analytic and bounded in $\mathbb{C}^{-}$.

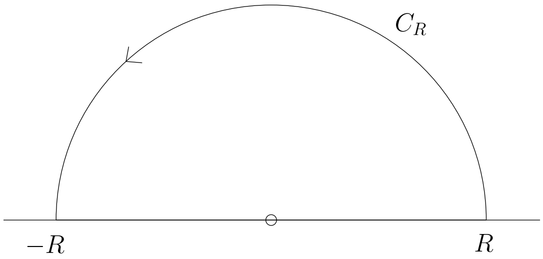


Figure 1.10: The graphical representation of Remark 1.3.12.

1.3.5 The Spectral Analysis of ODEs and the Fourier Transform

In this section, we present a simple application of the Riemann-Hilbert machinery, which yields an alternative, new derivation of the Fourier transform inversion formulae for smooth decaying functions, given by

$$\hat{q}(k) = \int_{-\infty}^{\infty} e^{-ikx} q(x)\, \mathrm{d}x\,, \qquad q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{q}(k)\, \mathrm{d}k\,, \qquad k \in \mathbb{R}\,.$$

This is obtained by performing the relevant *spectral analysis* of a particular ODE. (The Fourier transform formulae were given by (1.5), but with $q(x)$ replaced by $u(x)$).

We consider the first order eigenvalue ODE

$$\mu_x(x,k) - ik\mu(x,k) = q(x)\,, \quad k \in \mathbb{C}\,, \tag{1.16}$$

with $q(x)$ a smooth, decaying function, given above. Our goal is to find a solution $\mu(x,k)$ bounded in k for all $k \in \mathbb{C}$, and sectionally analytic. This process is called the *spectral analysis* of the ODE.

Proposition 1.3.13. *The function*

$$\mu(x,k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ik'x}\hat{q}(k')}{k'-k}\, \mathrm{d}k'\,, \quad k \in \mathbb{C}\,,$$

is a solution of the ODE (1.16), such that $\mu(x,k)$ is analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ separately, and $\mu(x,k) = O\left(1/k\right)$ as $k \to \infty$.

Proof. We define a solution $\mu(x,k)$, bounded for all $k \in \mathbb{C}$, by

$$\mu(x,k) = \begin{cases} \mu^+(x,k)\,, & \operatorname{Im} k \geqslant 0\,, \\ \mu^-(x,k)\,, & \operatorname{Im} k \leqslant 0\,, \end{cases}$$

where $\mu^{\pm}(x,k)$ are the two particular solutions of (1.16) given by

$$\mu^{+}(x,k) \;=\; \int_{-\infty}^{x} q(\xi)e^{ik(x-\xi)}\,\mathrm{d}\xi\,, \tag{1.17}$$

$$\mu^{-}(x,k) \;=\; -\int_{x}^{\infty} q(\xi)e^{ik(x-\xi)}\,\mathrm{d}\xi\,. \tag{1.18}$$

Since $x-\xi \geqslant 0$, the solution $\mu^{+}(x,k)$ is analytic in the upper half plane, and the solution $\mu^{-}(x,k)$ is analytic in the lower half plane. Subtracting the two equations we find

$$\mu^{+}(x,k) - \mu^{-}(x,k) = e^{ikx}\hat{q}(k)\,, \quad k \in \mathbb{R}\,, \tag{1.19}$$

where $\hat{q}(k)$ is defined by

$$\hat{q}(k) = \int_{-\infty}^{\infty} e^{-ik\xi}q(\xi)\,\mathrm{d}\xi\,, \quad k \in \mathbb{R}\,. \tag{1.20}$$

Using integration by parts, equations (1.17) and (1.18) imply that

$$\mu(x,k) = O\left(\frac{1}{k}\right)\,, \quad k \to \infty\,. \tag{1.21}$$

Equations (1.19) and (1.21) define an elementary Riemann-Hilbert problem whose unique solution, according to (1.15), is given by

$$\mu(x,k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ik'x}\hat{q}(k')}{k'-k}\,\mathrm{d}k'\,, \quad k \in \mathbb{C}\,. \tag{1.22}$$

□

Corollary 1.3.14. ***(The Fourier Transform Pair)*** *The function $q(x)$ can be represented as*

$$q(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{q}(k)\,\mathrm{d}k\,,$$

where $\hat{q}(k)$ is given by (1.20).

Proof. Differentiating (1.22) with respect to x yields

$$\mu_x(x,k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} ik'\left(\frac{e^{ik'x}\hat{q}(k')}{k'-k}\right)\mathrm{d}k'\,.$$

Hence,

$$\begin{aligned} q(x) \;&=\; \mu_x(x,k) - ik\mu(x,k) \\ &=\; \frac{1}{2\pi i}\int_{-\infty}^{\infty} i(k'-k)\left(\frac{e^{ik'x}\hat{q}(k')}{k'-k}\right)\mathrm{d}k' \\ &=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik'x}\hat{q}(k')\,\mathrm{d}k'\,. \end{aligned} \tag{1.23}$$

Equations (1.20) and (1.23) define the classical Fourier transform pair, and the proof is complete. □

1.3.6 Linear Differential Operators with Constant Coefficients

In this section we include some general results in the theory of linear differential operators, acting on the interval $[0, L]$, [35].

Definition 1.3.15. *A linear constant coefficients differential operator of order n, is an operator L whose action on a function $q(x) \in C^n([0, L])$ is given by an expression of the form*

$$Lq(x) = a_n q^{(n)}(x) + a_{n-1} q^{(n-1)}(x) + \cdots + a_0 q(x)\,, \tag{1.24}$$

where the coefficients $a_n, a_{n-1}, \dots, a_0$ are given complex constants.

Since the operator acts on functions defined on $[0, L]$, it is natural to consider the case when boundary conditions are considered. This restricts the domain of definition of the operator to differentiable functions which satisfy the given conditions.

We will only consider the case when exactly n boundary conditions are prescribed. Hence the domain of the operator is of the form

$$D(L) = \{q(x) \in C^n([0, L]) : \quad U_j(q(x)) = 0\,,\, j = 1, \dots, n\}\,,$$

where $U_j(q(x)) = 0$, $j = 1, \dots, n$ are the given (homogeneous) boundary conditions, in general of the form

$$U_j(q(x)) = \alpha_j q(0) + \beta_j q(L) = 0\,.$$

When the operator is self-adjoint, there exists a comprehensive theory describing its spectral structure. To define this notion, we need to consider all function spaces as contained in the Hilbert space $L^2([0, L])$, with the inner product structure inherited from it. Namely, we define the inner product of two functions $q(x)$ and $r(x)$ in $L^2([0, L])$ by

$$\langle q(x), r(x) \rangle = \int_0^L q(x)\, \overline{r}(x)\, \mathrm{d}x\,. \tag{1.25}$$

Using this inner product, we can define the adjoint operator:

Definition 1.3.16. *The linear differential operator L^* with constant coefficients, of order n, acting on the function $r(x) \in C^n([0, L])$ and given by*

$$L^* r(x) = (-1)^n \left(\overline{a}_n r(x)\right)^{(n)} + (-1)^{n-1} \left(\overline{a}_{n-1} r(x)\right)^{(n-1)} + \cdots + \overline{a}_0 r(x)\,,$$

is called the adjoint operator of L.

Definition 1.3.17. *If $D(L^*)$ is contained in $D(L)$ and $L^* = L$, then L is a self-adjoint operator.*

Self-adjoint operators are important because it is possible to characterise their spectra, and there exists a spectral theorem generalising the finite dimensional situation, and guaranteeing that all information about the operator is contained in the spectrum.

Definition 1.3.18. *The point spectrum of a linear differential operator L is defined as the set of eigenvalues λ for which the homogeneous boundary value problem*

$$Lq(x) = \lambda q(x), \quad U_j(q(x)) = 0, \quad j = 1, \dots, n\,,$$

has non-trivial solutions. Each of these non-trivial solutions is an eigenfunction belonging to λ.

In broad generalisation, the classical theory yields that if an operator has only point spectrum, and the spectral theorem holds for it in some form, then one can reconstruct the space from the corresponding eigenvalues, which then form a Riesz basis. Namely, we have the following result [35].

Theorem 1.3.19. *Any function which is continuous and has continuous derivatives up to the n^{th} order, and satisfies the boundary conditions associated with a self-adjoint differential operator of order n, can be expanded in terms of the eigenfunctions of this operator, in a uniformly convergent, generalised Fourier series.*

The circumstances under which a reasonably arbitrary function can be expanded as a series of eigenfunctions of a given boundary value operator is a central issue. The most difficult property to prove is the completeness of the family of eigenfunctions. The classical spectral theory of these operators yields completeness for even order cases, assuming that the given boundary conditions are 'reasonable' (a notion of *regular boundary conditions*, can be defined, but we will not go into that, see [9]). When the operator is even order but not self-adjoint, it is still possible to obtain an expansion theorem by considering the operator and its adjoint. This is the essence of the theorem we state below. All relevant definitions, can be found in [35].

Theorem 1.3.20. *Let L be an even order operator acting on the function $q(x)$, generated by regular boundary conditions and of the form (1.24), and for which the adjoint* *differential operator L^* exists, and assume that all eigenvalues of L are simple. Then any continuous function $q(x)$ which has continuous derivatives up to the n^{th} order, and satisfies the boundary conditions, can be expanded in a uniformly convergent series of the eigenfunctions:*

$$q(x) = \sum_{j=1}^{\infty} \alpha_j \, q_j(x), \quad \alpha_j = \int_0^L q(\xi) \, r_j(\xi) \, \mathrm{d}\xi \,,$$

where $q_j(x)$ and $r_j(x)$ are the eigenfunctions corresponding to the eigenvalues λ_j and $\overline{\lambda}_j$ of the operators L and L^* respectively.

A similar theorem applies to some special cases of odd order boundary operators. This is for example the case when boundary conditions are periodic (and this is trivial) or more generally they couple the endpoints 0 and L, [11]. In the general case of differential operators on $[0, L]$, although there is always a point spectrum, the corresponding set of eigenfunctions is not always guaranteed to be complete. For non self-adjoint examples, this is the hardest part of the construction of a series representation in the classical theory.

Remark 1.3.21. The method we present allows one to bypass the problem of proving completeness, and provides a constructive way to present the solution of a given boundary value problem.

1.3.7 The Distribution of the Zeros of Exponential Sums

The material summarised in this section is taken from [33]. We state a theorem which uses an explicit geometric construction to characterise the distribution of the zeros of exponential sums, of the form

$$\Delta(z) = \sum_{j=0}^{N-1} \alpha_j \, e^{\beta_j z} \,, \quad \alpha_j \,, \beta_j \in \mathbb{C} \,. \tag{1.26}$$

For the proof, see [33].

Theorem 1.3.22. ***(Levin)*** *Let $\Delta(z)$ be a function of the form (1.26). If P is the convex hull in the complex z-plane of the polygon whose N vertices are the β_j's, then all of the roots of $\Delta(z)$, except possibly for a set of zero density, lie inside arbitrarily small angles containing the normals to the sides of the polygon P.*

It follows from this result, that the argument of the zeros of the function $\Delta(z)$ depends on the exponents of the exponential terms, while the precise location of the zeros depends on the coefficients. In summary, the N points β_j, $j = 0, 1, \ldots, N-1$ are located in the complex z-plane and joined to form the convex hull P. The zeros then cluster along the rays, emanating from the origin with direction orthogonal to the sides of the polygon. As a simple demonstration, Figure 1.11 shows a typical example for which $N = 3$ and $\beta_j = e^{\frac{2\pi i j}{3}}$, $j = 0, 1, 2$.

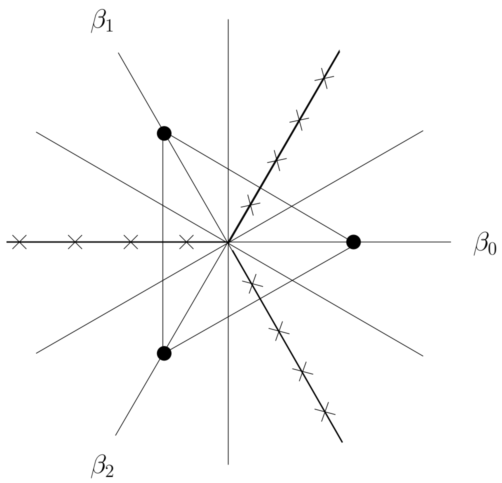


Figure 1.11: The graphical representation of Levin's Theorem (Theorem 1.3.22), for a typical example for which $N = 3$.

Remark 1.3.23. The density of the set of roots inside each of the angles can be calculated and is equal to $l_j/2\pi$, where l_j is the length of the corresponding side of the polygon P. Furthermore, the roots lie in the half-planes

$$|\operatorname{Im}(ze^{-i\theta_j})| < \gamma\,, \quad \operatorname{Re}(ze^{-i\theta_j}) \geqslant 0\,,$$

where θ_j defines the direction of the normal to the side of the polygon P, and γ is some positive number. However, for the application of Theorem 1.3.22 in the work that is to follow, the primary interest will be on the rays upon which the zeros lie, and not on their density. Therefore, no further comment regarding the density of the set of roots will be made.

Remark 1.3.24. Although it not always straightforward to find the exact location of the zeros, their asymptotic position in the complex plane is sufficient for our purposes.

1.4 Thesis Overview

In Chapter 2 we review the classical approaches for solving second and third order linear evolution PDEs and introduce the elements of the Fokas transform method for linear evolution PDEs, first for problems on the domain $[0, \infty)$, then on $[0, L]$. For comparison with the new method, the solution of several PDEs are derived by using the classical approach based on separation of variables.

The main difficulty posed by studying initial boundary value problems of the form (1.1), is the determination of the unknown boundary values in the representation of the solution. The transform method is used to derive the solution, in general, in the form of an integral representation in the complex plane. This representation which always exists, involves only the given initial and boundary data of the problem, and we describe the general algorithm to derive it. The method is illustrated by solving specific PDEs posed on the half-line, and the original derivation of Fokas, which is based on the formulation of a Riemann-Hilbert problem, is discussed along with the explicit derivation of the integral representation of the solution.

In Chapter 3 we focus on the spectral representation of two-point boundary value problems for second and third order linear evolution PDEs. In particular, we show that, in agreement with classical theory, the Fokas transform method can be successfully used for the derivation of the solution as an infinite discrete series, for all well-posed second order boundary value problems, and third order problems such that the boundary conditions couple the two end points of the interval. For both cases, the results are illustrated by examples. We also show that, in agreement with classical theory, for the third order problem with uncoupled boundary conditions, the integral representation of the solution cannot be expressed entirely as an infinite discrete series.

In Chapter 4 we consider the problem of solving higher order boundary value problems by the Fokas transform method. Initially we focus our analysis on fourth order linear evolution PDEs, then further develop the new method to be able to analyse even/odd higher order generalisations, and show that the derivation of the series representation in the non self-adjoint case can be obtained directly from the integral representation. Detailed examples illustrating the derivation of the solution for both self-adjoint and non self-adjoint problems are included.

Chapter 5 and Chapter 6 include the numerical results for linear and nonlinear differential equations, respectively. The focus of Chapter 5 is on the imposition of boundary conditions for linear differential equations, and we solve numerically a variety of boundary value problems. We also introduce the idea of the implicit imposition of the boundary conditions, using numerical transforms, and as illustrative examples, the cases of the discrete sine/cosine transform for the implicit imposition of boundary conditions of Dirichlet/Neumann type are included.

Spectral methods for PDEs in unbounded domains have received much attention, for example the third order problem on the real/half line. The interest of Chapter 6 is on the numerical study of the nonlinear KdV equation on a finite interval. It is well known that the nonlinear KdV equation with periodic boundary conditions, approximating the solution on $\mathbb{R}$, supports soliton solutions, and the numerical schemes of both Fornberg and Whitham [25], and Tappert [45] will be used in Chapter 6 to model this behaviour. Boundary value problems for the KdV equation have not been studied until very recently. In Chapter 6 we present several numerical schemes, based on spectral methods, for solving the nonlinear KdV equation posed on the bounded domain with periodic and nonperiodic boundary conditions.

Chapter 2

The Fokas Spectral Transform Method and Boundary Value Problems for Linear Evolution PDEs

In this chapter we introduce a new approach for studying boundary value problems for linear PDEs with constant coefficients and integrable nonlinear evolution PDEs in one space dimension. The method we describe is based on the fact that such equations are expressible as the compatibility condition of two linear ordinary differential equations (one in the spatial variable x and the other in the temporal variable t). In the integrable case, this pair of ODEs is called a Lax pair and its existence is a characterising property of the integrability.

Integrable nonlinear equations in one space dimension came to prominence when this general method, and the existence of Lax pairs, were discovered. The inverse scattering, or inverse spectral, transform method, first proposed in [26] for the KdV equation, was used for solving the pure initial value (Cauchy) problem with decaying initial data. The importance of this method was understood when the method was generalised from the KdV equation to any equation that could be written as the compatibility of a Lax pair (named after Peter Lax, who was the first to interpret the inverse scattering technique in this light) and can be considered as a nonlinearisation of the Fourier transform. The inverse scattering transform method consists of two steps: the spectral analysis of the

x-part of the Lax pair which yields a nonlinear Fourier transform, and the spectral analysis of the t-part which then determines the evolution of the associated nonlinear Fourier data, called the *inverse scattering data* or *spectral data*. The nonlinear Fourier transform cannot in general be expressed in closed form and is given through the solution of a matrix Riemann-Hilbert problem.

This approach can be used generally for integrable PDEs in two variables. However, here we focus on the case of linear PDEs with constant coefficients. Fokas [13] considered the problem of generalising the inverse scattering transform to a method for the solution of initial boundary value problems. For linear PDEs, Fokas and Gelfand [18] made the crucial observation that these PDEs can be regarded as a special case of integrable equations. In particular, linear PDEs possess a Lax pair formulation [32]. This suggests that one can use a linear inverse scattering transform; for the Cauchy problem this is just a Fourier transform method. The idea that was finally to yield positive results for the solution of boundary value problems was to treat the two ODEs in the Lax pair simultaneously. This is the basis of Fokas' general approach to solving boundary value problems for linear and integrable nonlinear PDEs [12].

To perform the simultaneous spectral analysis of the two equations of the Lax pair means to construct the solution of both ODEs in the pair, which is bounded in the auxiliary parameter k, for all k in $\mathbb{C}$. This leads to the formulation, in the complex k-plane, of a Riemann-Hilbert problem whose unique solution yields a spectral representation of the solution of the original problem.

In this chapter we describe the elements of the Fokas transform method for linear evolution PDEs on the domains $[0, \infty)$ and $[0, L]$. After a discussion of the steps involved in this transform method, we illustrate it concretely by solving a specific PDE posed on the half-line and constructing the integral representation of the solution, in terms of the given initial and boundary data. We present the original derivation based on the formulation of a Riemann-Hilbert problem in the complex k-plane as well as a simpler way that can be used (for linear PDEs) to derive the explicit integral representation of the solution. The main difficulty in solving boundary value problems is the characterisation of the boundary values that are not prescribed as boundary conditions. The derivation of an integral representation of the solution involving only the given data of the problem, always involves complex contours of integration, and relies on the analysis of the so-called *global relation*, which is an algebraic relation, defined in the complex k-plane, combining all the boundary values of the solution. This relation is at the heart of the method proposed, and it is presented in its general form. The analysis of the invariance properties of the global relation yields a certain system of equations whose solution characterises the unknown boundary values. We shall be particularly interested in two-point boundary value problems. In this case, one needs to characterise the *PDE discrete spectrum of the boundary value problem*, which we define as the set of zeros of the determinant of this system, and coincides with the discrete spectrum of an associated ordinary differential operator. In general this is a set of complex numbers, and we show that the location in the complex plane of the element of the PDE spectrum indicates whether or not an infinite discrete series representation of the solution can be derived from the integral representation. When a series solution exists, it can be realised by a new approach (different from the classical one based on separating variables), which we illustrate by several examples.

2.1 The Elements of the Fokas Transform Method for Linear Evolution PDEs

We introduce the new method of Fokas by describing how to solve a two-point boundary value problem for the n^{th} order linear evolution PDE of the form

$$q_t(x,t) + a(-i\partial_x)^n q(x,t) = 0\,, \quad t > 0\,, \quad x \in [0, L]\,, \tag{2.1a}$$

$$q(x,0) = q_0(x)\,, \quad x \in [0, L]\,, \tag{2.1b}$$

where L is a finite positive constant, n is an integer defining the order of the problem and $a \in \{\pm 1, \pm i\}$ is chosen such that the Cauchy problem is well-posed. By this we mean that the solution of the Cauchy problem with initial data $q(x,0) = q_0(x)$, which can be found by the Fourier transform, and is given by

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-ak^n t}\hat{q}_0(k)\,\mathrm{d}k\,,$$

does not grow as $t \to \infty$. i.e., $\mathrm{Re}(-ak^n) \leqslant 0$. Furthermore, it is required that the solution $q(x,t) \to 0$ as $|x| \to \infty$. Such examples include the following:

- $q_t(x,t) - q_{xx}(x,t)$, $(n = 2\,, a = 1)$,
- $iq_t(x,t) + q_{xx}(x,t) = 0$, $(n = 2\,, a = i)$,
- $q_t(x,t) + q_{xxx}(x,t) = 0$, $(n = 3\,, a = -i)$.

Furthermore, n boundary conditions must be prescribed, and it is assumed that these, along with the the initial condition, are compatible at $x = 0$ and $x = L$ and are sufficiently smooth.

The first problem is to determine how many boundary values must be prescribed as boundary conditions, and we use the following result established by Fokas and Sung [23].

Theorem 2.1.1. *A boundary value problem posed on* $[0, L]$ *for a PDE of the form (2.1a) is well-posed, hence it admits a unique solution, if in addition to the initial condition* $q(x, 0) = q_0(x)$*,* n *boundary conditions are prescribed.* N *of these boundary conditions should be prescribed at* $x = 0$ *and* $n - N$ *at* $x = L$ *where* N *is determined as follows:*

$$N = \begin{cases} \frac{n}{2}\,, & n \text{ even}\,, \\ \\ \frac{n\pm 1}{2}\,, & n \text{ odd}\,. \end{cases} \tag{2.2}$$

The sign in the latter equality is determined by the sign of the x-derivative. For example, if $q_t(x,t) + q_{xxx}(x,t) = 0$ then $N = 1$, whereas if $q_t(x,t) - q_{xxx}(x,t) = 0$ then $N = 2$.

The proof of this theorem is given in [22] by using the construction given by the method of Fokas. Hence this method also provides a rigorous characterisation of the boundary value problems that are well-posed. For such problems, we show how to construct an explicit integral representation of the solution $q(x,t)$, expressed as an integral in the complex k-plane involving the Fourier transform $\hat{q}_0(k)$ of the initial data $q_0(x)$ and some specific t-transforms of the given boundary data. Such problems, at least in some cases, can be solved also by the Fourier transform (with respect to x) or the Laplace transform (with respect to t) and we start with reviewing this classical solution approach.

Remark 2.1.2. There is no loss of generality in considering only equations of the form (2.1a). Indeed, consider the more general equation

$$q_t(x,t) + b_n\partial_x^n q(x,t) + b_{n-1}\partial_x^{n-1} q(x,t) + \cdots + b_0 q(x,t) = 0\,.$$

It turns out that the analysis of this more general equation depends on the highest order term, as it is this term that determines the asymptotic properties of the associated spectral problem [23].

The Classical Approaches

We now demonstrate how some problems can be solved by the classical Fourier/separation of variables method. We present the method for the third order problem on the half-line and a simple second order two-point boundary value problem, which will be revisited in Section 2.2 and Section 3.1 respectively, and solved using the new transform method.

The Half-Line Problem

We begin by considering the heat equation, posed on the half-line:

$$q_t(x,t) - q_{xx}(x,t) = 0\,, \quad t > 0\,, \quad 0 < x < \infty\,, \tag{2.3a}$$

$$q(x,0) = q_0(x)\,, \quad q(0,t) = f_0(t)\,, \tag{2.3b}$$

where $q_0(x)$ and $f_0(t)$ are some given functions. The appropriate x-transform for this initial boundary value problem is the sine transform pair given by

$$\begin{aligned} \hat{q}(t,k) &= \int_0^\infty \sin(kx)\, q(x,t)\, \mathrm{d}x\,, \quad k \in \mathbb{R}\,, \\ q(x,t) &= \frac{2}{\pi}\int_0^\infty \sin(kx)\, \hat{q}(t,k)\, \mathrm{d}k\,. \end{aligned}$$

Equation (2.3) and integration by parts yields

$$\hat{q}_t(t,k) + k^2 \hat{q}(t,k) = k f_0(t)\,,$$

and therefore the solution is given by

$$q(x,t) = \frac{2}{\pi}\int_0^\infty e^{-k^2 t}\left(\hat{q}_0^{(\sin)}(k) + \int_0^t e^{k^2 s} k f_0(s)\, \mathrm{d}s\right)\sin(kx)\, \mathrm{d}k\,, \tag{2.4}$$

where

$$\hat{q}_0^{(\sin)}(k) = \int_0^\infty \sin(kx)\, q_0(x)\, \mathrm{d}x\,. \tag{2.5}$$

For third order differential operators the same approach fails. For example it is not possible to solve the problem

$$q_t(x,t) + q_{xxx}(x,t) = 0\,, \quad t > 0\,, \quad 0 < x < \infty\,,$$

$$q(x,0) = q_0(x)\,, \quad q(0,t) = f_0(t)\,,$$

for some given functions $q_0(x)$ and $f_0(t)$, using the sine, cosine or Fourier transform. Indeed, it has been shown that there does not exist a transform in x that can yield a solution of this problem by separation of variables [20].

Remark 2.1.3. Expression (2.4) is not uniformly convergent as $x \to 0$, and therefore, one cannot compute $q(0,t)$ by simply setting $x = 0$ inside the integral.

The Two-Point Boundary Value Problem

It is well known that it is possible to expand any continuous function on $[0, L]$ in terms of the eigenfunctions of the operator ∂_x^2, with one boundary condition prescribed at each end of the interval. This is due to the fact that the operator is symmetric, and its eigenfunctions then form a Riesz basis for $L^2\left([0, L]\right)$.

The eigenfunction expansion can be used in the method of separation of variables to find the solution of a two-point boundary value problem for second order problems of the form (2.1a). The solution of the equation $q(x,t)$ is expressed in the form

$$q(x,t) = X(x)\, T(t)\,,$$

and is separated into the product of a function purely of x and a function purely of t. This is substituted into the PDE to achieve two ODEs for the single functions $X(x)$ and $T(t)$. The set of solutions that are obtained are then summed to give the general solution, and the boundary conditions applied to resolve the unknown coefficients of the series. It is trivial to use separation of variables to solve for example the heat equation with Dirichlet or Neumann boundary conditions in terms of a sine or a cosine series respectively. To demonstrate this we solve the heat equation

$$q_t(x,t) - q_{xx}(x,t) = 0\,, \quad t > 0\,, \quad x \in [0, L]\,,$$

$$q(x,0) = q_0(x)\,, \quad x \in [0, L]\,,$$

with non-homogeneous Dirichlet boundary conditions

$$q(0,t) = f_0(t)\,, \quad q(L,t) = g_0(t)\,,$$

where $q_0(x)$, $f_0(t)$ and $g_0(t)$ are given smooth functions. This problem is slightly more difficult than the analogous problem with $f_0(t) = 0$ and $g_0(t) = 0$ because of the non-homogeneous boundary conditions, and is solved by reducing it to a problem having homogeneous boundary conditions.

To obtain the solution, we begin by writing $q(x,t) = v(x,t) + w(x,t)$ where $v(x,t)$ is chosen as

$$v(x,t) = f_0(t) + \frac{x}{L}\left(g_0(t) - f_0(t)\right)\,,$$

so that $v(x,t)$ satisfies the boundary conditions

$$v(0,t) = f_0(t)\,, \quad v(L,t) = g_0(t)\,.$$

Substituting into the PDE gives

$$v_{xx}(x,t) + w_{xx}(x,t) = v_t(x,t) + w_t(x,t)\,.$$

Since, $v_{xx}(x,t) = 0$ and $v_t(x,t) = f_0'(t) + \frac{x}{L}\left(g_0'(t) - f_0'(t)\right)$ we find

$$w_t(x,t) - w_{xx}(x,t) = -F(x,t)\,, \qquad F(x,t) = f_0'(t) + \frac{x}{L}\left(g_0'(t) - f_0'(t)\right)\,.$$

The function $w(x,t)$ is subject to the initial condition

$$w(x,0) = q_0(x) - \Big(f_0(0) + \frac{x}{L}\left(g_0(0) - f_0(0)\right)\Big)\,,$$

which we shall refer to as $w_0(x)$, and the boundary conditions

$$w(0,t) = 0\,, \quad w(L,t) = 0\,.$$

The solution of the homogeneous equation $w_t(x,t) - w_{xx}(x,t) = 0$ is found by separating the variables and expressing the solution in the form

$$w(x,t) = X(x)\,T(t)\,,$$

and can be expressed as

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\,\sin\left(k_n x\right)\,, \qquad k_n = \frac{n\pi}{L}\,. \tag{2.6}$$

This expression ensures that the homogeneous boundary conditions for $w(x,t)$ are satisfied. The method relies on the assumption that $w(x,t)$ can be expressed in the form given by (2.6), and also that $F(x,t)$ can be expanded in the same way:

$$F(x,t) = \sum_{n=1}^{\infty} f_n(t)\,\sin(k_n x)\ , \tag{2.7}$$

where

$$\begin{aligned} f_n(t) &= \frac{2}{L}\int_0^L F(x,t)\,\sin(k_n x)\ \mathrm{d}x \\ &= \frac{2}{L}\int_0^L \Big(f_0'(t) + \frac{x}{L}\left(g_0'(t) - f_0'(t)\right)\Big)\sin(k_n x)\ \mathrm{d}x \\ &= \frac{2}{L}\frac{1}{k_n}\left(f_0'(t) - (-1)^n g_0'(t)\right)\ . \end{aligned} \tag{2.8}$$

Substituting the series expansions for $w(x,t)$ and $F(x,t)$ into the PDE leads to the first order ODE given by

$$\frac{\mathrm{d}}{\mathrm{d}t}w_n(t) + k_n^2 w_n(t) = -\frac{2}{L}\frac{1}{k_n}\left(f_0'(t) - (-1)^n g_0'(t)\right)\ .$$

The initial condition to be imposed is determined by putting $t=0$ into the expression $w(x,t) = \sum_{n=1}^{\infty} w_n(t)\sin(k_n x)$ and equating to $w(x,0) = w_0(x)$ expanded as a Fourier series. Hence

$$\begin{aligned} w_n(0) &= \frac{2}{L}\int_0^L w_0(x)\,\sin(k_n x)\ \mathrm{d}x \\ &= \frac{2}{L}\int_0^L \Big(q_0(x) - \Big(f_0(0) + \frac{x}{L}\left(g_0(0) - f_0(0)\right)\Big)\Big)\sin(k_n x)\ \mathrm{d}x \\ &= \frac{2}{L}\left\{\int_0^L \sin(k_n x)\ q_0(x)\,\mathrm{d}x - \frac{1}{k_n}\left(f_0(0) - (-1)^n g_0(0)\right)\right\}, \end{aligned}$$

and therefore

$$w_n(t) = -\frac{2}{L}\frac{1}{k_n}e^{-k_n^2 t}\int_0^t e^{k_n^2 s}\left(f_0'(s) - (-1)^n g_0'(s)\right)\,\mathrm{d}s + w_n(0)e^{-k_n^2 t}\,.$$

Finally, the solution for $w_n(t)$, given by

$$\begin{aligned} w_n(t) &= -\frac{2}{L}\frac{1}{k_n}\Bigg\{ (f_0(t)-(-1)^n g_0(t)) - e^{-k_n^2 t}(f_0(0)-(-1)^n g_0(0)) \\ &\qquad\qquad - \int_0^t e^{-k_n^2(t-s)}k_n^2 f_0(s)\,\mathrm{d}s + (-1)^n \int_0^t e^{-k_n^2(t-s)}k_n^2 g_0(s)\,\mathrm{d}s \Bigg\} \\ &\quad + w_n(0)e^{-k_n^2 t} \\ &= -\frac{2}{L}\frac{1}{k_n}\Bigg\{ (f_0(t)-(-1)^n g_0(t)) - \int_0^t e^{-k_n^2(t-s)}k_n^2 f_0(s)\,\mathrm{d}s \\ &\qquad\qquad + (-1)^n \int_0^t e^{-k_n^2(t-s)}k_n^2 g_0(s)\,\mathrm{d}s \Bigg\} + \frac{2}{L}e^{-k_n^2 t}\int_0^L \sin(k_n x)\, q_0(x)\mathrm{d}x\,, \end{aligned} \tag{2.9}$$

is inserted into

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\,\sin(k_n x)\ ,$$

and the series solution for $q(x,t)$ is obtained:

$$q(x,t) = f_0(t) + \frac{x}{L}(g_0(t)-f_0(t)) + \sum_{n=1}^{\infty} w_n(t)\,\sin(k_n x)\ . \tag{2.10}$$

We note that this method requires that the functions involved can be represented as Fourier series, at least in some sense. e.g., L^2 convergence. Also, in order to derive solution (2.10), one has to *assume* a sine series expansion. Assuming a cosine or exponential expansion would not lead to a closed formula. In contrast, the Fokas method constructs the basis requiring no arbitrary assumptions.

For third order differential operators, this same approach is not always possible, and abstract results guarantee that there exists a complete basis of eigenfunctions only for particular types of boundary conditions. It will be shown in Section 3.2 that, in agreement with classical results, it is not possible to expand a function in terms of a complete basis of eigenfunctions, using the Fokas transform method, when uncoupled boundary conditions are imposed. The separation of variables approach fails unless one can prove that the set of eigenfunctions forms a Riesz basis for $L^2([0,L])$, and this cannot be deduced for all boundary conditions from general results in the classical literature.

The Laplace transform in t

A particular solution of equation (2.1) is given by the function $E(x,t,k) = e^{ikx-\omega(k)t}$ where $\omega(k) = ak^n$ is the dispersion relation, which can be rewritten in the form $E(x,t,s) = e^{ik(s)x+st}$ where $s + \omega(k) = 0$. This change of variable allows PDEs of the form (2.1a) to be solved by a Laplace transform (or t-transform). However, the construction of the solution is simpler using the associated x-transform.

For equations of order equal or greater to three the Laplace transform is problematic. The approach fails to indicate how many boundary conditions must be prescribed at either end of the interval to guarantee well-posedness. If we assume however that this information is given, then provided the boundary conditions do not grow with t faster than linearly exponentially, the Laplace transform is applicable, but its application is not straightforward and the computations involved are cumbersome [22]. If the boundary conditions do have sufficiently rapid growth then the method fails.

2.1.1 The Steps of the Fokas Method

The transform method of Fokas for solving boundary value problems for linear evolution PDEs yields an explicit integral representation of the solution in the general form

$$q(x,t) = \int_\Gamma e^{ikx-\omega(k)t} R(k)\,\mathrm{d}k\,,$$

where Γ is a contour in the complex k-plane and the function $R(k)$, called the *spectral function*, is explicitly determined in terms of the given initial and boundary conditions. This representation of the solution is called the *spectral representation* of the solution because of the explicit x and t-exponential dependence and offers a number of advantages. For example, this representation is suitable for studying large t asymptotics by the steepest descent method or Watson's Lemma and related methods [1].

The existence of the integral representation of the solution (at least under certain conditions) can be inferred, in some cases, from the so-called Ehrenpreis fundamental principle. An implication of this result, is that for equation (2.1) there exists a measure $\mathrm{d}\mu(k)$ and a contour Γ such that

$$q(x,t) = \int_\Gamma e^{ikx-\omega(k)t}\,\mathrm{d}\mu(k)\,.$$

This result is not constructive, in particular the measure $\mathrm{d}\mu(k)$ and the contour Γ are not given explicitly. The new method provides a constructive approach for finding the appropriate measure $d\mu(k)$ and complex contour Γ.

In quick summary, the method for representing the solution $q(x,t)$ of the given boundary value problem, comprises the following steps:

i.) Given a PDE, construct a Lax pair

The first step consists of writing the PDE in an alternative form. Namely, it is possible to realise the PDE as the compatibility condition of two linear ODEs, one in x and one in t. This is called a Lax pair and the two equations that form the Lax pair can be constructed algorithmically and are referred to as the x-part and the t-part of the Lax pair.

ii.) Simultaneous spectral analysis of the Lax pair and the global relation

Given the domain where the PDE is defined, the second step consists of performing the simultaneous spectral analysis of the two ODEs in the Lax pair for (x,t) in the domain. This analysis is on the domain on which the PDE is considered. We note that the spectral analysis of the x-part of the Lax pair corresponds to constructing an x-transform and the spectral analysis of the t-part corresponds to constructing a t-transform. The advantage of the Lax pair formulation is that it allows us to consider both equations in the Lax pair simultaneously, hence in a sense, the new method provides the synthesis of separation of variables. The spectral analysis of the Lax pair yields formally an integral representation of the solution of the problem in terms of all its boundary values, much in the spirit of the Fourier transform. However, this spectral analysis also yields one additional relation, called the *global relation*. This is a fundamental algebraic expression combining all the initial and boundary data of the solution of the problem, and it is the crucial novel relation introduced by this method.

iii.) Given appropriate boundary conditions, analyse the global relation and its invariance properties

This step is the most difficult one of the procedure, and it is here that the boundary conditions come into play. To obtain the solution, in terms only of the given data of the problem, one needs to determine the unknown boundary values of the solution, and this can be achieved by analysing the global relation. For evolution equations with simple boundary conditions, this involves the analysis of a system of algebraic equations obtained from the study of the global relation.

We now consider the above steps in some more detail.

Step i.) We begin with giving the definition and algorithmic derivation of the Lax pair we associate with a given linear PDE of the form (2.1).

Definition 2.1.4. *A Lax pair, associated with a linear evolution PDE $q_t(x,t) + D_x^{(n)}q(x,t) = 0$, where $D_x^{(n)}$ is an n^{th} order x-differential operator, is a pair of linear ODE's*

$$D_x^1\,\mu(x,t,k) = f_1(k,q)\,, \quad D_t^2\,\mu(x,t,k) = f_2(k,q)\,, \qquad k \in \mathbb{C}\,,$$

where D_x^1 and D_t^2 are x and t linear differential operators respectively, and $f_1(k,q)$ and $f_2(k,q)$ are linear functions of k and $\partial_x^j q(x,t)$, $j = 0,1,...,n-1$ such that $D_t^2\,D_x^1\,\mu(x,t,k) = D_x^1\,D_t^2\,\mu(x,t,k)$ if and only if $q(x,t)$ solves the PDE.

Proposition 2.1.5. *A Lax pair for the equation $q_t(x,t) + D_x^{(n)}q(x,t) = 0$,where $D_x^{(n)}$ is an n^{th} order x-differential operator, is given by*

$$\mu_x - ik\mu \;=\; q(x,t)\,, \tag{2.11a}$$

$$\mu_t + D_x^{(n)}\mu \;=\; 0\,, \tag{2.11b}$$

where $\mu = \mu(x,t,k)$ is a scalar function and $k \in \mathbb{C}$ is a parameter referred to as the spectral parameter.

Proof. Equations (2.11) are compatible provided that $q(x,t)$ satisfies $q_t(x,t) + D_x^{(n)}q(x,t) = 0$:

$$\begin{aligned} q_t(x,t) + D_x^{(n)}q(x,t) &= \left(\partial_t + D_x^{(n)}\right) q(x,t) \\ &= \left(\partial_t + D_x^{(n)}\right)\left(\mu_x(x,t,k) - ik\mu(x,t,k)\right) \\ &= \left(\partial_x - ik\right)\left(\partial_t + D_x^{(n)}\right)\mu(x,t,k) \\ &= \left(\partial_x - ik\right)\left(\mu_t(x,t,k) + D_x^{(n)}\mu(x,t,k)\right) \\ &= 0\,. \end{aligned}$$

□

We note that linear equations possess several Lax pair formulations and amongst these there exist two that correspond to the traditional x and t-transforms. In Proposition 2.1.5 the first equation in the Lax pair is selected because the spectral analysis yields the classical Fourier transform pair (the proof of this was given in Section 1.3.5). In general it is most convenient to use the Lax pair involving only first order derivatives of $\mu(x,t,k)$ and this can be done by substituting for the x-derivatives in the t-part using $\mu_x(x,t,k) = ik\mu(x,t,k) + q(x,t)$. Thus the Lax pair (2.11) for equation (2.1) takes the form

$$\mu_x - ik\mu \;=\; q(x,t)\,, \tag{2.12a}$$

$$\mu_t + \omega(k)\mu \;=\; X(x,t,k)\,, \tag{2.12b}$$

where $\mu = \mu(x,t,k)$, $k \in \mathbb{C}$ is a complex parameter called the *spectral parameter*,

$$\omega(k) = ak^n \tag{2.13}$$

and $X(x,t,k)$ is a function involving $q(x,t)$ and its derivatives up to order $n-1$:

$$\begin{aligned} X(x,t,k) \;&=\; iak^{n-1}q + ak^{n-2}q_x + \cdots - (-i)^n a q_x^{(n-1)} \\ &=\; \sum_{j=0}^{n-1} c_j(k)\,\partial_x^j q(x,t)\,, \end{aligned} \tag{2.14}$$

where the coefficients $c_j(k)$ are known polynomials in k:

$$c_j(k) = -ak^n(ik)^{-(j+1)}\,. \tag{2.15}$$

Example: *As an example, we construct the Lax pair of the third order PDE $q_t(x,t) + q_{xxx}(x,t) = 0$. According to (2.11), this Lax pair is given by*

$$\mu_x - ik\mu \;=\; q\,,$$

$$\mu_t + \mu_{xxx} \;=\; 0\,,$$

where $\mu = \mu(x,t,k)$. The first of these equations yields $\mu_x = ik\mu + q$, hence it can be used to express μ_{xxx} in terms of μ and q and its derivatives. This yields $\mu_{xxx} = -ik^3\mu - k^2 q + ikq_x + q_{xx}$ and therefore the first order Lax pair we use is the one given by

$$\mu_x - ik\mu \;=\; q\,,$$

$$\mu_t - ik^3\mu \;=\; k^2 q - ikq_x - q_{xx}\,.$$

In this case,

$$\omega(k) \;=\; -ik^3\,, \tag{2.16}$$

$$X(x,t,k) \;=\; k^2q - ikq_x - q_{xx}\,. \tag{2.17}$$

Equations (2.12) can be rewritten as

$$\left(\mu(x,t,k)e^{-ikx+\omega(k)t}\right)_x \;=\; e^{-ikx+\omega(k)t}q(x,t)\,, \tag{2.18a}$$

$$\left(\mu(x,t,k)e^{-ikx+\omega(k)t}\right)_t \;=\; e^{-ikx+\omega(k)t}X(x,t,k)\,, \tag{2.18b}$$

where $\omega(k)$ and $X(x,t,k)$ are given by (2.13) and (2.14) respectively. We shall see how in order to derive the integral representation of the solution, it is convenient to rewrite (2.18) as a differential form:

$$\left(e^{-ikx+\omega(k)t}q(x,t)\right)_t - \left(e^{-ikx+\omega(k)t}X(x,t,k)\right)_x = 0\,. \tag{2.19}$$

Step ii.) The second step of the method is the spectral analysis of equation (2.19). This means finding a solution $\mu(x,t,k)$ of equation (2.18), bounded in $k \in \mathbb{C}$ for all $(x,t) \in D_R = \{[0,L]\times[0,t]\}$, and in fact sectionally analytic. This yields an integral representation of the solution in terms of the initial data and all the boundary values of $q(x,t)$. Actually, the solution representation depends on some appropriate t-transforms of these boundary values, which we denote by $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$ respectively. These are defined as follows:

$$f_j(t) = \partial_x^j q(0,t)\,, \quad g_j(t) = \partial_x^j q(L,t)\,, \qquad j = 0,1,\ldots,n-1\,, \quad t>0\,,$$

and

$$\tilde{f}_j(t,k) = \int_0^t e^{\omega(k)s} f_j(s)\,\mathrm{d}s\,, \qquad k\in\mathbb{C}\,,\ t>0\,, \tag{2.20}$$

$$\tilde{g}_j(t,k) = \int_0^t e^{\omega(k)s} g_j(s)\,\mathrm{d}s\,, \qquad k\in\mathbb{C}\,,\ t>0\,. \tag{2.21}$$

It turns out that in order to achieve an effective integral representation of the solution, it is necessary to express the integral in terms of specific deformed contours of integration. We shall motivate this step later, but for now we define the domain D by

$$D = \{k\in\mathbb{C} : \mathrm{Re}\,\omega(k) \leqslant 0\}\,, \quad D^\pm = D\cap\mathbb{C}^\pm\,, \tag{2.22}$$

where $\omega(k)$ is given by (2.13) and D has boundaries given by $\partial D^{\pm}$, where the orientation is such that the interior of the domain is always to the left.

We define

$$\begin{aligned}
\tilde{f}(t,k) &= \int_0^t e^{\omega(k)s} X(0,s,k)\,\mathrm{d}s \\
&= iak^{n-1}\tilde{f}_0(t,k) + ak^{n-2}\tilde{f}_1(t,k) + \cdots - (-i)^n a\tilde{f}_{n-1}(t,k)\,, \qquad (2.23) \\
\tilde{g}(t,k) &= \int_0^t e^{\omega(k)s} X(L,s,k)\,\mathrm{d}s \\
&= iak^{n-1}\tilde{g}_0(t,k) + ak^{n-2}\tilde{g}_1(t,k) + \cdots - (-i)^n a\tilde{g}_{n-1}(t,k)\,. \qquad (2.24)
\end{aligned}$$

For the half-line problem on $[0,\infty)$, the integral representation is given by

$$q(x,t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx-\omega(k)t}\tilde{f}(t,k)\,\mathrm{d}k\right\},$$

where

$$\hat{q}_0(k) = \int_0^{\infty} e^{-ikx} q_0(x)\,\mathrm{d}x$$

denotes the Fourier transform of the given initial condition $q_0(x)$, and for the problem on the bounded domain $[0,L]$, the integral representation is given by

$$\begin{aligned}
q(x,t) = \frac{1}{2\pi}\bigg\{ &\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx-\omega(k)t}\tilde{f}(t,k)\,\mathrm{d}k \\
&- \int_{\partial D^-} e^{ik(x-L)-\omega(k)t}\tilde{g}(t,k)\,\mathrm{d}k\bigg\}, \qquad (2.25)
\end{aligned}$$

where

$$\hat{q}_0(k) = \int_0^{L} e^{-ikx} q_0(x)\,\mathrm{d}x\,. \qquad (2.26)$$

We now derive the global relation, which is an algebraic expression relating the initial and boundary data, by applying Green's Theorem (Theorem 1.3.2) to the domain $D_R = \{[0,L]\times[0,t]\}$, and derive explicitly the integral representation of the solution, given by (2.25). We begin by rewriting (2.19) in the form

$$\frac{\partial P(x,t)}{\partial t} - \frac{\partial Q(x,t)}{\partial x} = 0\,,$$

where $P(x,t) = e^{-ikx+\omega(k)t}q(x,t)$ and $Q(x,t) = e^{-ikx+\omega(k)t}X(x,t,k)$. Therefore, applying Green's Theorem to the domain $D_R = \{[0,L]\times[0,t]\}$, consisting of points interior to and on the simple closed contour ∂D_R in the x-t plane, gives

$$\int_{\partial D_R} (Q(x,t)\,\mathrm{d}t + P(x,t)\,\mathrm{d}x) = 0\,.$$

Substituting for $P(x,t)$ and $Q(x,t)$ implies

$$e^{\omega(k)t}\hat{q}(t,k) = \int_0^t e^{-ikL+\omega(k)s}X(L,s,k)\,\mathrm{d}x - \int_0^t e^{\omega(k)s}X(0,s,k)\,\mathrm{d}s + \hat{q}_0(k)\,, \tag{2.27}$$

where $\hat{q}_0(k)$ is given by (2.26) and

$$\hat{q}(t,k) = \int_0^L e^{-ikx}q(x,t)\,\mathrm{d}x$$

denotes the Fourier transform of $q(x,t)$. Expression (2.27) can be rewritten explicitly in the form

$$\sum_{j=0}^{n-1} c_j(k)\left(\tilde{f}_j(t,k) - e^{-ikL}\tilde{g}_j(t,k)\right) = \hat{q}_0(k) - e^{\omega(k)t}\hat{q}(t,k)\,, \qquad k\in\mathbb{C}\,, \tag{2.28}$$

where $c_j(k)$, $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$ are defined by (2.15), (2.20) and (2.21) respectively. The algebraic relation, given by (2.28), is called the *global relation* and relates all the boundary values of the solution. The global relation can be written concisely as

$$\tilde{f}(t,k) - e^{-ikL}\tilde{g}(t,k) = \hat{q}_0(k) - e^{\omega(k)t}\hat{q}(t,k), \quad k\in\mathbb{C}\,, \tag{2.29}$$

where $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$ are given by (2.23) and (2.24) respectively. Taking the inverse Fourier transform of (2.29), with respect to x, yields

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{q}(t,k)\,\mathrm{d}k \\ &= \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)\,\mathrm{d}k - \int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\tilde{f}(t,k)\,\mathrm{d}k \right. \\ &\qquad \left. + \int_{-\infty}^{\infty} e^{ik(x-L)-\omega(k)t}\tilde{g}(t,k)\,\mathrm{d}k\right\}. \end{aligned}$$

We observe that the function $e^{ikx-\omega(k)t}\tilde{f}(t,k)$ is analytic and bounded for $k\in D_c^+$, whilst the function $e^{ik(x-L)-\omega(k)t}\tilde{g}(t,k)$ is analytic and bounded for $k\in D_c^-$, where

$$D_c^\pm = \mathbb{C}^\pm \backslash D^\pm\,.$$

Therefore, an application of Jordan's Lemma (Lemma 1.3.3) implies that

$$\int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\tilde{f}(t,k)\,\mathrm{d}k = \int_{\partial D^+} e^{ikx-\omega(k)t}\tilde{f}(t,k)\,\mathrm{d}k$$

and

$$\int_{-\infty}^{\infty} e^{ik(x-L)-\omega(k)t}\tilde{g}(t,k)\,\mathrm{d}k = -\int_{\partial D^-} e^{ik(x-L)-\omega(k)t}\tilde{g}(t,k)\,\mathrm{d}k\,.$$

Hence the solution can alternatively be written, in terms of the complex contours $\partial D^{\pm}$, and given by (2.25).

Step iii.) This step is the most difficult one of the analysis. For the linear evolution PDEs we are considering, it can be carried out explicitly using only algebraic methods, however in general (e.g. for elliptic PDEs) this step requires analytical tools and does not always yield an explicit expression for the unknown boundary data.

The idea of this step is to exploit the properties of invariance of the global relation. The functions $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$, $j = 0,1,\ldots,n-1$ are functions of k only through $\omega(k)$, hence the transformations $\lambda(k)$ that leave $\omega(k)$ invariant, where $\omega(k)$ is given by (2.13), determined by the equation

$$\lambda(k) : \mathbb{C} \to \mathbb{C} \text{ such that } \omega(k) = \omega(\lambda)\,,$$

also leave $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$ invariant:

$$\omega(k) = \omega(\lambda(k)) \quad \Rightarrow \quad \tilde{f}_j(k) = \tilde{f}_j(\lambda(k))\,, \quad \tilde{g}_j(k) = \tilde{g}_j(\lambda(k))\,.$$

The equation $\omega(k) - \omega(\lambda) = 0$ is a polynomial equation in $\lambda(k)$. The roots of this polynomial $k^n = \lambda^n$ are distinct and given by

$$\lambda_0(k) = k\,, \quad \lambda_1(k) = \zeta k\,, \quad \lambda_2(k) = \zeta^2 k\,, \quad \ldots, \quad \lambda_{n-1}(k) = \zeta^{n-1}k\,,$$

where $\zeta = e^{\frac{2\pi i}{n}}$. Evaluating the global relation at the n roots $\lambda_l(k)$, $l = 0,1,\ldots,n-1$, yields a system of n equations involving the $2n$ spectral functions $\tilde{f}_j(t,k)$ and

$\tilde{g}_j(t,k)$, $j = 0, 1, \ldots, n-1$:

$$\begin{aligned}
\sum_{j=0}^{n-1} c_j(\lambda_0(k))\left(\tilde{f}_j(t,k) - e^{-i\lambda_0(k)L}\tilde{g}_j(t,k)\right) &= \hat{q}_0(\lambda_0(k)) - e^{\omega(k)t}\hat{q}(t,\lambda_0(k))\,,\\
\sum_{j=0}^{n-1} c_j(\lambda_1(k))\left(\tilde{f}_j(t,k) - e^{-i\lambda_1(k)L}\tilde{g}_j(t,k)\right) &= \hat{q}_0(\lambda_1(k)) - e^{\omega(k)t}\hat{q}(t,\lambda_1(k))\,,\\
\vdots \qquad\qquad & \qquad \vdots \qquad\qquad \vdots\\
\sum_{j=0}^{n-1} c_j(\lambda_{n-1}(k))\left(\tilde{f}_j(t,k) - e^{-i\lambda_{n-1}(k)L}\tilde{g}_j(t,k)\right) &= \hat{q}_0(\lambda_{n-1}(k)) - e^{\omega(k)t}\hat{q}(t,\lambda_{n-1}(k))\,.
\end{aligned} \tag{2.30}$$

If n boundary conditions are prescribed, hence n of these $2n$ functions are known, then the remaining n unknown functions can be obtained by solving the system of n equations. However, this is not possible if *any* n boundary conditions are prescribed and this is the origin of Theorem 2.1.1. Recall that, in order to determine a well-posed problem, N of the boundary conditions must be prescribed at $x = 0$ and $n - N$ at $x = L$ where N is determined by (2.2).

This simple observation is the basis of the analysis. Rather than doing this in general, we now look at several examples - to illustrate all steps we start from the example of $q_t(x,t) + q_{xxx}(x,t) = 0$ on $[0,\infty)$. In this case, one boundary value at $x = 0$ is required, hence one of the $f_j(t)$'s is imposed - we derive these results in the next section.

2.2 The Spectral Representation of a Third Order Linear Evolution Equation on the Half-Line

In this section, we consider as an illustrative example, a boundary value problem posed on the half-line $[0,\infty)$. Namely, we analyse the third order linear evolution initial boundary value problem

$$q_t(x,t) + q_{xxx}(x,t) = 0\,, \quad 0 < t < T\,, \quad 0 < x < \infty\,. \tag{2.31}$$

with initial condition $q(x,0) = q_0(x)$, which decays as $x \to \infty$ (for simplicity we take $q_0(x) \in S(0,\infty)$). In order to have a well-posed initial boundary value problem we must prescribe one boundary condition at $x = 0$ [1], and the boundary condition that we shall impose is $q(0,t) = f_0(t)$ for some smooth function $f_0(t)$.

As we mentioned already, this example is interesting because odd order linear evolution PDEs on the half line cannot be solved by an x-transform and separation of variables.

We now follow, for this example, the steps outlined in Section 2.1.

Step i.) The Lax pair for equation (2.31) is given by

$$\mu_x - ik\mu = q\,,$$

$$\mu_t - ik^3\mu = k^2 q - ikq_x - q_{xx}\,.$$

where $\mu = \mu(x,t,k)$, $q = q(x,t)$, and $\omega(k)$ and $X(x,t,k)$ are given by (2.16) and (2.17) respectively. We set

$$\tilde{f}(t,k) = \int_0^t e^{-ik^3 s} X(0,s,k)\,\mathrm{d}s = k^2 \tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\,, \tag{2.32}$$

where

$$\tilde{f}_0(t,k) = \int_0^t e^{-ik^3 s} q(0,s)\,\mathrm{d}s\,, \tag{2.33}$$

$$\tilde{f}_1(t,k) = \int_0^t e^{-ik^3 s} q_x(0,s)\,\mathrm{d}s\,, \tag{2.34}$$

$$\tilde{f}_2(t,k) = \int_0^t e^{-ik^3 s} q_{xx}(0,s)\,\mathrm{d}s\,. \tag{2.35}$$

Step ii.) This involves solving the Lax pair for $\mu(x,t,k)$ and then solving the global relation via a certain Riemann-Hilbert problem. We shall do this in the next chapter but here we use a simple constructive algorithm to arrive at the same result. This simpler approach, based on contour deformation, was derived after the Riemann-Hilbert approach indicated what the integral representation should look like.

We consider the x-Fourier transform of $q(x,t)$, for $x \in [0,\infty)$, which we denote by $\hat{q}(t,k)$:

$$\hat{q}(t,k) = \int_0^\infty e^{-ikx} q(x,t)\,\mathrm{d}x\,, \quad \operatorname{Im} k \leqslant 0\,.$$

[1] We remark that for the PDE $q_t(x,t) - q_{xxx}(x,t) = 0$, two boundary conditions are required at $x = 0$ for well-posedness.

This is used to compute the time evolution of $\hat{q}(t,k)$:

$$\begin{aligned}\left(e^{-ik^3t}\hat{q}(t,k)\right)_t &= \left(\int_0^\infty e^{-ikx-ik^3t}q(x,t)\,\mathrm{d}x\right)_t\\ &= \int_0^\infty \left(e^{-ikx-ik^3t}X(x,t,k)\right)_x \mathrm{d}x\\ &= -e^{-ik^3t}X(0,t,k)\,.\end{aligned}$$

Therefore the *global relation* is given by

$$e^{-ik^3t}\hat{q}(t,k) = \hat{q}_0(k) - \tilde{f}(t,k)\,, \quad \operatorname{Im} k \leqslant 0\,. \tag{2.36}$$

We note that whilst $\tilde{f}(t,k)$ is defined $\forall k$, $\hat{q}(t,k)$ and $\hat{q}_0(k)$ are defined only for $\operatorname{Im} k \leqslant 0$ and hence expression (2.36) is defined only for $\operatorname{Im} k \leqslant 0$. Substituting for $\tilde{f}(t,k)$ into (2.36) yields the expression

$$k^2\tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k) = \hat{q}_0(k) - e^{-ik^3t}\hat{q}(t,k)\,, \tag{2.37}$$

whereby it is evident that the *global relation* relates all the boundary values of the solution. We now take the inverse Fourier transform. This yields

$$q(x,t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{-\infty}^{\infty} e^{ikx+ik^3t}\tilde{f}(t,k)\,\mathrm{d}k\right\}. \tag{2.38}$$

So far we have only used Fourier transforms, hence (2.38) contains the functions $\tilde{f}_1(t,k)$ and $\tilde{f}_2(t,k)$ which are unknown. In order to achieve an integral representation of the solution, in terms only of the known initial and boundary data, it is necessary to deform the contour of the second integral to the boundary ∂D^+ of the region D^+ within which the function $\tilde{f}(t,k)$ is analytic and bounded $\forall k$.

According to (2.22), since the dispersion relation is defined by $\omega(k) = -ik^3$, the region $D = D^+ \cup D_1^- \cup D_2^-$ is comprised of the three regions,

$$D^+ = \left\{k \in \mathbb{C} : \tfrac{\pi}{3} \leqslant \arg(k) \leqslant \tfrac{2\pi}{3}\right\}$$

$$\left.\begin{aligned} D_1^- &= \left\{k \in \mathbb{C} : \pi \leqslant \arg(k) \leqslant \tfrac{4\pi}{3}\right\}\\ D_2^- &= \left\{k \in \mathbb{C} : \tfrac{5\pi}{3} \leqslant \arg(k) \leqslant 2\pi\right\}\end{aligned}\right\} \quad D^- = D_1^- \cup D_2^-\,,$$

given in Figure 2.1.

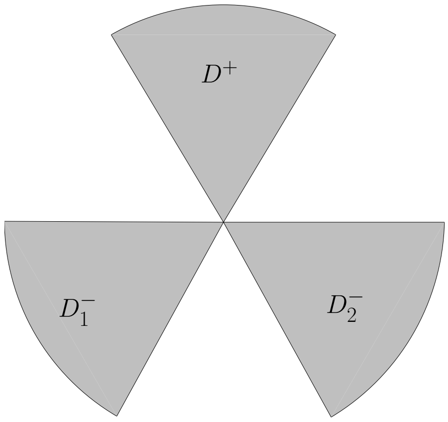


Figure 2.1: The regions $D^+ = \left\{k \in \mathbb{C} : \frac{\pi}{3} \leqslant \arg(k) \leqslant \frac{2\pi}{3}\right\}$, $D_1^- = \left\{k \in \mathbb{C} : \pi \leqslant \arg(k) \leqslant \frac{4\pi}{3}\right\}$ and $D_2^- = \left\{k \in \mathbb{C} : \frac{5\pi}{3} \leqslant \arg(k) \leqslant 2\pi\right\}$ for the third order problem $q_t(x,t) + q_{xxx}(x,t) = 0$.

In order to use the invariance properties of the global relation to characterise the unknown boundary values, we need to deform the contours of integration. This can be done by using Cauchy's Theorem (Theorem 1.3.1) and yields

$$\int_{-\infty}^{\infty} e^{ikx+ik^3t}\tilde{f}(t,k)\,\mathrm{d}k = \int_{\partial D^+} e^{ikx+ik^3t}\tilde{f}(t,k)\,\mathrm{d}k\,.$$

Therefore, substituting expression (2.32) for $\tilde{f}(t,k)$, we obtain

$$\begin{aligned} q(x,t) \;=\; & \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k \right. \\ & \left. - \int_{\partial D^+} e^{ikx+ik^3t}\left(k^2\tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right)\,\mathrm{d}k \right\}, \end{aligned}$$

which is the general form of the integral representation of the solution.

Step iii.) The final step is the determination of the unknown boundary values in terms of the given initial and boundary data, and is the first time where we use the prescribed boundary condition. Since $q(0,t) = f_0(t)$, the integral representation of the solution is given by

$$\begin{aligned} q(x,t) \;=\; & \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx+ik^3t}k^2\tilde{f}_0(t,k)\,\mathrm{d}k \right. \\ & \left. + \int_{\partial D^+} e^{ikx+ik^3t}\left(ik\tilde{f}_1(t,k) + \tilde{f}_2(t,k)\right)\,\mathrm{d}k \right\}, \end{aligned} \tag{2.39}$$

where $\tilde{f}_j(t,k)$, $j = 0,1,2$ are given by (2.33), (2.34) and (2.35) respectively. The second of the integrals around ∂D^+ is unknown, and it indicates that $\tilde{f}(t,k)$ must be computed for $k \in \partial D^+$.

A crucial aspect of the new approach, is the determination of the unknown boundary values. Hence we need to evaluate $\tilde{f}_1(t,k)$ and $\tilde{f}_2(t,k)$ for $k \in \partial D^+$. To do this we now exploit the invariance properties of these functions. The functions $\tilde{f}_j(t,k)$ are entire functions of k for $j = 0,1,2$ and depend on k only through $\omega(k)$. Hence these functions are *invariant* under any transformation of the complex k-plane that leaves $\omega(k)$ invariant. These transformations are determined by the equation

$$\lambda(k) : \mathbb{C} \to \mathbb{C} \ \text{ such that } \ \omega(k) = \omega(\lambda)\,.$$

The three distinct roots are denoted by

$$\lambda_0(k) = k\,, \quad \lambda_1(k) = \zeta k\,, \quad \lambda_2(k) = \zeta^2 k\,,$$

where $\zeta = e^{\frac{2\pi i}{3}}$. The function $\tilde{f}(t,k)$ is analytic and bounded in D^+ and the global relation is valid for $\operatorname{Im} k \leqslant 0$. Therefore, to use this relation to compute $\tilde{f}(t,k)$ in D^+, we must transform the global relation from the lower half complex k-plane to the domain $D^+ = \{k \in \mathbb{C}^+ : \operatorname{Re}\omega(k) \leqslant 0\}$. To do this we evaluate (2.37) at ζk and $\zeta^2 k$ to obtain the following expressions which are valid for $k \in D^+$:

$$\begin{aligned} -i\zeta k \tilde{f}_1(t,k) - \tilde{f}_2(t,k) &= N(t,\zeta k) - e^{-ik^3 t}\hat{q}(t,\zeta k)\,, \\ -i\zeta^2 k \tilde{f}_1(t,k) - \tilde{f}_2(t,k) &= N(t,\zeta^2 k) - e^{-ik^3 t}\hat{q}(t,\zeta^2 k)\,, \end{aligned}$$

where

$$N(t,k) = \hat{q}_0(k) - k^2 \tilde{f}_0(t,k)\,,$$

and these two equations can be written in matrix form as

$$\begin{pmatrix} \zeta & 1 \\ \zeta^2 & 1 \end{pmatrix} \begin{pmatrix} -ik\tilde{f}_1(t,k) \\ -\tilde{f}_2(t,k) \end{pmatrix} = \begin{pmatrix} N(t,\zeta k) \\ N(t,\zeta^2 k) \end{pmatrix} - \begin{pmatrix} e^{-ik^3 t}\hat{q}(t,\zeta k) \\ e^{-ik^3 t}\hat{q}(t,\zeta^2 k) \end{pmatrix}.$$

We solve for the two unknowns using Cramer's Rule, to give

$$\begin{aligned} -ik\tilde{f}_1(t,k) &= \frac{1}{\Delta(k)}\Big(N(t,\zeta k) - N(t,\zeta^2 k) - e^{-ik^3 t}\left(\hat{q}(t,\zeta k) - \hat{q}(t,\zeta^2 k)\right)\Big)\,, \\ -\tilde{f}_2(t,k) &= \frac{1}{\Delta(k)}\Big(-\zeta^2 N(t,\zeta k) + \zeta N(t,\zeta^2 k) + e^{-ik^3 t}\left(\zeta^2 \hat{q}(t,\zeta k) - \zeta \hat{q}(t,\zeta^2 k)\right)\Big)\,, \end{aligned}$$

where $\Delta(k) = \zeta - \zeta^2$ is the determinant of the system. Hence

$$ik\tilde{f}_1(t,k) + \tilde{f}_2(t,k) = \zeta N(t,\zeta k) + \zeta^2 N(t,\zeta^2 k) + \text{unknown terms}\,. \tag{2.40}$$

Proposition 2.2.1. *The unknown functions $\hat{q}(t,\zeta k)$ and $\hat{q}(t,\zeta^2 k)$, in expression (2.40), when multiplied by the factor e^{ikx+ik^3t}, are analytic and bounded as $k \to \infty$ in D^+ and do not contribute to the integral representation of the solution, given by (2.39).*

Proof. Let $k \in D^+$. Then

- e^{ikL}, $e^{-i\zeta kL}$ and $e^{-i\zeta^2 kL}$ are bounded,
- e^{-ikL}, $e^{i\zeta kL}$ and $e^{i\zeta^2 kL}$ are unbounded.

The contribution from the unknown terms, to the integral representation of the solution, is given by

$$\int_{\partial D^+} e^{ikx}\left(\zeta\hat{q}(t,\zeta k)+\zeta^2\hat{q}(t,\zeta^2 k)\right)\mathrm{d}k = \int_{\partial D^+} e^{ikx}\left(\int_0^L \zeta e^{-i\zeta kx}q(x,t)\,\mathrm{d}x\right)\mathrm{d}k + \int_{\partial D^+} e^{ikx}\left(\int_0^L \zeta^2 e^{-i\zeta^2 kx}q(x,t)\,\mathrm{d}x\right)\mathrm{d}k\,.$$

All of the terms in this expression are bounded for $k \in D^+$, and it follows, by Jordan's Lemma (Lemma 1.3.3) that this contribution vanishes. □

Therefore the integral representation of the solution is given by

$$q(x,t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx+ik^3t}k^2\tilde{f}_0(t,k)\,\mathrm{d}k + \int_{\partial D^+} e^{ikx+ik^3t}\left(\zeta N(t,\zeta k)+\zeta^2 N(t,\zeta^2 k)\right)\mathrm{d}k\right\},$$

which is given explicitly in terms of the initial and boundary data as

$$q(x,t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx+ik^3t}3k^2\tilde{f}_0(t,k)\,\mathrm{d}k + \int_{\partial D^+} e^{ikx+ik^3t}\left(\zeta\hat{q}_0(\zeta k)+\zeta^2\hat{q}_0(\zeta^2 k)\right)\mathrm{d}k\right\}. \tag{2.41}$$

Remark 2.2.2. The domain D has always as many connected components in $\mathbb{C}^-$ as unknown boundary values, and this is the essence of the theorem by Fokas and Sung [23].

Remark 2.2.3. In even order cases, an integral representation on the real line can be derived by a transform in x. To stress the difference between this and the complex integral representation derived here, we consider the example of the heat equation posed on the half-line (2.3), for which the use of the sine transform yields the solution representation given by (2.4). The steps outlined above, can be used to derive the integral representation of the solution, and it is straightforward to derive the solution, given by

$$q(x,t) = \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx-k^2t}\left(\hat{q}_0(-k) + 2ik\tilde{f}_0(t,k)\right)\,\mathrm{d}k \right\},$$

where the domain $D^+ = \left\{k \in \mathbb{C} : \frac{\pi}{4} \leqslant \arg(k) \leqslant \frac{3\pi}{4}\right\}$. To show that this is equivalent to the classical solution, given by (2.4), the contours of integration must be deformed to the real line.

Via Cauchy's Theorem (Theorem 1.3.1)

$$\int_{\partial D^+} e^{ikx-k^2t}\left(\hat{q}_0(-k) + 2ik\tilde{f}_0(t,k)\right)\,\mathrm{d}k = \int_{-\infty}^{\infty} e^{ikx-k^2t}\left(\hat{q}_0(-k) + 2ik\tilde{f}_0(t,k)\right)\,\mathrm{d}k\,,$$

and therefore

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\left(\hat{q}_0(k) - \hat{q}_0(-k) - 2ik\tilde{f}_0(t,k)\right)\,\mathrm{d}k \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{ikx-k^2t}\left(-i\hat{q}_0^{(\sin)}(k) - ik\tilde{f}_0(t,k)\right)\,\mathrm{d}k\,, \end{aligned}$$

where $\hat{q}_0^{(\sin)}(k)$ is given by (2.5). This can be written as

$$\begin{aligned} q(x,t) &= \frac{2}{\pi}\int_0^{\infty} e^{-k^2t}\left(\hat{q}_0^{(\sin)}(k) + k\tilde{f}_0(t,k)\right)\sin(kx)\,\mathrm{d}k \\ &= \frac{2}{\pi}\int_0^{\infty} e^{-k^2t}\left(\hat{q}_0^{(\sin)}(k) + \int_0^t e^{k^2s}kf_0(s)\,\mathrm{d}s\right)\sin(kx)\,\mathrm{d}k\,, \end{aligned}$$

which concurs with (2.4), and the proof is complete.

This representation is *uniformly convergent* as $x \to 0$, so that to prove that it satisfies the given boundary condition at $x = 0$ is sufficient to evaluate it at the boundary point. This is to be contrasted with the sine representation which is not uniformly convergent, so that the proof that this representation satisfies the given boundary condition cannot be obtained by simply setting $x = 0$ inside the integral.

Remark 2.2.4. In the special case that $q_0(x) = 0$, the integral representation of the solution can be realised on the real line. Indeed, the imposition of the smooth initial condition $q(x,0) = 0$, simplifies the integral representation of the solution given by (2.41) to the following:

$$\begin{aligned} q(x,t) &= -\frac{1}{2\pi}\int_{\partial D^+} e^{ikx+ik^3t} 3k^2 \tilde{f}_0(t,k)\,\mathrm{d}k \\ &= -\frac{1}{2\pi}\int_{\partial D^+} e^{ikx+ik^3t} 3k^2 \left(\int_0^t e^{-ik^3s} q(0,s)\,\mathrm{d}s\right)\,\mathrm{d}k\,. \end{aligned}$$

Since e^{ikx} is analytic and bounded in the upper half complex k-plane, and $e^{ik^3(t-s)}$ is bounded and analytic in D_c, the contour of integration can be deformed to any contour within D_c. In particular, one can write the representation as a real integral of the form

$$q(x,t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+ik^3t} 3k^2 \tilde{f}_0(t,k)\,\mathrm{d}k\,.$$

It is not clear how this real representation can be obtained by using the usual real Fourier transform.

This fact has also important consequences for the numerical evaluation of the solution $q(x,t)$, using the integral formula. Indeed, since the contour of integration can be chosen anywhere inside D_c, it can in particular be selected in such a way that the t-exponential is *decreasing* as $k \to \infty$, rather than oscillatory as on ∂D. This implies that the numerical evaluation of this integral by a straightforward quadrature method is both fast and accurate. Preliminary results of Fornberg and Flyer [24] confirm that this method of numerical evaluation of the solution is faster and more accurate than a numerical computation of the solution by time stepping.

2.3 The Riemann-Hilbert Problem

In this section, we present the original derivation of the complex integral representation of the solution, by associating to the PDE a Riemann-Hilbert problem. This is the approach used to derive the inverse spectral transform method and is the methodology that generalises to the integrable nonlinear case. We also stress that the deformation technique of the previous section was only described after the Riemann-Hilbert formulation indicated that the complex representation derived by it, can be realised effectively in terms only of the given data of the problem.

To illustrate this methodology, we now consider equation (2.1), and show that the formulation of a Riemann-Hilbert problem in the complex k-plane yields the complex integral representation of the solution $q(x,t)$ in terms of the spectral functions $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$ given by (2.23) and (2.24) respectively.

Example 1: We demonstrate the derivation of the integral representation of the solution by example, and begin by considering the second order linear evolution PDE, for $q = q(x,t)$, given by

$$iq_t(x,t) + q_{xx}(x,t) = 0\,, \quad 0 < t < T\,, \quad x \in [0,L]\,,$$

$$q(x,0) = q_0(x)\,, \quad x \in [0,L]\,.$$

The Lax pair is given by

$$\begin{aligned} \mu_x - ik\mu &= q(x,t)\,, \\ \mu_t + ik^2\mu &= -kq(x,t) + iq_x(x,t)\,, \end{aligned}$$

where $\mu = \mu(x,t,k)$, hence

$$\begin{aligned} \omega(k) &= ik^2\,, \\ X(x,t,k) &= -kq(x,t) + iq_x(x,t)\,. \end{aligned}$$

We define the domains $D^{\pm}$ and $D_c^{\pm}$ by

$$D = \{k \in \mathbb{C} : \operatorname{Re}\omega(k) \leqslant 0\}\,, \quad D^{\pm} = D \cap \mathbb{C}^{\pm}\,,$$

$$D_c = \{k \in \mathbb{C} : \operatorname{Re}\omega(k) > 0\}\,, \quad D_c^{\pm} = D_c \cap \mathbb{C}^{\pm}\,.$$

Thus

$$D^+ = \left\{k \in \mathbb{C}^+ : 0 \leqslant \arg(k) \leqslant \tfrac{\pi}{2}\right\}\,, \quad D_c^+ = \mathbb{C}^+ \backslash D^+\,,$$

$$D^- = \left\{k \in \mathbb{C}^- : \pi \leqslant \arg(k) \leqslant \tfrac{3\pi}{2}\right\}\,, \quad D_c^- = \mathbb{C}^- \backslash D^-\,.$$

We show that a solution of (2.18), bounded in $k \in \mathbb{C}$ has the form

$$\mu(x,t,k) = \begin{cases} \mu_1(x,t,k), & k \in D_c^+\,, \\ \mu_2(x,t,k), & k \in D_c^-\,, \\ \mu_3(x,t,k), & k \in D^-\,, \\ \mu_4(x,t,k), & k \in D^+\,, \end{cases} \tag{2.42}$$

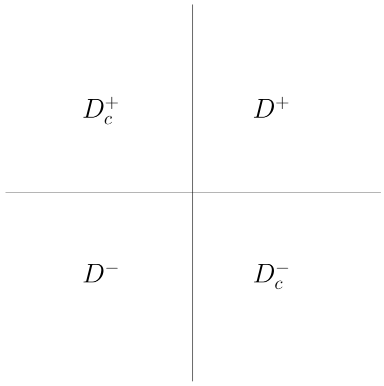


Figure 2.2: The regions $D^{\pm}$ and $D_c^{\pm}$ in the complex k-plane for the second order linear evolution PDE given by $iq_t(x,t) + q_{xx}(x,t) = 0$.

where $\mu_j(x,t,k)$ are defined below.

Let $z = (x,t)$. The domain $0 \leqslant x \leqslant L$, $0 \leqslant t \leqslant T$, represented by Figure 2.3, is a polygon in the z-plane with corners $z_1 = (0,0)$, $z_2 = (L,0)$, $z_3 = (L,T)$ and $z_4 = (0,T)$.

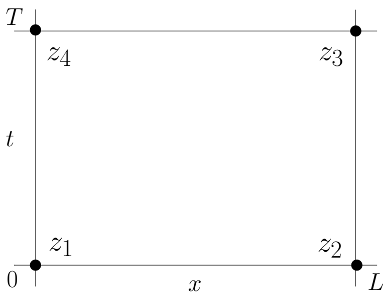


Figure 2.3: The domain $0 \leqslant x \leqslant L$, $0 \leqslant t \leqslant T$ with corners $z_1 = (0,0)$, $z_2 = (L,0)$, $z_3 = (L,T)$ and $z_4 = (0,T)$.

Let $z_\dagger$ be an arbitrary point in the polygon and let $\int_{z_\dagger}^{z}$ denote the line integral from $z_\dagger$ to $z = (x,t)$. The function

$$\mu_\dagger(x,t,k) = \int_{z_\dagger}^{z} e^{ik(x-x')-ik^2(t-t')} \Big\{ q(x',t')\,\mathrm{d}x' + X(x',t',k)\,\mathrm{d}t' \Big\} \tag{2.43}$$

is a particular solution of (2.18). Furthermore, the definition of $\mu_\dagger(x,t,k)$ is independent of the path from $z_\dagger$ to z.

We now choose the point $z_\dagger$ in such a way that this function is holomorphic in k. It is shown in [15] that the $z_\dagger$'s must be chosen to be the corners of the polygon. We therefore define $\mu_j(x,t,k)$ by (2.43) where $z_\dagger = z_j$. Hence

$$\begin{aligned}
\mu_1(x,t,k) &= \int_0^x e^{ik(x-x')}q(x',t)\,\mathrm{d}x' + e^{ikx}\int_0^t e^{-ik^2(t-t')}X(0,t',k)\,\mathrm{d}t', && k \in D_c^+,\\
\mu_2(x,t,k) &= \int_L^x e^{ik(x-x')}q(x',t)\,\mathrm{d}x' + e^{ik(x-L)}\int_0^t e^{-ik^2(t-t')}X(L,t',k)\,\mathrm{d}t', && k \in D_c^-,\\
\mu_3(x,t,k) &= \int_L^x e^{ik(x-x')}q(x',t)\,\mathrm{d}x' + e^{ik(x-L)}\int_T^t e^{-ik^2(t-t')}X(L,t',k)\,\mathrm{d}t', && k \in D^-,\\
\mu_4(x,t,k) &= \int_0^x e^{ik(x-x')}q(x',t)\,\mathrm{d}x' + e^{ikx}\int_T^t e^{-ik^2(t-t')}X(0,t',k)\,\mathrm{d}t', && k \in D^+,
\end{aligned} \tag{2.44}$$

where for example $\int_{z_1}^{z}$ is split into two integrals (one along the t-axis and one parallel to the x-axis) to give the functions $\mu_j(x,t,k)$ which are entire functions of k. The general theory implies that the functions are bounded as $k \to \infty$ provided that as $k \to \infty$, $\mu_j(x,t,k)$ are defined according to (2.42). The contours associated with $\mu_j(x,t,k)$ are given in Figure 2.4. Hence the functions $\mu_1(x,t,k), \mu_2(x,t,k), \mu_3(x,t,k)$ and $\mu_4(x,t,k)$ are bounded and analytic in the domains D_c^+, D_c^-, D^- and D^+ respectively.

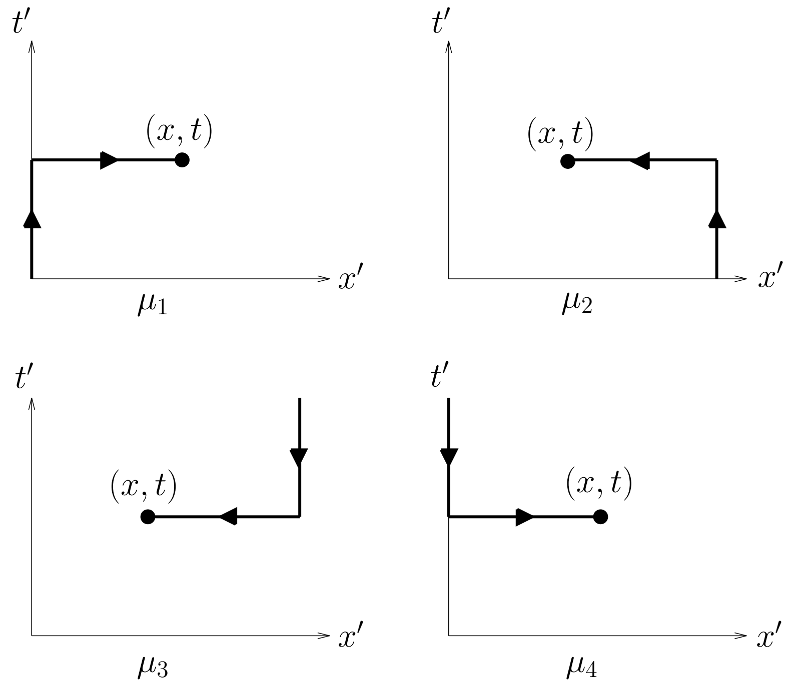


Figure 2.4: The contours associated with μ_1, μ_2, μ_3 and μ_4.

Using the representation (2.18), the jump of $\mu(x,t,k)$ can be computed in terms of line integrals along the boundary of the polygon, for example $\mu_4(x,t,k) - \mu_1(x,t,k) = \int_{z_4}^{z_1}$.

Equation (2.43) implies that

$$\begin{aligned}\mu_i(x,t,k)-\mu_j(x,t,k) &= e^{ikx-ik^2t}\rho_{ij}(k)\,, \quad i\neq j\,,\\ \rho_{ij}(k) &= \int_{z_i}^{z_j} e^{-ikx+ik^2t}\left(q(x,t)\,\mathrm{d}x+X(x,t,k)\,\mathrm{d}t\right)\,.\end{aligned}$$

The integrals, computed along paths parallel to the x and t axes, yield the following:

$$\begin{aligned}\mu_1(x,t,k)-\mu_3(x,t,k) &= e^{ikx-ik^2t}\left(\hat{q}_0(k)+e^{-ikL}\tilde{g}(T,k)\right)\,, && k\in D_c^+\cap D^-\,,\\ \mu_1(x,t,k)-\mu_4(x,t,k) &= e^{ikx-ik^2t}\tilde{f}(T,k)\,, && k\in D_c^+\cap D^+\,,\\ \mu_2(x,t,k)-\mu_3(x,t,k) &= e^{ikx-ik^2t}e^{-ikL}\tilde{g}(T,k)\,, && k\in D_c^-\cap D^-\,,\\ \mu_2(x,t,k)-\mu_4(x,t,k) &= e^{ikx-ik^2t}\left(-\hat{q}_0(k)+\tilde{f}(T,k)\right)\,, && k\in D_c^-\cap D^+\,,\end{aligned} \tag{2.45}$$

where $\hat{q}_0(k)$, $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$ are given by (2.26), (2.23) and (2.24) respectively. Via expressions (2.42) and (2.44) and integration by parts, the expression for $\mu(x,t,k)$ yields the estimate

$$\mu(x,t,k)=O\left(\frac{1}{k}\right)\,, \quad k\to\infty\,. \tag{2.46}$$

Also, since each of the $\mu_j(x,t,k)$ is holomorphic, $\mu(x,t,k)$ is a sectionally holomorphic function of k. Equations (2.45) represent the jumps of $\mu(x,t,k)$ along the curve separating the domains of analyticity of the known $\mu_j(x,t,k)$. Therefore equations (2.45) along with the estimate (2.46) determine a well defined Riemann-Hilbert problem for $\mu(x,t,k)$ whose unique solution is given by

$$\mu(x,t,k)=\frac{1}{2\pi i}\sum_{a,b}\int_{C_{a,b}}\frac{(\mu_a-\mu_b)(k')}{k'-k}\,\mathrm{d}k'\,,$$

where $C_{a,b}$ is the straight line contour at the intersection of the two regions within which the solutions $\mu_a(x,t,k)$ and $\mu_b(x,t,k)$ lie, given in Figure 2.5.

Substitution and simplification yields the following unique solution

$$\begin{aligned}\mu(x,t,k) = \frac{1}{2\pi i}\Bigg\{&\int_{-\infty}^{\infty} e^{ik'x-ik'^2t}\left(\frac{\hat{q}_0(k')}{k'-k}\right)\mathrm{d}k' - \int_{\partial D^+} e^{ik'x-ik'^2t}\left(\frac{\tilde{f}(T,k')}{k'-k}\right)\mathrm{d}k'\\ &-\int_{\partial D^-} e^{ik'(x-L)-ik'^2t}\left(\frac{\tilde{g}(T,k')}{k'-k}\right)\mathrm{d}k'\Bigg\}\,.\end{aligned}$$

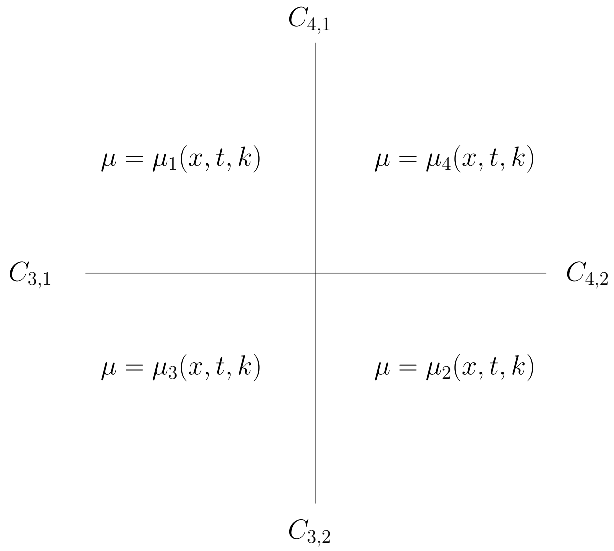


Figure 2.5: The straight line contours $C_{a,b}$ at the intersection of the regions within which the solutions $\mu_a(x,t,k)$ and $\mu_b(x,t,k)$ lie.

The general solution $q(x,t)$ is then found from substituting $\mu(x,t,k)$ into the equation $\mu_x(x,t,k) - ik\mu(x,t,k) = q(x,t)$ to give

$$\begin{aligned} q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ & \int_{-\infty}^{\infty} e^{ik'x-ik'^2t}\hat{q}_0(k')\,\mathrm{d}k' - \int_{\partial D^+} e^{ik'x-ik'^2t}\tilde{f}(T,k')\,\mathrm{d}k' \\ & - \int_{\partial D^-} e^{ik'(x-L)-ik'^2t}\tilde{g}(T,k')\,\mathrm{d}k' \Bigg\}, \end{aligned}$$

where the functions $\tilde{f}(T,k')$ and $\tilde{g}(T,k')$ are expressions involving the transforms of all the boundary values.

Example 2: As another example we consider the third order problem

$$q_t(x,t) + q_{xxx}(x,t) = 0\,, \quad 0 < t < T\,, \quad x \in [0,L]\,, \tag{2.47a}$$

$$q(x,0) = q_0(x)\,, \quad x \in [0,L]\,. \tag{2.47b}$$

The Lax pair for the third order problem (2.47) is given by

$$\begin{aligned} \mu_x - ik\mu \;&=\; q\,, \\ \mu_t - ik^3\mu \;&=\; k^2q - ikq_x - q_{xx}\,, \end{aligned}$$

where $q = q(x,t)$, $\mu = \mu(x,t,k)$ and

$$\begin{aligned} \omega(k) &= -ik^3 \,, \\ X(x,t,k) &= k^2 q - ikq_x - q_{xx} \,. \end{aligned}$$

The domains D^+ and $D^-_{1,2}$ in the complex k-plane are given by

$$D^+ = \left\{k \in \mathbb{C}^+ : \tfrac{\pi}{3} \leqslant \arg(k) \leqslant \tfrac{2\pi}{3}\right\} \,, \qquad D^+_{c,1} \cap D^+_{c,2} = \mathbb{C}^+ \backslash D^+ \,,$$

$$\left.\begin{aligned} D^-_1 &= \left\{k \in \mathbb{C}^- : \pi \leqslant \arg(k) \leqslant \tfrac{4\pi}{3}\right\} \\ D^-_2 &= \left\{k \in \mathbb{C}^- : \tfrac{5\pi}{3} \leqslant \arg(k) \leqslant 2\pi\right\} \end{aligned}\right\} \qquad D^-_c = \mathbb{C}^- \backslash \left(D^-_1 \cap D^-_2\right) \,,$$

and are given in Figure 2.6.

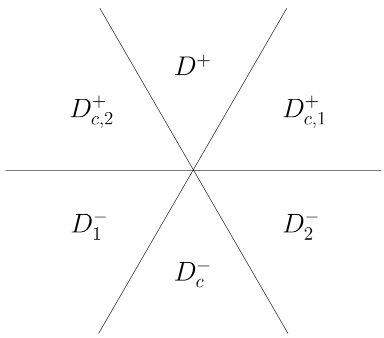


Figure 2.6: The regions $D^+, D^-_{1,2}, D^-_c$, $D^+_{c,1}$ and $D^+_{c,2}$ in the complex k-plane for the third order linear evolution PDE given by $q_t(x,t) + q_{xxx}(x,t) = 0$.

The solution, bounded for all $k \in \mathbb{C}$ has the form

$$\mu(x,t,k) = \begin{cases} \mu_1(x,t,k) \,, & k \in D^+_{c,1} \cup D^+_{c,2} \,, \\ \mu_2(x,t,k) \,, & k \in D^-_c \,, \\ \mu_3(x,t,k) \,, & k \in D^-_1 \cup D^-_2 \,, \\ \mu_4(x,t,k) \,, & k \in D^+ \,. \end{cases}$$

The formulation of the polygon with corners z_1, z_2, z_3 and z_4, and the function (2.43) that was constructed for the second order problem, is independent of the order of the problem. The difference that arises, due to the order of the problem under consideration, is given by the number of domains $D^\pm$ in the complex k-plane. For the problem given by (2.47) the functions $\mu_j(x,t,k)$ are therefore given by

$$\begin{aligned}
\mu_1(x,t,k) &= \int_0^x e^{ik(x-x')}q(x',t)\,\mathrm{d}x' + e^{ikx}\int_0^t e^{ik^3(t-t')}X(0,t',k)\,\mathrm{d}t'\,, && k\in D_c^+\,,\\
\mu_2(x,t,k) &= \int_L^x e^{ik(x-x')}q(x',t)\,\mathrm{d}x' + e^{ik(x-L)}\int_0^t e^{ik^3(t-t')}X(L,t',k)\,\mathrm{d}t'\,, && k\in D_c^-\,,\\
\mu_3(x,t,k) &= \int_L^x e^{ik(x-x')}q(x',t)\,\mathrm{d}x' + e^{ik(x-L)}\int_T^t e^{ik^3(t-t')}X(L,t',k)\,\mathrm{d}t'\,, && k\in D^-\,,\\
\mu_4(x,t,k) &= \int_0^x e^{ik(x-x')}q(x',t)\,\mathrm{d}x' + e^{ikx}\int_T^t e^{ik^3(t-t')}X(0,t',k)\,\mathrm{d}t'\,, && k\in D^+\,,
\end{aligned}$$

where $D_c^+ = D_{c,1}^+\cup D_{c,2}^+$ and $D^- = D_1^-\cup D_2^-$. The integrals, computed along paths parallel to the x and t axes are therefore given by (2.45), and the remainder of the formulation of the integral representation of the solution follows identically as for the second order problem. Hence

$$\mu(x,t,k) = \frac{1}{2\pi i}\sum_{a,b}\int_{C_{a,b}}\frac{(\mu_a-\mu_b)(k')}{k'-k}\,\mathrm{d}k'\,,$$

which is given explicitly as

$$\begin{aligned}
\mu(x,t,k) &= \frac{1}{2\pi i}\left\{\int_{-\infty}^{\infty} e^{ik'x+ik'^3t}\left(\frac{\hat q_0(k')}{k'-k}\right)\mathrm{d}k' - \int_{\partial D^+} e^{ik'x+ik'^3t}\left(\frac{\tilde f(T,k')}{k'-k}\right)\mathrm{d}k'\right.\\
&\qquad\left. - \int_{\partial D^-} e^{ik'(x-L)+ik'^3t}\left(\frac{\tilde g(T,k')}{k'-k}\right)\mathrm{d}k'\right\}.
\end{aligned}$$

We conclude that for the linear evolution PDE given by equation (2.1), the explicit integral representation of the solution $q(x,t)$ is based on the formulation of a Riemann-Hilbert problem, and is given by

$$\begin{aligned}
q(x,t) &= \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat q_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx+ik^3t}\tilde f(T,k)\,\mathrm{d}k\right.\\
&\qquad\left. - \int_{\partial D^-} e^{ik(x-L)+ik^3t}\tilde g(T,k)\,\mathrm{d}k\right\}. \qquad (2.48)
\end{aligned}$$

2.4 The Global Relation and its Analysis

We now return to the final stage of the method and derive the global relations for the third order two-point boundary value problem given by

$$q_t(x,t) + q_{xxx}(x,t) = 0\,,\quad t>0\,,\quad x\in[0,L]\,,$$

$$q(x,0) = q_0(x)\,,\quad x\in[0,L]\,,$$

with the boundary conditions

$$q(0,t)=0\,,\quad q(L,t)=0\,,\quad q_x(L,t)=0\,,$$

chosen according to (2.2) to guarantee well-posedness. We remark that we shall consider more general boundary conditions in the next chapter.

The functions $c_0(k)=k^2$, $c_1(k)=-ik$ and $c_2=-1$, and the boundary conditions imply that $\tilde{f}_0(t,k)=0$, $\tilde{g}_0(t,k)=0$ and $\tilde{g}_1(t,k)=0$. Hence the functions $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$ are given by

$$\begin{aligned}\tilde{f}(t,k) &= -ik\tilde{f}_1(t,k)-\tilde{f}_2(t,k)\,,\\ \tilde{g}(t,k) &= -\tilde{g}_2(t,k)\,,\end{aligned}$$

and the global relation is therefore an algebraic expression relating the three unknown spectral functions $\tilde{f}_1(t,k)$, $\tilde{f}_2(t,k)$ and $\tilde{g}_2(t,k)$:

$$-ik\tilde{f}_1(t,k)-\Big(\tilde{f}_2(t,k)-e^{-ikL}\tilde{g}_2(t,k)\Big)=\hat{q}_0(k)-e^{-ik^3t}\hat{q}(t,k)\,. \tag{2.49}$$

We need to express $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$ in terms only of the known functions $\hat{q}_0(\lambda_l(k))$, $l=0,1,2$. The transformations that leave $\omega(k)$ invariant are determined by the roots of the equation $\omega(k)=\omega(\lambda)$. Hence

$$\lambda_0(k)=k\,,\quad \lambda_1(k)=\zeta k\,,\quad \lambda_2(k)=\zeta^2 k\,,\qquad \zeta=e^{\frac{2\pi i}{3}}\,.$$

If we evaluate the global relation (2.49) at $\lambda_1(k)$ and $\lambda_2(k)$ we obtain two additional equations

$$-i\zeta k\tilde{f}_1(t,k)-\Big(\tilde{f}_2(t,k)-e^{-i\zeta kL}\tilde{g}_2(t,k)\Big) = \hat{q}_0(\zeta k)-e^{-ik^3t}\hat{q}(t,\zeta k)\,, \tag{2.50}$$

$$-i\zeta^2 k\tilde{f}_1(t,k)-\Big(\tilde{f}_2(t,k)-e^{-i\zeta^2 kL}\tilde{g}_2(t,k)\Big) = \hat{q}_0(\zeta^2 k)-e^{-ik^3t}\hat{q}(t,\zeta^2 k)\,. \tag{2.51}$$

Hence equations (2.49), (2.50) and (2.51) form a system of three equations in terms of the three unknown functions.

We observe that the system involves the unknown functions $\hat{q}(t,\lambda_l(k))$. However, it is shown in general in [23, 38] that provided the boundary conditions are chosen according to (2.2), the resulting system is a set of n functions with the correct boundedness and analyticity properties for which the contribution of these unknown functions can be determined. If N is chosen according to (2.2), such that when solving for the unknown functions in the set $\{\tilde{f}_j(t, \lambda_l(k))\}$, the expression involving $\hat{q}(t, \lambda_l(k))$ is bounded as $k \to \infty$ when $k \in D^+$, and when solving for the unknown functions in the set $\{\tilde{g}_j(t, \lambda_l(k))\}$, the expression is bounded as $k \to \infty$ when $k \in D^-$, then the contribution of the additional unknown functions $\hat{q}(t, \lambda_l(k))$ vanishes. Hence the system does *not* depend on $\hat{q}(t, \lambda_l(k))$. This significant result implies that the integral representation of the solution is *effective*, expressed only in terms of the given initial and boundary data, and is discussed in further detail in the next section.

2.4.1 The PDE Discrete Spectrum of a Boundary Value Problem

We begin with the following definition:

Definition 2.4.1. *The set of zeros of the determinant function $\Delta(k)$ of the system (2.30) obtained from the global relation is called the PDE discrete spectrum of the boundary value problem.*

Remark 2.4.2. In the cases for which the solution can be represented in the form of an infinite discrete series, we refer to the set of zeros as the *effective discrete spectrum*.

The PDE discrete spectrum of the boundary value problem is uniquely determined by the PDE and by the boundary conditions. We shall show in Chapter 4 that this spectrum corresponds to the discrete spectrum of the differential operator $D = \partial_x^n$. This spectrum is always of the form of a finite sum of exponentials, whose zeros cluster asymptotically in a neighbourhood of specific rays in the complex k-plane, passing through the origin, and whose direction is uniquely determined by the PDE and the given boundary conditions.

We now indicate how the contribution of the unknown functions $\hat{q}(t, \lambda_l(k))$, appearing in (2.49), (2.50) and (2.51), can be determined.

Consider the solution $\left\{\tilde{f}_1(t,k), \tilde{f}_2(t,k), \tilde{g}_2(t,k)\right\}$ of the system given by (2.49), (2.50) and (2.51). Each of these functions is easily seen to be of the general form

$$\frac{1}{\Delta(k)} \left\{H\left(\hat{q}_0(k), \hat{q}_0(\zeta k), \dots, \hat{q}_0(\zeta^{n-1}k)\right) - e^{\omega(k)t} H\left(\hat{q}(t,k), \hat{q}(t,\zeta k), \dots, \hat{q}(t,\zeta^{n-1}k)\right)\right\}, \tag{2.52}$$

where H is a linear combination with coefficients $\lambda_l(k)$ and $e^{-i\lambda_l(k)L}$, $l = 0, 1, \ldots, n-1$, (see Section 2.5). In the sequel we shall use the following:

Proposition 2.4.3. *The integral of the function $H\left(\hat{q}(t,k), \hat{q}(t,\zeta k), \ldots, \hat{q}(t,\zeta^{n-1}k)\right)$ has the following property:*

i.) If $\Delta(k) \neq 0$ for all $k \in D$ then

$$\begin{aligned}
&\int_{\partial D} (\textit{terms involving } \hat{q}(t, \lambda_l(k)))\, \mathrm{d}k \\
&\quad = \int_{\partial D^+} \frac{e^{ikx}}{\Delta(k)} H(\hat{q}(t,k), \hat{q}(t,\zeta k), \ldots, \hat{q}(t,\zeta^{n-1}k))\, \mathrm{d}k \\
&\qquad + \int_{\partial D^-} \frac{e^{ik(x-L)}}{\Delta(k)} H(\hat{q}(t,k), \hat{q}(t,\zeta k), \ldots, \hat{q}(t,\zeta^{n-1}k))\, \mathrm{d}k \\
&\quad = 0\,.
\end{aligned}$$

ii.) If $\Delta(k) = 0$ for $k \in D$ then

$$\begin{aligned}
&\int_{\partial D} (\textit{terms involving } \hat{q}(t, \lambda_l(k)))\, \mathrm{d}k \\
&\quad = \int_{\partial D^+} \frac{e^{ikx}}{\Delta(k)} H(\hat{q}(t,k), \hat{q}(t,\zeta k), \ldots, \hat{q}(t,\zeta^{n-1}k))\, \mathrm{d}k \\
&\qquad + \int_{\partial D^-} \frac{e^{ik(x-L)}}{\Delta(k)} H(\hat{q}(t,k), \hat{q}(t,\zeta k), \ldots, \hat{q}(t,\zeta^{n-1}k))\, \mathrm{d}k \\
&\quad = \sum_{\substack{k_n \in D: \\ \Delta(k_n)=0}} e^{ik_n x - \omega(k_n)t} \left\{ \frac{H\left(\hat{q}_0(k_n), \hat{q}_0(\zeta k_n), \ldots, \hat{q}_0(\zeta^{n-1}k_n)\right)}{\Delta'(k_n)} \right\}.
\end{aligned}$$

Proof. The proof is included for some specific examples that are to follow (see for example Section 3.2.3). For the general proof, see [21]. □

In Section 2.5 we formulate the effective integral representation of the solution, but first we derive the integral representation of the solution using the classical approach and contour deformation.

2.4.2 The Classical Approach and Contour Deformation

We consider now the integral representation of the solution for the third order problem

$$q_t(x,t) + q_{xxx}(x,t) = 0\,, \quad t > 0\,, \quad x \in [0,L]\,,$$

$$q(x,0) = q_0(x)\,, \quad x \in [0,L]\,,$$

achieved from classical Fourier analysis, in terms of *all* boundary values, and show that the solution that results is equivalent to equation (2.48).

The equation is solved by the transformation into Fourier space and the repeated application of integration by parts. This results in the expression

$$\hat{q}_t(t,k) - ik^3\hat{q}(t,k) = -\left(k^2 f_0(t) - ikf_1(t) - f_2(t)\right) + e^{-ikL}\left(k^2 g_0(t) - ikg_1(t) - g_2(t)\right)\,,$$

where $f_j(t) = \partial_x^j q(0,t)$ and $g_j(t) = \partial_x^j q(L,t)$ for $j = 0,1,2$. Solving with respect to $\hat{q}(t,k)$ and imposing the initial condition $q(x,0) = q_0(x)$, gives the solution

$$\begin{aligned} q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ & \int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{-\infty}^{\infty} e^{ikx+ik^3t}\left(k^2\tilde{f}_0 - ik\tilde{f}_1 - \tilde{f}_2\right)(t,k)\,\mathrm{d}k \\ & + \int_{-\infty}^{\infty} e^{ik(x-L)+ik^3t}\left(k^2\tilde{g}_0 - ik\tilde{g}_1 - \tilde{g}_2\right)(t,k)\,\mathrm{d}k \Bigg\}\,, \end{aligned}$$

where $\{\tilde{f}_j(t,k)\}_0^2$ and $\{\tilde{g}_j(t,k)\}_0^2$ denote certain t-transforms of $f_j(t)$ and $g_j(t)$, given explicitly by (2.20) and (2.21) respectively. The solution can therefore be written in the form

$$\begin{aligned} q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ & \int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{-\infty}^{\infty} e^{ikx+ik^3t}\tilde{f}(t,k)\,\mathrm{d}k \\ & + \int_{-\infty}^{\infty} e^{ik(x-L)+ik^3t}\tilde{g}(t,k)\,\mathrm{d}k \Bigg\}\,. \end{aligned} \tag{2.53}$$

We now prove that this representation is equivalent to expression (2.48). We begin by deforming the contours of integration using Cauchy's Theorem (Theorem 1.3.1). Recall the definition of the domain D given by

$$D = \{k \in \mathbb{C} : \operatorname{Re}\omega(k) \leqslant 0\}\,, \quad D^{\pm} = D \cap \mathbb{C}^{\pm}\,. \tag{2.54}$$

The function $e^{ikx+ik^3t}\tilde{f}(t,k)$ is analytic and bounded for $k \in D_c^+$ and the function $e^{ik(x-L)+ik^3t}\tilde{g}(t,k)$ is analytic and bounded for $k \in D_c^-$. Therefore an application of

Cauchy's Theorem (Theorem 1.3.1) implies that

$$\int_{\partial D_c^+} e^{ikx+ik^3t}\tilde{f}(t,k)\,\mathrm{d}k = 0\,, \qquad \int_{\partial D_c^-} e^{ik(x-L)+ik^3t}\tilde{g}(t,k)\,\mathrm{d}k = 0\,,$$

where ∂D_c^+ and ∂D_c^- are the boundaries of D_c^+ and D_c^- respectively, oriented such that the interior of the regions is always on the right. Hence the solution can alternatively be written, in terms of the complex contours of integration $\partial D^{\pm}$, as

$$\begin{aligned} q(x,t) \;&=\; \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx+ik^3t}\tilde{f}(t,k)\,\mathrm{d}k \right. \\ &\qquad \left. - \int_{\partial D^-} e^{ik(x-L)+ik^3t}\tilde{g}(t,k)\,\mathrm{d}k \right\}. \end{aligned} \tag{2.55}$$

Consider now the integral of the function $e^{ikx+ik^3t}\tilde{f}(T,k)$ around ∂D^+. By definition (2.23) we have that

$$\tilde{f}(T,k) = \tilde{f}(t,k) + \int_t^T e^{-ik^3s}X(0,s,k)\,\mathrm{d}s\,, \quad 0 < t < T\,,$$

and when substituted into expression (2.55), we see that the second of the terms here vanishes since the integrand that results is analytic and bounded in D^+. Hence

$$\int_{\partial D^+} e^{ikx+ik^3t}\tilde{f}(t,k)\,\mathrm{d}k \equiv \int_{\partial D^+} e^{ikx+ik^3t}\tilde{f}(T,k)\,\mathrm{d}k\,.$$

A similar argument applies to the integrand around ∂D^-. This concludes the proof that the integral representation of the solution, given by (2.53), is equivalent to (2.48).

2.5 The Effective Integral Representation of the Solution

In this section we show how the integral representation of the solution, given by

$$\begin{aligned} q(x,t) \;&=\; \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx+ik^3t}\tilde{f}(t,k)\,\mathrm{d}k \right. \\ &\qquad \left. - \int_{\partial D^-} e^{ik(x-L)+ik^3t}\tilde{g}(t,k)\,\mathrm{d}k \right\}, \end{aligned} \tag{2.56}$$

which depends on all the boundary values, can be written in terms only of the given initial and boundary data of the problem. This can be achieved by characterising the functions $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$, in terms only of the given data.

To do this we use the global relation, the analyticity properties of the functions $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$ and Cauchy's Theorem (Theorem 1.3.1).

The system of global relations given by (2.49), (2.50) and (2.51) can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & e^{-ikL} \\ \zeta & 1 & e^{-i\zeta kL} \\ \zeta^2 & 1 & e^{-i\zeta^2 kL} \end{pmatrix} \begin{pmatrix} -ik\tilde{f}_1(t,k) \\ -\tilde{f}_2(t,k) \\ \tilde{g}_2(t,k) \end{pmatrix} = \begin{pmatrix} \hat{q}_0(k) \\ \hat{q}_0(\zeta k) \\ \hat{q}_0(\zeta^2 k) \end{pmatrix} - \begin{pmatrix} e^{-ik^3t}\hat{q}(t,k) \\ e^{-ik^3t}\hat{q}(t,\zeta k) \\ e^{-ik^3t}\hat{q}(t,\zeta^2 k) \end{pmatrix}. \tag{2.57}$$

Solving this system, we obtain expressions for the three unknown functions $\tilde{f}_1(t,k)$, $\tilde{f}_2(t,k)$ and $\tilde{g}_2(t,k)$, and correspondingly the functions $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$, given by

$$\begin{aligned} \tilde{f}(t,k) &= -ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k) \\ &= \frac{1}{\tilde{\Delta}(k)}\Big\{\hat{q}_0(k)(\zeta^2 e^{-i\zeta^2 kL} + \zeta e^{-i\zeta kL}) - \hat{q}_0(\zeta k)\zeta e^{-ikL} - \hat{q}_0(\zeta^2 k)\zeta^2 e^{-ikL} \\ &\qquad - e^{-ik^3t}\Big(\hat{q}(t,k)(\zeta^2 e^{-i\zeta^2 kL} + \zeta e^{-i\zeta kL}) - \hat{q}(t,\zeta k)\zeta e^{-ikL} \\ &\qquad\qquad - \hat{q}(t,\zeta^2 k)\zeta^2 e^{-ikL}\Big)\Big\}, \end{aligned}$$

$$\begin{aligned} \tilde{g}(t,k) &= -\tilde{g}_2(t,k) \\ &= \frac{1}{\tilde{\Delta}(k)}\Big\{ - \big(\hat{q}_0(k) + \zeta\hat{q}_0(\zeta k) + \zeta^2\hat{q}_0(\zeta^2 k)\big) \\ &\qquad + e^{-ik^3t}\left(\hat{q}(t,k) + \zeta\hat{q}(t,\zeta k) + \zeta^2\hat{q}(t,\zeta^2 k)\right)\Big\}, \end{aligned}$$

where the determinant function $\Delta(k)$ is given by

$$\Delta(k) = (\zeta - \zeta^2)\tilde{\Delta}(k), \qquad \tilde{\Delta}(k) = e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL}. \tag{2.58}$$

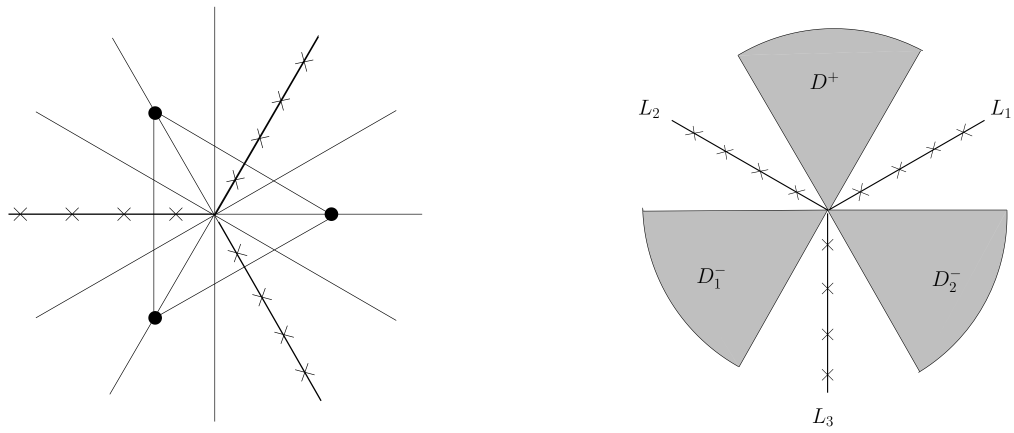
The important observation to be made now, is that if $\Delta(k)$ has zeros, then $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$ are meromorphic. Hence a central issue in the construction of the solution is the location in the complex k-plane of the zeros of the determinant function $\Delta(k)$.

A general result in the theory of such finite exponential sums [33] implies that the argument of the zeros of the determinant function $\Delta(k)$ depends on the exponents, while their location depends on the coefficients of the exponentials. In general, this result, known as Levin's Theorem (Theorem 1.3.22), given in Section 1.3.7, does not always determine the exact location of the zeros, however the knowledge of their asymptotic position in the complex k-plane is sufficient.

To demonstrate the ease with which Levin's Theorem can be applied, we substitute $z = -ikL$ into the determinant function $\Delta(k)$, given by (2.58), and locate the points 1 ζ and ζ^2 in the complex z-plane. These are then joined to form a triangle, and the zeros found to lie along the three rays, that emanate from the origin and are orthogonal to the sides of the triangle, (Figure 2.7(a)). In the complex k-plane these rays correspond to L_1, L_2 and L_3, given by

$$L_1 = \left\{ k : \arg(k) = \frac{\pi}{6} \right\}, \quad L_2 = \left\{ k : \arg(k) = \frac{5\pi}{6} \right\}, \quad L_3 = \left\{ k : \arg(k) = \frac{3\pi}{2} \right\}.$$

It can be seen in Figure 2.7(b) that the zeros cluster along the three bisecting rays of the complement regions D_c of $D = D^+ \cup D^-$.



(a) z-plane ($z = -ikL$).

(b) k-plane.

Figure 2.7: The regions $D^\pm$ for the third order problem $q_t(x,t) + q_{xxx}(x,t) = 0$ with the boundary conditions $q(0,t) = 0$, $q(L,t) = 0$ and $q_x(L,t) = 0$ and the location of the zeros of the determinant function $\Delta(k) = (\zeta - \zeta^2)(e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL})$ found using Levin's Theorem (Theorem 1.3.22).

Remark 2.5.1. For this example the rays upon which the zeros lie can be found explicitly and the detailed computation is given in [39].

For this example the zeros are located outside of the domain D and therefore $\Delta(k) \neq 0$ for $k \in D$. After multiplication by the factors e^{ikx+ik^3t} (or $e^{ik(x-L)+ik^3t}$) the terms involving $\hat{q}(t, \lambda_l(k))$, $l = 0, 1, 2$ are bounded as $k \to \infty$ in D^+ (or D^-). An application of Jordan's Lemma (Lemma 1.3.3) implies that their integral vanishes and hence these terms give a zero contribution.

Therefore the effective integral representation of the solution is given in terms of the given initial data by

$$
\begin{aligned}
q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ & \int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k \\
& - \int_{\partial D^+} e^{ikx+ik^3t}\left(\frac{\hat{q}_0(k)(\zeta^2e^{-i\zeta^2kL}+\zeta e^{-i\zeta kL})-\hat{q}_0(\zeta k)\zeta e^{-ikL}-\hat{q}_0(\zeta^2k)\zeta^2e^{-ikL}}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2e^{-i\zeta^2kL}}\right)\mathrm{d}k \\
& + \int_{\partial D^-} e^{ik(x-L)+ik^3t}\left(\frac{\hat{q}_0(k)+\zeta\hat{q}_0(\zeta k)+\zeta^2\hat{q}_0(\zeta^2k)}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2e^{-i\zeta^2kL}}\right)\mathrm{d}k\Bigg\}.
\end{aligned}
$$

Chapter 3

Two-Point Boundary Value Problems

In this chapter we analyse in detail two-point boundary value problems for second and third order PDES of the form (2.1a). For concreteness, we consider two illustrative examples, the heat equation

$$q_t(x,t) - q_{xx}(x,t) = 0\,, \quad t > 0\,, \quad x \in [0,L]\,, \tag{3.1a}$$

$$q(x,0) = q_0(x)\,, \quad x \in [0,L]\,, \tag{3.1b}$$

and the linear KdV equation

$$q_t(x,t) + q_{xxx}(x,t) = 0\,, \quad t > 0\,, \quad x \in [0,L]\,,$$

$$q(x,0) = q_0(x)\,, \quad x \in [0,L]\,,$$

with a variety of boundary conditions.

We start by considering the second order example (3.1) posed on the domain $[0,L]$ and show that the nature of the effective discrete spectrum indicates the existence of the series solution. We construct the integral representation of the solution, in terms of the given initial and boundary data and then derive the equivalent infinite series representation of the solution as the explicit residue contribution at the poles which coincide with the zeros of the determinant function $\Delta(k)$. This series solution is then shown to coincide with the classical result of Section 2.1.

We then present the transform method for third order linear evolution boundary value problems. We show that the integral representation is not always equivalent to a series representation, and that the property that characterises when this is the case is the asymptotic location of the zeros of the determinant function, inside or outside D. We illustrate the method for a variety of boundary conditions and show that a series representation of the solution only exists if the boundary conditions couple the two endpoints of the interval.

3.1 The Spectral Representation of Two-Point Boundary Value Problems for Second Order Linear Evolution PDEs

For second order linear evolution PDEs, it is well known that separation of variables can be used to solve boundary value problems on the finite interval using Fourier series, and an example of this was given in Section 2.1. This is based on the fact that the associated x-differential operator is symmetric (or symmetrisable) hence it has a discrete spectrum, and the corresponding eigenfunctions form a complete orthogonal basis of L^2. Therefore it is always possible to expand a function in terms of the complete set of the associated eigenfunctions provided that one boundary condition is prescribed at each end of the interval [47].

In this chapter we discuss the relation of this classical theory with the integral Fokas representation. Indeed, the Fokas transform method presents an alternative derivation of the classical series representation of the solution through the algorithmic construction of the eigenfunctions of the associated linear ordinary differential x-operator, as well as providing an alternative equivalent integral representation of the solution, generally involving complex contours. Unlike the classical representation of the solution, this complex integral representation of the solution is always uniformly convergent at the boundary points.

One advantage of the method we use, is that it does not rely on separation of variables or the knowledge of eigenfunctions and eigenvalues of the x-operator, therefore it can be used to obtain an integral representation of the solution for PDEs for which the spectral theory of the associated linear differential operator fails. Also, the integral representation does not depend on the explicit knowledge of the eigenvalues, which cannot always be computed exactly. The benefits of the method manifest themselves mainly for PDEs of order greater than two, however even for the second order case, this method has certain advantages over the classical approaches, particularly concerning the imposition of more complicated boundary conditions, for example Robin conditions.

Another advantage of the Fokas method over the classical approaches, is that the method works in the same way for homogeneous boundary conditions, as it does for non-homogeneous boundary conditions. This is to be contrasted with the classical method. For example, the formula (2.10) given in Section 2.1, shows the complications that arise from using the approach of separation of variables for the imposition of non-homogeneous boundary conditions.

The remainder of the chapter contains the analysis of second and third order problems in this spirit. In this first part we present the rigorous derivation, using the new approach, of the series representation of the solution of the two-point boundary value problem for the heat equation with the boundary conditions $q(0,t) = f_0(t)$ and $q(L,t) = g_0(t)$, for some given functions $f_0(t)$ and $g_0(t)$, along with the Fokas transform method for deriving the effective integral representation of the solution. The detailed proof that the series representation of the solution can be reproduced by the explicit computation at the zeros of the determinant function $\Delta(k)$ of the principal value contributions in the integral representation is also presented. To conclude the section, we generalise the results for general second order two-point linear evolution PDEs.

The Transform Method

For simplicity, we illustrate the method for the heat equation

$$q_t(x,t) - q_{xx}(x,t) = 0\,, \quad t > 0\,, \quad x \in [0,L]\,,$$

$$q(x,0) = q_0(x)\,, \quad x \in [0,L]\,,$$

with the non-homogeneous Dirichlet boundary conditions

$$q(0,t) = f_0(t), \quad q(L,t) = g_0(t)\,,$$

for some given functions $q_0(x)$, $f_0(t)$ and $g_0(t)$. The Lax pair is given by

$$\begin{aligned}\mu_x - ik\mu &= q(x,t)\,,\\ \mu_t + \omega(k)\mu &= X(x,t,k)\,,\end{aligned}$$

where $\mu = \mu(x,t,k)$ and

$$\begin{aligned}\omega(k) &= k^2\,,\\ X(x,t,k) &= ikq(x,t) + q_x(x,t)\,.\end{aligned}$$

Hence $c_0(k) = ik$ and $c_1(k) = 1$. The boundary conditions imply that

$$\tilde{f}(t,k) = ik\tilde{f}_0(t,k) + \tilde{f}_1(t,k)\,, \qquad \tilde{g}(t,k) = ik\tilde{g}_0(t,k) + \tilde{g}_1(t,k)\,,$$

where $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$, $j = 0, 1$ are given by (2.20) and (2.21) respectively. Therefore the global relation, according to (2.29), is given by

$$\tilde{f}_1(t,k) - e^{-ikL}\tilde{g}_1(t,k) = N(t,k) - e^{k^2t}\hat{q}(t,k)\,, \tag{3.2}$$

where

$$N(t,k) = \hat{q}_0(k) - ik\tilde{f}_0(t,k) + ike^{-ikL}\tilde{g}_0(t,k)\,. \tag{3.3}$$

By the invariance properties of the functions $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$, the equation $\omega(k) = \omega(\lambda)$ implies

$$k^2 - \lambda^2 = (k-\lambda)(k+\lambda) = 0\,,$$

hence $\lambda_1(k) = -k$. The global relation, given by (3.2), is therefore supplemented with the additional equation

$$\tilde{f}_1(t,k) - e^{ikL}\tilde{g}_1(t,k) = N(t,-k) - e^{k^2t}\hat{q}(t,-k)\,. \tag{3.4}$$

Expressions (3.2) and (3.4) form a system of two equations in terms of the two unknowns $\tilde{f}_1(t,k)$ and $\tilde{g}_1(t,k)$ that can be written in matrix form as follows:

$$\begin{pmatrix} 1 & e^{-ikL} \\ 1 & e^{ikL} \end{pmatrix}\begin{pmatrix} \tilde{f}_1(t,k) \\ -\tilde{g}_1(t,k) \end{pmatrix} = \begin{pmatrix} N(t,k) \\ N(t,-k) \end{pmatrix} - \begin{pmatrix} e^{k^2t}\hat{q}(t,k) \\ e^{k^2t}\hat{q}(t,-k) \end{pmatrix}. \tag{3.5}$$

This system can be solved to give expressions for $\tilde{f}_1(t,k)$ and $\tilde{g}_1(t,k)$ and used to achieve both the effective integral representation of the solution and the equivalent discrete series representation of the solution.

3.1.1 The Integral Representation

The integral representation of the solution is given by equation (2.56) where the domains $D^{\pm}$ in the complex k-plane are defined by (2.54). Hence,

$$
\begin{aligned}
q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ \int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx-k^2t}\left(ik\tilde{f}_0(t,k)+\tilde{f}_1(t,k)\right)\mathrm{d}k \\
- \int_{\partial D^-} e^{ik(x-L)-k^2t}\left(ik\tilde{g}_0(t,k)+\tilde{g}_1(t,k)\right)\mathrm{d}k \Bigg\},
\end{aligned}
$$

where

$$
D^+ = \left\{k \in \mathbb{C} : \frac{\pi}{4} \leqslant \arg(k) \leqslant \frac{3\pi}{4}\right\}, \quad D^- = \left\{k \in \mathbb{C} : \frac{5\pi}{4} \leqslant \arg(k) \leqslant \frac{7\pi}{4}\right\},
$$

and the oriented boundaries of $D^{\pm}$ are such that the interior of the domain D is always on the left of the positive direction.

To achieve explicit expressions for $\tilde{f}_1(t,k)$ and $\tilde{g}_1(t,k)$ we solve system (3.5) for the two unknown boundary terms using Cramer's rule:

$$
\begin{aligned}
\tilde{f}_1(t,k) \;&=\; \frac{1}{\Delta(k)}\left\{e^{ikL}\left(N(t,k)-e^{k^2t}\hat{q}(t,k)\right) - e^{-ikL}\left(N(t,-k)-e^{k^2t}\hat{q}(t,-k)\right)\right\},\\
\tilde{g}_1(t,k) \;&=\; \frac{1}{\Delta(k)}\left\{\left(N(t,k)-e^{k^2t}\hat{q}(t,k)\right) - \left(N(t,-k)-e^{k^2t}\hat{q}(t,-k)\right)\right\},
\end{aligned}
$$

where $\Delta(k) = e^{ikL} - e^{-ikL}$ is the determinant function of system (3.5) with zeros

$$
k = k_n = \frac{n\pi}{L}, \qquad n \in \mathbb{Z}.
$$

Hence, the explicit integral representation of the solution is given by

$$
\begin{aligned}
q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ &\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx-k^2t} ik\tilde{f}_0(t,k)\,\mathrm{d}k \\
&- \int_{\partial D^-} e^{ik(x-L)-k^2t} ik\tilde{g}_0(t,k)\,\mathrm{d}k \\
&- \int_{\partial D^+} e^{ikx-k^2t}\left(\frac{e^{ikL}\left(N(t,k)-e^{k^2t}\hat{q}(t,k)\right)-e^{-ikL}\left(N(t,-k)-e^{k^2t}\hat{q}(t,-k)\right)}{e^{ikL}-e^{-ikL}}\right)\mathrm{d}k \\
&- \int_{\partial D^-} e^{ik(x-L)-k^2t}\left(\frac{\left(N(t,k)-e^{k^2t}\hat{q}(t,k)\right)-\left(N(t,-k)-e^{k^2t}\hat{q}(t,-k)\right)}{e^{ikL}-e^{-ikL}}\right)\mathrm{d}k\Bigg\}. \quad (3.6)
\end{aligned}
$$

The zeros of the function $\Delta(k) = e^{ikL} - e^{-ikL}$ are situated on the real axis, which bisects the complement region D_c. The domains $D^{\pm}$ and the location of the zeros k_n are given in Figure 3.1.

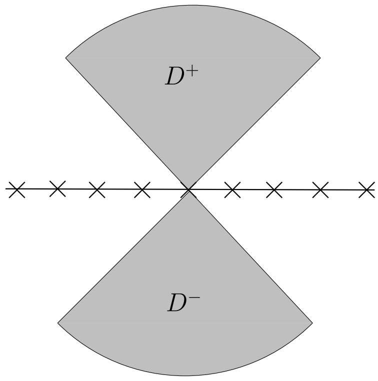


Figure 3.1: The regions $D^+ = \left\{k \in \mathbb{C} : \frac{\pi}{4} \leqslant \arg(k) \leqslant \frac{3\pi}{4}\right\}$ and $D^- = \left\{k \in \mathbb{C} : \frac{5\pi}{4} \leqslant \arg(k) \leqslant \frac{7\pi}{4}\right\}$ for the second order heat equation $q_t(x,t) - q_{xx}(x,t) = 0$ with the boundary conditions $q(0,t) = f_0(t)$ and $q(L,t) = g_0(t)$ and the location of the zeros k_n of the determinant function $\Delta(k) = e^{ikL} - e^{-ikL}$.

Proposition 3.1.1. *The unknown terms $\hat{q}(t,k)$ and $\hat{q}(t,-k)$, in the integral representation of the solution, given by (3.6), when multiplied by the factors e^{ikx-k^2t} (or $e^{ik(x-L)-k^2t}$), are analytic and bounded as $k \to \infty$ in D^+ (or D^-) and do not contribute to the integral representation of the solution.*

Proof. We prove the case rigorously for the integral of the unknown terms around ∂D^+ given by

$$\int_{\partial D^+} e^{ikx} \left(\frac{-e^{ikL}\hat{q}(t,k) + e^{-ikL}\hat{q}(t,-k)}{e^{ikL} - e^{-ikL}} \right) \mathrm{d}k \,.$$

If $k \in D^+$ then $\frac{\pi}{4} \leqslant \arg(k) \leqslant \frac{3\pi}{4}$ and hence e^{ikL} is bounded and e^{-ikL} is unbounded. To establish the asymptotic behaviour of the denominator we look at the real part of each of the exponents.

$$\mathrm{Re}(-ikL) = k_I L \,, \quad \mathrm{Re}(ikL) = -k_I L \,,$$

where $k = k_R + ik_I$, and conclude that asymptotically the denominator behaves like e^{-ikL}. Therefore, the asymptotic behaviour of the integrand is given by

$$\begin{aligned} e^{ikx}\left(-e^{2ikL}\hat{q}(t,k) + \hat{q}(t,-k)\right) \;&=\; -e^{ikx}e^{ikL}\int_0^L e^{ik(L-x')}q(x',t)\,\mathrm{d}x' \\ &\quad + e^{ikx}\int_0^L e^{ikx'}q(x',t)\,\mathrm{d}x' \,. \end{aligned}$$

We conclude that the terms involving the unknown functions $\hat{q}(t,k)$ and $\hat{q}(t,-k)$ are analytic and bounded in D^+ and therefore by Jordan's Lemma (Lemma 1.3.3) do not contribute to the integral representation of the solution around ∂D^+. $\square$

Remark 3.1.2. An analogous calculation proves that the integral around ∂D^- of the terms involving $\hat{q}(t,k)$ and $\hat{q}(t,-k)$ also vanishes.

Remark 3.1.3. To explain the use of Jordan's Lemma in the proof of Proposition 3.1.1, it is necessary only to observe the following:

$$\int_{\partial D^+} e^{ikx} f(k)\,\mathrm{d}k = \lim_{R\to\infty} \int_{\vee} f(k)\,\mathrm{d}k = -\lim_{R\to\infty} \int_{\frown} f(k)\,\mathrm{d}k = 0$$

where $f(k)$ is a suitably bounded function for $k \in D^+$, R represents the radius length of the wedge $\vee$, and the orientation of the closed contour $\widehat{\vee}$ is such that interior is always to the left.

We conclude that the effective integral representation of the solution is given by

$$\begin{aligned} q(x,t) \;=\; & \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k)\,\mathrm{d}k \right. \\ & - \int_{\partial D^+} e^{ikx-k^2t}\left(\frac{e^{ikL}N(t,k)-e^{-ikL}N(t,-k)}{e^{ikL}-e^{-ikL}} + ik\tilde{f}_0(t,k)\right)\mathrm{d}k \\ & \left. - \int_{\partial D^-} e^{ik(x-L)-k^2t}\left(\frac{N(t,k)-N(t,-k)}{e^{ikL}-e^{-ikL}} + ik\tilde{g}_0(t,k)\right)\mathrm{d}k \right\}. \end{aligned} \tag{3.7}$$

3.1.2 A Derivation of the Series Representation using the Global Relation

It is well known that it is always possible to expand a function in terms of the complete basis of eigenfunctions for second order PDEs of the form (2.1) and thus obtain the solution as an infinite discrete series. Indeed, this classical approach of obtaining the infinite discrete series solution, was demonstrated in Section 2.1, where we solved the second order heat equation with the boundary conditions $q(0,t) = f_0(t)$ and $q(L,t) = g_0(t)$.

We now show how the series representation of the solution can be obtained from the analysis of the pair of global relations given by

$$\begin{aligned} \tilde{f}_1(t,k) - e^{-ikL}\tilde{g}_1(t,k) \;&=\; N(t,k) - e^{k^2t}\hat{q}(t,k)\,, \\ \tilde{f}_1(t,k) - e^{ikL}\tilde{g}_1(t,k) \;&=\; N(t,-k) - e^{k^2t}\hat{q}(t,-k)\,, \end{aligned}$$

where $N(t,k)$ is given by (3.3). Subtracting the two equations and evaluating the resulting expression at an arbitrary positive real zero $k = k_n = \frac{n\pi}{L}$ of the determinant function $\Delta(k)$ yields

$$e^{-k_n^2 t}\left(N(t,k_n) - N(t,-k_n)\right) = \hat{q}(t,k_n) - \hat{q}(t,-k_n)\,.$$

Hence

$$\int_0^L \left(e^{-ik_n x} - e^{ik_n x}\right) q(x,t)\,\mathrm{d}x = e^{-k_n^2 t}(N(t,k_n) - N(t,-k_n))\,,$$

which implies that the eigenfunctions are given by

$$e_n(x) = e^{ik_n x} - e^{-ik_n x}\,.$$

Indeed, the functions $e_n(x)$ are eigenfunctions of the linear differential operator $D = \frac{\partial^2}{\partial x^2}$ and satisfy the homogeneous boundary conditions $e_n(0) = 0$ and $e_n(L) = 0$, and the orthogonality condition

$$\int_0^L e_n(x)\, e_m(x)\,\mathrm{d}x = \begin{cases} 0\,, & n \neq m\,, \\ -2L\,, & n = m\,. \end{cases}$$

To achieve the series representation of the solution, we suppose $q(x,t)$ takes the form of an infinite discrete series:

$$q(x,t) = \sum_{n=1}^{\infty} c_n(t)\, e_n(x)$$

where the coefficients $c_n(t)$ are determined as follows:

$$\int_0^L e_m(x)\, q(x,t)\,\mathrm{d}x = \int_0^L \left(\sum_{n=1}^{\infty} c_n(t)\, e_n(x)\right) e_m(x)\,\mathrm{d}x = -2Lc_m(t)\,.$$

Hence

$$c_n(t) = -\frac{1}{2L}\int_0^L \left(e^{ik_n x} - e^{-ik_n x}\right) q(x,t)\,\mathrm{d}x = \frac{1}{2L} e^{-k_n^2 t}(N(t,k_n) - N(t,-k_n))\,.$$

Therefore the series representation of the solution is given by

$$q(x,t) = \frac{1}{2L}\sum_{n=1}^{\infty} e^{-k_n^2 t}\left(N(t,k_n) - N(t,-k_n)\right)\left(e^{ik_n x} - e^{-ik_n x}\right)\,, \qquad (3.8)$$

and this is equivalent to the series solution given by (2.10), obtained using the classical approach.

Proof. To prove the equivalence of expressions (2.10) and (3.8), we rewrite (3.8) in the form

$$
\begin{aligned}
q(x,t) &= \frac{1}{L}\sum_{n=1}^{\infty} e^{-k_n^2 t}\Big(\hat{q}_0(k_n) - \hat{q}_0(-k_n) - 2ik_n\tilde{f}_0(t,k_n) + 2(-1)^n ik_n\tilde{g}_0(t,k_n)\Big)\, i\sin(k_n x) \\
&= \frac{2}{L}\sum_{n=1}^{\infty} e^{-k_n^2 t}\Big(\hat{q}_0^{(\sin)}(k_n) + k_n\tilde{f}_0(t,k_n) - (-1)^n k_n\tilde{g}_0(t,k_n)\Big)\sin(k_n x)\,,
\end{aligned}
$$

where

$$
\hat{q}_0^{(\sin)}(k_n) = \int_0^L \sin(k_n x)\, q_0(x)\,\mathrm{d}x\,. \tag{3.9}
$$

The expression for $w_n(t)$, given by (2.9), can be written in terms of the functions $\hat{q}_0^{(\sin)}(k_n)$, $\tilde{f}_0(t,k_n)$ and $\tilde{g}_0(t,k_n)$, according to (3.9), (2.20) and (2.21) respectively, to give

$$
\begin{aligned}
w_n(t) = \frac{2}{L}\Bigg\{ &-\frac{1}{k_n}\left(f_0(t) - (-1)^n g_0(t)\right) \\
&+ e^{-k_n^2 t}\Big(\hat{q}_0^{(\sin)}(k_n) + k_n\tilde{f}_0(t,k_n) - (-1)^n k_n\tilde{g}_0(t,k_n)\Big)\Bigg\}\,.
\end{aligned}
$$

Therefore, the series representation of the solution, obtained from the classical approach, given by (2.10), is explicitly given by

$$
\begin{aligned}
q(x,t) = {}& f_0(t) + \frac{x}{L}\left(g_0(t) - f_0(t)\right) - \frac{2}{L}\sum_{n=1}^{\infty}\frac{1}{k_n}\left(f_0(t) - (-1)^n g_0(t)\right)\sin(k_n x) \\
&+ \frac{2}{L}\sum_{n=1}^{\infty} e^{-k_n^2 t}\Big(\hat{q}_0^{(\sin)}(k_n) + k_n\tilde{f}_0(t,k_n) - (-1)^n k_n\tilde{g}_0(t,k_n)\Big)\sin(k_n x)\,,
\end{aligned}
$$

and by (2.7) and (2.8) it follows that

$$
f_0(t) + \frac{x}{L}\left(g_0(t) - f_0(t)\right) = \frac{2}{L}\sum_{n=1}^{\infty}\frac{1}{k_n}\left(f_0(t) - (-1)^n g_0(t)\right)\sin(k_n x)\,.
$$

This concludes the proof that (2.10) is equivalent to (3.8). □

Remark 3.1.4. The classical approach relies on the assumption that the functions $w(x,t)$ and $F(x,t)$, given by (2.6) and (2.7) respectively, can be expanded in terms of the associated eigenfunctions, whereas no such assumptions are required for the Fokas derivation.

3.1.3 Equivalence of the Series and Integral Representations

We now prove that the series representation of the solution, given by (3.8) is equivalent to the effective integral representation of the solution given by (3.7) by showing that the series representation can be reproduced by the explicit computation, at the zeros k_n of $\Delta(k)$, of the principal value contributions in the integral representation.

The zeros of $\Delta(k)$ lie on the real axis which perpendicularly bisects the domain D_c, hence $\Delta(k) \neq 0$ for $k \in D$ and therefore the contours of integration must be deformed from ∂D to the real axis.

Proposition 3.1.5. *The integrands of the integrals around ∂D^+ (and ∂D^-), in the integral representation of the solution, given by (3.7), are bounded as $k \to \infty$ in D_c^+ (and D_c^-).*

Proof. We prove the case only for the integral around ∂D^+, (the proof for the integral around ∂D^- follows analogously), and recall the integral around ∂D^+ given explicitly by

$$\int_{\partial D^+} e^{ikx-k^2t}\left(\frac{e^{ikL}N(t,k)-e^{-ikL}N(t,-k)}{e^{ikL}-e^{-ikL}}+ik\tilde{f}_0(t,k)\right)\mathrm{d}k\,.$$

We show that the integrand is bounded for k such that $0 < \arg(k) < \frac{\pi}{4}$ and conclude that the contour of integration can be deformed from where $\arg(k) = \frac{\pi}{4}$ to the positive real axis so that the principal value contributions at the zeros $k = k_n = \frac{n\pi}{L}$, $n \geqslant 0$ can be computed.

Consider the wedge such that $0 < \arg(k) < \frac{\pi}{4}$. For k in this wedge, $-k$ will be such that $\pi < \arg(-k) < \frac{5\pi}{4}$. Hence k will lie in the upper half of the complex plane and $-k$ will lie in the lower half of the complex plane (Figure 3.2).

Hence, for k in this wedge

- e^{ikx-k^2t} is bounded,
- e^{ikL} is bounded,
- e^{-ikL} is unbounded.

Therefore, asymptotically the denominator behaves like e^{-ikL} and hence the asymptotic

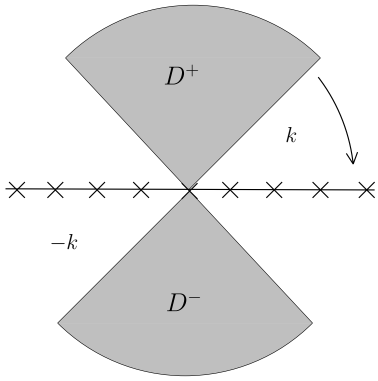


Figure 3.2: The wedge such that $0 < \arg(k) < \frac{\pi}{4}$ in the complex k-plane.

behaviour of the integrand is given by

$$
\begin{aligned}
&e^{ikx-k^2t}\left(e^{2ikL}N(t,k) - N(t,-k) + (e^{2ikL}-1)ik\tilde{f}_0(t,k)\right) \\
&= e^{ikx-k^2t}\left(e^{2ikL}\hat{q}_0(k) - \hat{q}_0(-k) - 2ik\tilde{f}_0(t,k) + 2ike^{ikL}\tilde{g}_0(t,k)\right) \\
&= e^{ikx-k^2t}\left(e^{ikL}\int_0^L e^{ik(L-x')}q_0(x')\,\mathrm{d}x' - \int_0^L e^{ikx'}q_0(x')\,\mathrm{d}x'\right) \\
&\quad - 2ike^{ikx}\left(\int_0^t e^{k^2(s-t)}f_0(s)\,\mathrm{d}s - e^{ikL}\int_0^t e^{k^2(s-t)}g_0(s)\,\mathrm{d}s\right).
\end{aligned}
$$

All of the terms in this expression are bounded as $k \to \infty$ and the proof is complete. □

Remark 3.1.6. We remark that a similar argument can be used for the proof that the contour of integration can be deformed from where $\arg(k) = \frac{3\pi}{4}$ to the negative real axis.

An application of Cauchy's Theorem (Theorem 1.3.1) is now used to deform the contours of integration and the explicit computation at the zeros of the determinant function $\Delta(k)$ of the principal value contributions in the integral representation is shown to be equivalent to the discrete series representation of the solution.

Recall the integral representation of the solution given by (3.7). The contours are deformed to the real axis and the residue contributions from all of the poles k_n, such that $\Delta(k_n) = 0$, are computed using Theorem 1.3.8 as follows:

$$
\int_{\partial D^+} \frac{p^+(k)}{r(k)}\,\mathrm{d}k = \int_{-\infty}^{\infty} \frac{p^+(k)}{r(k)}\,\mathrm{d}k + \pi i \sum_{\substack{k_n: \\ \Delta(k_n)=0}} \frac{p^+(k_n)}{r'(k_n)},
$$

$$\int_{\partial D^-} \frac{p^-(k)}{r(k)}\,\mathrm{d}k = -\int_{-\infty}^{\infty} \frac{p^-(k)}{r(k)}\,\mathrm{d}k + \pi i \sum_{\substack{k_n:\\ \Delta(k_n)=0}} \frac{p^-(k_n)}{r'(k_n)}\,,$$

where

$$\begin{aligned}
p^+(k) &= e^{ikx-k^2t}\left(e^{ikL}N(t,k) - e^{-ikL}N(t,-k)\right)\,,\\
p^-(k) &= e^{ik(x-L)-k^2t}\left(N(t,k) - N(t,-k)\right)\,,\\
r(k) &= \Delta(k) = e^{ikL} - e^{-ikL}\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\partial D^+} \frac{p^+(k)}{r(k)}\,\mathrm{d}k &= \int_{-\infty}^{\infty} \frac{p^+(k)}{r(k)}\,\mathrm{d}k + \pi i \sum_{\substack{k_n:\\ \Delta(k_n)=0}} e^{ik_nx-k_n^2t}\left(\frac{e^{ik_nL}N(t,k_n) - e^{-ik_nL}N(t,-k_n)}{\Delta'(k_n)}\right)\,,\\
\int_{\partial D^-} \frac{p^-(k)}{r(k)}\,\mathrm{d}k &= -\int_{-\infty}^{\infty} \frac{p^-(k)}{r(k)}\,\mathrm{d}k + \pi i \sum_{\substack{k_n:\\ \Delta(k_n)=0}} e^{ik_n(x-L)-k_n^2t}\left(\frac{N(t,k_n) - N(t,-k_n)}{\Delta'(k_n)}\right)\,.
\end{aligned}$$

Therefore the integral representation of the solution, given by (3.7) can be written in the form

$$\begin{aligned}
q(x,t) &= \frac{1}{2\pi}\Bigg\{\int_{-\infty}^{\infty} e^{ikx-k^2t}\hat{q}_0(k)\,\mathrm{d}k\\
&\qquad - \int_{-\infty}^{\infty} e^{ikx-k^2t}\left(\frac{e^{ikL}N(t,k) - e^{-ikL}N(t,-k)}{e^{ikL} - e^{-ikL}} + ik\tilde{f}_0(t,k)\right)\,\mathrm{d}k\\
&\qquad + \int_{-\infty}^{\infty} e^{ik(x-L)-k^2t}\left(\frac{N(t,k) - N(t,-k)}{e^{ikL} - e^{-ikL}} + ik\tilde{g}_0(t,k)\right)\,\mathrm{d}k\Bigg\}\\
&\quad + \frac{1}{2L}\Bigg\{\sum_{\substack{k_n:\\ \Delta(k_n)=0}} e^{ik_nx-k_n^2t}\left(\frac{e^{ik_nL}N(t,k_n) - e^{-ik_nL}N(t,-k_n)}{e^{ik_nL} + e^{-ik_nL}}\right)\\
&\qquad + \sum_{\substack{k_n:\\ \Delta(k_n)=0}} e^{ik_n(x-L)-k_n^2t}\left(\frac{N(t,k_n) - N(t,-k_n)}{e^{ik_nL} + e^{-ik_nL}}\right)\Bigg\}\,.
\end{aligned}$$

The integrals sum to zero and all that remains are the summation terms which are simplified to yield

$$q(x,t) = \frac{1}{2L}\sum_{\substack{k_n:\\ \Delta(k_n)=0}} e^{ik_nx-k_n^2t}\left(N(t,k_n) - N(t,-k_n)\right)\,.$$

This can trivially be rewritten in the form

$$q(x,t) = \frac{1}{2L}\sum_{n=1}^{\infty} e^{-k_n^2 t}\left(N(t,k_n) - N(t,-k_n)\right)\left(e^{ik_n x} - e^{-ik_n x}\right) ,$$

where the index now indicates that the series is summed over the zeros that lie on the positive real axis, and concurs with the series representation of the solution given by (3.8).

In summary, the complex contours of integration $\partial D^{\pm}$ can be deformed to the real axis to achieve the integral representation entirely on $\mathbb{R}$ and in doing so the residue contribution from each of the zeros lying on the real axis is acquired. A simple manipulation of the resulting expression for the solution $q(x,t)$ shows that the integral terms cancel and all that remains is the infinite sum which is equivalent to the series representation of the solution, achieved using the classical Fourier approach.

For second order problems the complex contours of integration in the integral representation of the solution, can always be deformed to the real line at the price of computing the residues/principal value contributions at the zeros of the determinant function $\Delta(k)$.

Remark 3.1.7. In the case that the contours of the integral representation can be realised on the real line, it is possible to show formally that all integral terms *must* cancel. To show this, we begin with the formal integral representation of the solution, given by (2.25), and assume that the contours of integration can be deformed to the real line. We rearrange the global relation, and write

$$\tilde{f}(t,k) = e^{-ikL}\tilde{g}(t,k) + \hat{q}_0(k) - e^{\omega(k)t}\hat{q}(t,k) ,$$

and substitute into (2.25) to yield

$$\begin{aligned} q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ & \int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)\,\mathrm{d}k - \int_{-\infty}^{\infty} e^{ik(x-L)-\omega(k)t}\tilde{g}(t,k)\,\mathrm{d}k \\ & - \int_{-\infty}^{\infty} e^{ikx-\omega(k)t}\hat{q}_0(k)\,\mathrm{d}k + \int_{-\infty}^{\infty} e^{ikx}\hat{q}(t,k)\,\mathrm{d}k \\ & + \int_{-\infty}^{\infty} e^{ik(x-L)-\omega(k)t}\tilde{g}(t,k)\,\mathrm{d}k \Bigg\} . \end{aligned}$$

Of course we have not given the effective expression for the spectral functions $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$. However this identity shows formally that, in this case, the integral terms must cancel.

3.1.4 General Boundary Conditions

We now prove that for the second order problem of the form (2.1), the zeros of the determinant function $\Delta(k)$ always lie on the real axis [22]. For the proof, we consider (3.1) with general uncoupled boundary conditions

$$q(0,t) + \alpha q_x(0,t) = f_0(t)\,, \quad q(L,t) + \beta q_x(L,t) = g_0(t)\,,$$

for constants α and β and given functions $f_0(t)$ and $g_0(t)$. Applying the boundary conditions, the global relation, according to (2.28) is given by

$$(1 - ik\alpha)\tilde{f}_1(t,k) - e^{-ikL}(1 - ik\beta)\tilde{g}_1(t,k) = N(t,k) - e^{k^2t}\hat{q}(t,k)\,,$$

where

$$N(t,k) = \hat{q}_0(k) - ik(\tilde{f}_0(t,k) - e^{-ikL}\tilde{g}_0(t,k))\,,$$

and this is complemented by the equation

$$(1 + ik\alpha)\tilde{f}_1(t,k) - e^{ikL}(1 + ik\beta)\tilde{g}_1(t,k) - N(t,-k) - e^{k^2t}\hat{q}(t,-k)\,.$$

Therefore, the determinant function is given by

$$\Delta(k) = (1 - ik\alpha)(1 + ik\beta)e^{ikL} - (1 + ik\alpha)(1 - ik\beta)e^{-ikL}\,,$$

and the zeros of this function lie on the real axis, and their exact location depends on the values of α and β.

Therefore, for the heat equation, Figure 3.3 shows that $\Delta(k) \neq 0$ for $k \in D$ and that the zeros lie outside of D and cluster along the real axis which perpendicularly bisects the region D_c. The integrand containing the unknown terms is analytic and bounded as $k \to \infty$ in D_c and therefore, according to Proposition 2.4.3, there is a zero contribution from the unknown terms. The contours of integration can be deformed from ∂D to the real axis and in doing so we realise the contribution of the residues at the zeros of the determinant function $\Delta(k)$. All of the integral terms sum to zero and all that remains is the discrete series representation of the solution.

Remark 3.1.8. We remark that the zeros of the determinant functions of the two other well-posed second order PDEs $q_t(x,t) \mp iq_{xx}(x,t) = 0$ also cluster asymptotically along the real axis. The domains are given by $D^+ = \left\{k \in \mathbb{C} : 0 \leqslant \arg(k) \leqslant \frac{\pi}{2}\right\}$ and $D^- =$

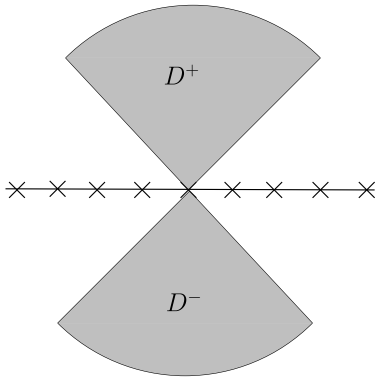
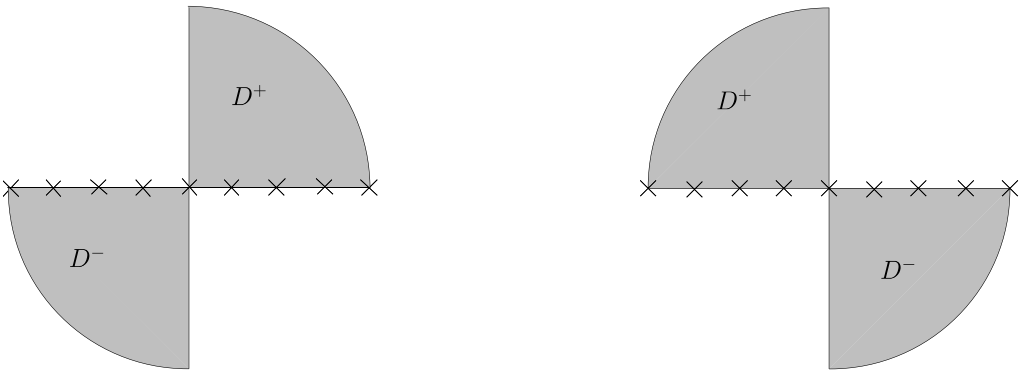


Figure 3.3: The regions $D^+ = \left\{k \in \mathbb{C} : \frac{\pi}{4} \leqslant \arg(k) \leqslant \frac{3\pi}{4}\right\}$ and $D^- = \left\{k \in \mathbb{C} : \frac{5\pi}{4} \leqslant \arg(k) \leqslant \frac{7\pi}{4}\right\}$ for the heat equation $q_t(x,t) - q_{xx}(x,t) = 0$ with the general uncoupled boundary conditions $q(0,t) + \alpha q_x(0,t) = f_0(t)$ and $q(L,t) + \beta q_x(L,t) = g_0(t)$ and the location of the zeros of the determinant function $\Delta(k) = (1 - ik\alpha)(1 + ik\beta)e^{ikL} - (1 + ik\alpha)(1 - ik\beta)e^{-ikL}$.

$\left\{k \in \mathbb{C} : \pi \leqslant \arg(k) \leqslant \frac{3\pi}{2}\right\}$ for the second order PDE $q_t(x,t) - iq_{xx}(x,t) = 0$, and $D^+ = \left\{k \in \mathbb{C} : \frac{\pi}{2} \leqslant \arg(k) \leqslant \pi\right\}$ and $D^- = \left\{k \in \mathbb{C} : \frac{3\pi}{2} \leqslant \arg(k) \leqslant 2\pi\right\}$ for the second order PDE $q_t(x,t) + iq_{xx}(x,t) = 0$, and are given in Figure 3.4(a) and Figure 3.4(b) respectively. Therefore, the zeros of $\Delta(k)$ lie on the boundary rays ∂D of D.



(a) The regions $D^\pm$ for the PDE $q_t(x,t) - iq_{xx}(x,t) = 0$.

(b) The regions $D^\pm$ for the PDE $q_t(x,t) + iq_{xx}(x,t) = 0$.

Figure 3.4: The regions $D^\pm$ for the second order PDEs $q_t(x,t) \mp iq_{xx}(x,t) = 0$ with the general uncoupled boundary conditions $q(0,t) + \alpha q_x(0,t) = f_0(t)$ and $q(L,t) + \beta q_x(L,t) = g_0(t)$ and the location of the zeros of the determinant function $\Delta(k)$.

To prove the equivalence of the series and integral representations of the solution, one must indent the contours so that they pass above the zeros in $\mathbb{R}^+$ (in ∂D^+) and below the zeros in $\mathbb{R}^-$ (in ∂D^-), inside any complex disc of radius R. The limit as $R \to \infty$

defines the principal value integral at infinity. Inside the contour, the unknown terms $\hat{q}(t,k)$ and $\hat{q}(t,-k)$ are analytic and bounded and hence the integrals of these terms give a zero contribution.

Therefore, as for the heat equation, the only nonzero contribution to the solution representation is due to the residue contribution at the zeros, and this is equivalent to an infinite discrete series.

Remark 3.1.9. The index, in expression (3.8), indicates that the summation is taken over all of the positive zeros k_n. The Fokas transform method offers an algorithmic derivation of the series representation of the solution which, as opposed to the classical derivation, is not complicated by the imposition of non-homogeneous boundary conditions.

3.2 The Spectral Representation of Two-Point Boundary Value Problems for Third Order Linear Evolution PDEs

In this section we present the transform method for third order linear evolution boundary value problems of the form (2.1) and we show that the integral representation of the solution is not always equivalent to a discrete series representation.

We illustrate the case for the third order problem

$$q_t(x,t) + q_{xxx}(x,t) = 0\,, \quad t > 0\,, \quad x \in [0,L]\ , \tag{3.10a}$$

$$q(x,0) = q_0(x)\,, \quad x \in [0,L]\ , \tag{3.10b}$$

for some given function $q_0(x)$, for a variety of boundary conditions and show that a series representation is not obtainable when all of the singularities of the determinant function $\Delta(k)$ are outside of the domain D, which is identified only by the equation.

It is well known that for third order linear evolution PDEs, if the boundary conditions yield a non self-adjoint operator, then separation of variables cannot be used to solve the boundary value problem. The failure of the classical approach is due to the lack of a general proof of the completeness of the set of eigenfunctions for non self-adjoint operators. However, it can be shown by classical methods that when the boundary conditions couple the boundary points, the set of eigenfunctions of the problem form a Riesz basis.

It appears that one can make a general distinction between coupled and uncoupled boundary conditions. We shall refer to coupled boundary conditions of the form

$$\begin{aligned} &q_x^{(j)}(0,t) = f_j(t)\,, \quad q_x^{(j)}(L,t) = g_j(t)\,, \\ &q_x^{(k)}(L,t) = \alpha q_x^{(k)}(0,t)\,, \quad \alpha \in \mathbb{R}\,, \end{aligned} \qquad j,k \in \{0,1,2\}\;, \quad j \neq k\,, \tag{3.11}$$

for some given functions $f_j(t)$ and $g_j(t)$, whereas uncoupled boundary conditions lack any form of symmetry and take the form

$$q_x^{(j)}(0,t) = f_j(t)\,, \quad q_x^{(k)}(L,t) = g_k(t)\,, \quad q_x^{(l)}(L,t) = g_l(t)\,, \qquad j,k,l \in \{0,1,2\}\;, \quad k \neq l\,, \tag{3.12}$$

for some given functions $f_j(t)$, $g_k(t)$ and $g_l(t)$. The superscripts in expressions (3.11) and (3.12) denote the order of the derivative imposed.

When coupled boundary conditions are prescribed, classical theory guarantees the existence of a complete basis of eigenfunctions corresponding to a discrete spectrum, and the transform approach, in agreement with classical theory, yields a representation that can be explicitly computed in the form of an infinite discrete series. It is shown that the zeros of the determinant function $\Delta(k)$ lie on the six boundary rays of D in the complex k-plane implying $\Delta(k) = 0$ for $k \in D$. The contours of integration, $\partial D^{\pm}$, can be indented to avoid these zeros and the solution given by the explicit computation of the principal value contributions. In fact, the only case for which the solution admits an entirely discrete series representation is when the prescribed boundary conditions are of the form (3.11). A special case of such conditions are when the boundary conditions are periodic, and in this case it is well known that the series representation of the solution is obtainable using Fourier transforms.

In comparison, when the boundary conditions are uncoupled, and of the form (3.12), the series representation of the solution cannot be obtained by the same route, and we are not able to obtain any such representation. We stress that no results in classical theory imply that such a representation should exist. We show that $\Delta(k) \neq 0$ for $k \in D$ and the zeros of the determinant function $\Delta(k)$ lie on three rays emanating from the origin. However, the contours of integration cannot be deformed throughout D_c, and there always appear integral terms in the representation of the solution and hence the solution cannot be expressed entirely as an infinite discrete series.

For the case of coupled conditions, we present the example of periodic boundary conditions in the context of the Fokas method, along with the case where the boundary conditions are coupled and of the form

$$q(0,t) = 0\,, \quad q(L,t) = 0\,, \quad q_x(L,t) = \alpha q_x(0,t)\,, \quad \alpha \in \mathbb{R}\,.$$

We also consider the imposition of the quasi-periodic boundary conditions of the form

$$q(L,t) = \alpha q(0,t)\,, \quad q_x(L,t) = \alpha q_x(0,t)\,, \quad q_{xx}(L,t) = \alpha q_{xx}(0,t)\,, \quad \alpha > 0\,,$$

and show that in this special example, unlike the cases of periodic and coupled boundary conditions where the zeros are found to lie on the six boundary rays of ∂D, the zeros of the determinant function $\Delta(k)$ lie on three rays. However, unlike the case of uncoupled boundary conditions, these rays do not emanate from the origin, but instead lie parallel to the boundary rays of ∂D. There are two cases that arise, depending on the value of α. In one case, it is shown that infinitely many zeros lie outside of the domain D, and in the other, it is shown that only finitely many lie outside of D. However, in both cases, it is possible to obtain a discrete series representation of the solution.

3.2.1 The Formulation of the Problem

The first step of the transform approach is the algorithmic derivation of the Lax pair for the third order PDE given by (3.10). Recall that (3.10a) is equivalent to the pair of equations

$$\begin{aligned}
\left(\mu(x,t,k)e^{-ikx-ik^3t}\right)_x &= e^{-ikx-ik^3t}q(x,t)\,, \\
\left(\mu(x,t,k)e^{-ikx-ik^3t}\right)_t &= e^{-ikx-ik^3t}\left(k^2q(x,t) - ikq_x(x,t) - q_{xx}(x,t)\right)\,.
\end{aligned}$$

The functions $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$ are given by

$$\tilde{f}(t,k) = k^2\tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\,, \tag{3.13}$$

$$\tilde{g}(t,k) = k^2\tilde{g}_0(t,k) - ik\tilde{g}_1(t,k) - \tilde{g}_2(t,k)\,. \tag{3.14}$$

Hence $c_0(k) = k^2$, $c_1(k) = -ik$ and $c_2(k) = -1$. The global relation, given in the general form by equation (2.28), is therefore the following algebraic expression relating the six spectral functions $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$, $j = 0,1,2$:

$$\begin{aligned}\Big(k^2\tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\Big) &- e^{-ikL}\left(k^2\tilde{g}_0(t,k) - ik\tilde{g}_1(t,k) - \tilde{g}_2(t,k)\right) \\ &= \hat{q}_0(k) - e^{-ik^3t}\hat{q}(t,k)\,. \qquad (3.15)\end{aligned}$$

The transformations that leave $\omega(k)$ invariant are determined by the roots of the equation $\omega(k) = \omega(\lambda)$ and are given explicitly by $\lambda_l(k)$ for $l = 0,1,2$. Hence $\lambda_0(k) = k$, $\lambda_1(k) = \zeta k$ and $\lambda_2(k) = \zeta^2 k$ where $\zeta = e^{\frac{2\pi i}{3}}$. The global relation, given by (3.15), is therefore supplemented by the two additional equations

$$\begin{aligned}\Big(\zeta^2k^2\tilde{f}_0(t,k) - i\zeta k\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\Big) &- e^{-i\zeta kL}\left(\zeta^2k^2\tilde{g}_0(t,k) - i\zeta k\tilde{g}_1(t,k) - \tilde{g}_2(t,k)\right) \\ &= \hat{q}_0(\zeta k) - e^{-ik^3t}\hat{q}(t,\zeta k)\,, \qquad (3.16)\end{aligned}$$

$$\begin{aligned}\Big(\zeta k^2\tilde{f}_0(t,k) - i\zeta^2 k\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\Big) &- e^{-i\zeta^2 kL}\left(\zeta k^2\tilde{g}_0(t,k) - i\zeta^2 k\tilde{g}_1(t,k) - \tilde{g}_2(t,k)\right) \\ &= \hat{q}_0(\zeta^2 k) - e^{-ik^3t}\hat{q}(t,\zeta^2 k)\,. \qquad (3.17)\end{aligned}$$

Expressions (3.15), (3.16) and (3.17) form a system of three equations involving the six boundary functions $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$, $j = 0,1,2$, which can be written as follows:

$$\sum_{j=0}^{2} c_j(k)\left(\tilde{f}_j(t,k) - e^{-ikL}\tilde{g}_j(t,k)\right) = \hat{q}_0(k) - e^{-ik^3t}\hat{q}(t,k)\,, \qquad (3.18a)$$

$$\sum_{j=0}^{2} c_j(\zeta k)\left(\tilde{f}_j(t,k) - e^{-i\zeta kL}\tilde{g}_j(t,k)\right) = \hat{q}_0(\zeta k) - e^{-ik^3t}\hat{q}(t,\zeta k)\,, \qquad (3.18b)$$

$$\sum_{j=0}^{2} c_j(\zeta^2 k)\left(\tilde{f}_j(t,k) - e^{-i\zeta^2 kL}\tilde{g}_j(t,k)\right) = \hat{q}_0(\zeta^2 k) - e^{-ik^3t}\hat{q}(t,\zeta^2 k)\,. \qquad (3.18c)$$

The knowledge of three of these values will yield a system of three equations, solvable for the three remaining unknown boundary functions. However, as we discussed briefly, and is shown in [22], not *any* three boundary conditions can be prescribed.

3.2.2 Periodic Boundary Conditions

It is well known that if the boundary conditions are periodic then the Fourier transform yields the solution as an infinite discrete series over the eigenvalues of the differential operator. In this section we outline the classical derivation and verify that this is the same as the series representation obtained via the Fokas transform method.

We consider the third order linear evolution PDE

$$q_t(x,t) + q_{xxx}(x,t) = 0\,, \quad t>0\,, \quad x\in[0,L]\,, \tag{3.19a}$$

$$q(x,0) = q_0(x)\,, \quad x\in[0,L]\,, \tag{3.19b}$$

with the periodic boundary conditions $q_x^{(j)}(0,t) = q_x^{(j)}(L,t)$ for $j=0,1,2$.

Solution by Separation of Variables

The solution $q(x,t)$ of the PDE is found straightforwardly using separation of variables. We begin by assuming that the solution can be expressed in the form $q(x,t) = X(x)\,T(t)$, where $X(x)$ is a function purely of x and $T(t)$ is a function purely of t. Substitution into the PDE yields the two ODEs given by

$$\frac{\mathrm{d}^3X(x)}{\mathrm{d}x^3} + \lambda X(x) = 0\,, \qquad \frac{\mathrm{d}T(t)}{\mathrm{d}t} - \lambda T(t) = 0\,. \tag{3.20}$$

for some constant λ. Substituting $\lambda = -p^3$, and solving for $X(x)$ we find

$$X(x) = Ae^{px} + Be^{\zeta px} + Ce^{\zeta^2 px}\,, \qquad \zeta = e^{\frac{2\pi i}{3}}\,,$$

for some constants A, B and C to be determined from the imposition of the periodic boundary conditions $X_x^{(j)}(0) = X_x^{(j)}(L)$, $j=0,1,2$. The system of equations that results is solvable for A, B and C and given by

$$A\left(1-e^{pL}\right) + B\left(1-e^{\zeta pL}\right) + C\left(1-e^{\zeta^2 pL}\right) \;=\; 0\,, \tag{3.21}$$

$$A\left(1-e^{pL}\right) + \zeta B\left(1-e^{\zeta pL}\right) + \zeta^2 C\left(1-e^{\zeta^2 pL}\right) \;=\; 0\,, \tag{3.22}$$

$$A\left(1-e^{pL}\right) + \zeta^2 B\left(1-e^{\zeta pL}\right) + \zeta C\left(1-e^{\zeta^2 pL}\right) \;=\; 0\,,$$

and adding all three equations yields the expression

$$3A\left(1-e^{pL}\right) = 0\,.$$

Hence either $A=0$ or $1-e^{pL}=0$. Let us assume that $A\neq 0$. Then $p = ik_n$ where $k_n = \frac{2n\pi}{L}$, $n\in\mathbb{Z}$ which implies that $\lambda = ik_n^3$. It follows, from (3.21) and (3.22) that $B=0$ and $C=0$ and therefore the solution $X_n(x)$ is given by

$$X_n(x) = A_n e^{ik_n x}\,, \qquad k_n = \frac{2n\pi}{L}\,,$$

for arbitrary A_n. Finally, the value $\lambda = ik_n^3$ is substituted into the ODE for $T(t)$, given by (3.20), and solved to give

$$T_n(t) = D_n e^{ik_n^3 t}, \qquad n \in \mathbb{Z},$$

for some arbitrary D_n. The solution for $q(x,t)$ is therefore given by the infinite discrete series

$$q(x,t) = \sum_{n=1}^{\infty} a_n e^{ik_n x + ik_n^3 t},$$

where $a_n = A_n D_n$. Since the functions $e^{ik_n x}$ form a complete set of orthogonal functions on $[0, L]$ we conclude that

$$a_n = \frac{1}{L}\int_0^L e^{-ik_n x} q_0(x)\,\mathrm{d}x = \frac{1}{L}\hat{q}_0(k_n).$$

Therefore the solution $q(x,t)$ is given by

$$q(x,t) = \frac{1}{L}\sum_{n=1}^{\infty} e^{ik_n x + ik_n^3 t}\hat{q}_0(k_n). \tag{3.23}$$

The Transform Approach

We now use the transform approach to derive the solution (3.23) of the PDE given by (3.19). The periodic boundary conditions imply that $\tilde{f}_j(t,k) = \tilde{g}_j(t,k)$, $j = 0, 1, 2$, and therefore the system of global relations is given by

$$\left(1 - e^{-ikL}\right)\left(k^2\tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right) = \hat{q}_0(k) - e^{-ik^3t}\hat{q}(t,k), \tag{3.24}$$

$$\left(1 - e^{-i\zeta kL}\right)\left(\zeta^2k^2\tilde{f}_0(t,k) - i\zeta k\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right) = \hat{q}_0(\zeta k) - e^{-ik^3t}\hat{q}(t,\zeta k), \tag{3.25}$$

$$\left(1 - e^{-i\zeta^2 kL}\right)\left(\zeta k^2\tilde{f}_0(t,k) - i\zeta^2 k\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right) = \hat{q}_0(\zeta^2 k) - e^{-ik^3t}\hat{q}(t,\zeta^2 k), \tag{3.26}$$

which is a system of three equations in terms of the three unknowns $\tilde{f}_0(t,k)$, $\tilde{f}_1(t,k)$ and $\tilde{f}_2(t,k)$. The determinant function $\Delta(k)$ is given by

$$\Delta(k) = 3\left(\zeta - \zeta^2\right)\left(1 - e^{-ikL}\right)\left(1 - e^{-i\zeta kL}\right)\left(1 - e^{-i\zeta^2 kL}\right), \tag{3.27}$$

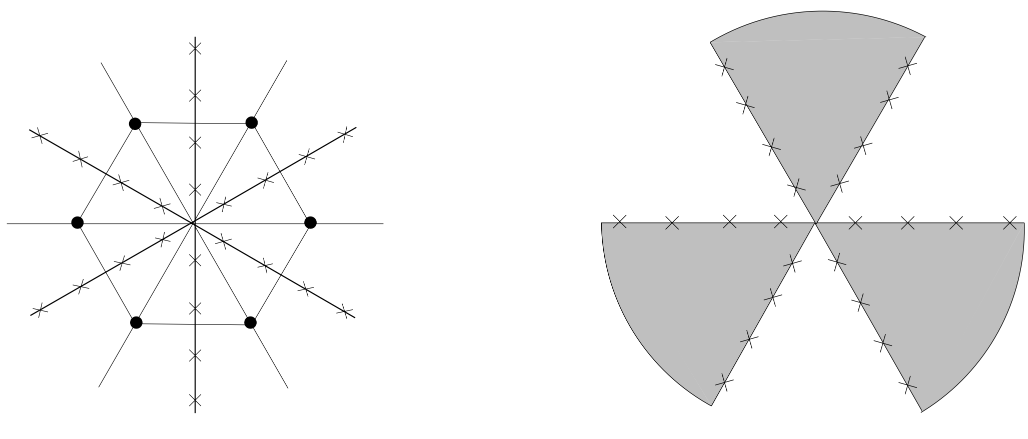
and has zeros at the points k_n, ζk_n and $\zeta^2 k_n$ where $k_n = \frac{2n\pi}{L}$, $n \in \mathbb{Z}$. The domain D is defined by the dispersion relation $\omega(k) = -ik^3$, which is governed only by the PDE.

Hence the domain D is comprised of the three connected components given by

$$
\begin{aligned}
&D^+ = \left\{k \in \mathbb{C} : \tfrac{\pi}{3} \leqslant \arg(k) \leqslant \tfrac{2\pi}{3}\right\} \\
&\left.\begin{aligned}
D_1^- &= \left\{k \in \mathbb{C} : \pi \leqslant \arg(k) \leqslant \tfrac{4\pi}{3}\right\} \\
D_2^- &= \left\{k \in \mathbb{C} : \tfrac{5\pi}{3} \leqslant \arg(k) \leqslant 2\pi\right\}
\end{aligned}\right\} \quad D^- = D_1^- \cup D_2^- \,.
\end{aligned}
\tag{3.28}
$$

It follows immediately that the zeros of the determinant function lie on the six boundary rays of ∂D.

We remark that for this example, the location of the zeros follows immediately from the factorisation of the determinant function, which yields a sum of six exponential terms in k (with no constant term). However, if the terms are expanded, the general theory given in [33] can be used to establish the location of the zeros. This is illustrated in Figure 3.5 where it is shown that the six exponential terms, of the expanded determinant function $\Delta(k)$, imply a hexagonal convex hull in the complex z-plane (where the substitution $z = -ikL$ is applied to $\Delta(k)$). The zeros lie on the six rays that emanate from the origin and perpendicularly bisect the six sides of the hexagon, (Figure 3.5(a)). This in turn implies that the zeros cluster asymptotically along the six boundary rays of D in the complex k-plane, (Figure 3.5(b)).



(a) z-plane ($z = -ikL$). (b) k-plane.

Figure 3.5: The regions $D^\pm$ for the third order problem $q_t(x,t) + q_{xxx}(x,t) = 0$ with the periodic boundary conditions $q_x^{(j)}(0,t) = q_x^{(j)}(L,t)$ for $j = 0,1,2$, and the location of the zeros of the determinant function $\Delta(k) = 3\left(\zeta - \zeta^2\right)\left(1 - e^{-ikL}\right)\left(1 - e^{-i\zeta kL}\right)\left(1 - e^{-i\zeta^2 kL}\right)$.

The Series Representation of the Solution

To obtain the discrete series representation of the solution, we observe that solving equation (3.24) at $k = k_n$, equation (3.25) at $k = \zeta^2 k_n$ or equation (3.26) at $k = \zeta k_n$, for which the left hand side vanishes, we obtain

$$\hat{q}(t, k_n) = e^{ik_n^3 t}\hat{q}_0(k_n)\,. \tag{3.29}$$

To obtain the infinite series representation of the solution, we suppose that $q(x,t)$ can be expanded in terms of the eigenfunctions $e^{ik_n x}$ and written in the form

$$q(x,t) = \sum_{n=1}^{\infty} c_n(t)\, e^{ik_n x}\,. \tag{3.30}$$

Since the set of functions $e^{ik_n x}$ form a complete set of orthogonal functions on $[0, L]$ and satisfy the relationship

$$\int_0^L e^{ik_n x} e^{-ik_m x}\,\mathrm{d}x = \begin{cases} 0\,, & n \neq m\,, \\ L\,, & n = m\,, \end{cases}$$

we conclude that

$$\begin{aligned} c_n(t) &= \frac{1}{L}\int_0^L e^{-ik_n x} q(x,t)\,\mathrm{d}x \\ &= \frac{1}{L}\, e^{ik_n^3 t}\hat{q}_0(k_n)\,. \end{aligned} \tag{3.31}$$

Therefore, the contribution of the integrals involving the unknown functions takes the form of the infinite discrete series

$$q(x,t) = \frac{1}{L}\sum_{n=1}^{\infty} e^{ik_n x + ik_n^3 t}\hat{q}_0(k_n)\,, \tag{3.32}$$

which is consistent with the solution, given by (3.23), obtained from the classical Fourier series analysis.

The Integral Representation of the Solution

The integral representation of the solution is given by

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\Bigg\{ &\int_{-\infty}^{\infty} e^{ikx+ik^3 t}\hat{q}_0(k)\,\mathrm{d}k \\ &- \int_{\partial D^+} e^{ikx+ik^3 t}\left(k^2 \tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right)\mathrm{d}k \\ &- \int_{\partial D^-} e^{ik(x-L)+ik^3 t}\left(k^2 \tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right)\mathrm{d}k \Bigg\} \end{aligned}$$

and expression (3.24) yields the following:

$$k^2 \tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k) = \frac{\hat{q}_0(k)}{1-e^{-ikL}} - e^{-ik^3t}\frac{\hat{q}(t,k)}{1-e^{-ikL}}\,.$$

Hence the integral representation of the solution is given by

$$\begin{aligned} q(x,t) \;=\; & \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx+ik^3t}\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}} - e^{-ik^3t}\frac{\hat{q}(t,k)}{1-e^{-ikL}}\right)\mathrm{d}k \right. \\ & \left. - \int_{\partial D^-} e^{ik(x-L)+ik^3t}\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}} - e^{-ik^3t}\frac{\hat{q}(t,k)}{1-e^{-ikL}}\right)\mathrm{d}k \right\}. \end{aligned}$$

We note that the unknown terms involving the function $\hat{q}(t,k)$ are bounded in D and define meromorphic functions with poles at the zeros of the determinant function $\Delta(k)$ that belong to D. Hence the contributions of these terms is only given by the residues of the function at these points.

Equivalence of the Series and Integral Representations

The integrands of the integrals around ∂D^+ and ∂D^- are analytic and bounded in the domains D_c^+ and D_c^- respectively, implying that the contours of integration around ∂D^+ and ∂D^- can be deformed to the real line. This yields the following expression

$$\begin{aligned} q(x,t) \;=\; & \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{-\infty}^{\infty} e^{ikx+ik^3t}\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}}\right)\mathrm{d}k \right. \\ & \left. + \int_{-\infty}^{\infty} e^{ik(x-L)+ik^3t}\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}}\right)\mathrm{d}k \right\} \\ & + \sum_{\substack{k_n:\\ \Delta(k_n)=0}} \textit{residue contributions}\,. \end{aligned}$$

The integral terms trivially sum to zero. Therefore, the only contribution to the solution is from the series term, that results from the explicit computation of the principal value contributions at the zeros k_n of the determinant function $\Delta(k)$. We let

$$\begin{aligned} p^+(k) \;&=\; e^{ikx+ik^3t}\hat{q}_0(k)\,, \\ p^-(k) \;&=\; e^{ik(x-L)+ik^3t}\hat{q}_0(k)\,, \\ r(k) \;&=\; 1-e^{-ikL}\,. \end{aligned}$$

Therefore the series solution is given by

$$
\begin{aligned}
q(x,t) &= \frac{1}{2\pi}\left\{\pi i\sum_{n=1}^{\infty}\frac{p^{+}(k_n)}{r'(k_n)} + \pi i\sum_{n=1}^{\infty}\frac{p^{-}(k_n)}{r'(k_n)}\right\} \\
&= \frac{1}{2\pi}\left\{\pi i\sum_{n=1}^{\infty}\frac{e^{ik_nx+ik_n^3t}(1+e^{-ik_nL})\hat{q}_0(k_n)}{iLe^{-ik_nL}}\right\} \\
&= \frac{1}{L}\sum_{n=1}^{\infty}e^{ik_nx+ik_n^3t}\hat{q}_0(k_n)\,, \qquad (3.33)
\end{aligned}
$$

which concurs with (3.32).

Remark 3.2.1. The index used in the summation of (3.33), indicates that the residue contributions arise from *all* of the zeros k_n of the determinant function $\Delta(k)$.

3.2.3 Quasi-Periodic Boundary Conditions

We consider now the third order linear evolution PDE, given by (3.19), with the quasi-periodic boundary conditions given by

$$
q(L,t) = \alpha q(0,t)\,, \quad q_x(L,t) = \alpha q_x(0,t)\,, \quad q_{xx}(L,t) = \alpha q_{xx}(0,t)\,,
$$

for some real $\alpha > 0$.

The boundary conditions imply that $\tilde{g}_0(t,k) = \alpha\tilde{f}_0(t,k)$, $\tilde{g}_1(t,k) = \alpha\tilde{f}_1(t,k)$ and $\tilde{g}_2(t,k) = \alpha\tilde{f}_2(t,k)$ and therefore the global relation can be written entirely in terms of the functions $\tilde{f}_j(t,k)$, $j = 0,1,2$, in the form

$$
\left(k^2\tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right)\left(1 - e^{-ikL}\alpha\right) = \hat{q}_0(k) - e^{-ik^3t}\hat{q}(t,k)\,. \qquad (3.34)
$$

This is supplemented by the two additional equations

$$
\begin{aligned}
\left(\zeta^2k^2\tilde{f}_0(t,k) - i\zeta k\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right)\left(1 - e^{-i\zeta kL}\alpha\right) &= \hat{q}_0(\zeta k) - e^{-ik^3t}\hat{q}(t,\zeta k)\,, \qquad (3.35) \\
\left(\zeta k^2\tilde{f}_0(t,k) - i\zeta^2k\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right)\left(1 - e^{-i\zeta^2kL}\alpha\right) &= \hat{q}_0(\zeta^2k) - e^{-ik^3t}\hat{q}(t,\zeta^2k)\,, \qquad (3.36)
\end{aligned}
$$

where $\zeta = e^{\frac{2\pi i}{3}}$ and form the matrix system given by

$$\begin{pmatrix} 1-e^{-ikL}\alpha & 1-e^{-ikL}\alpha & 1-e^{-ikL}\alpha \\ \zeta^2(1-e^{-i\zeta kL}\alpha) & \zeta(1-e^{-i\zeta kL}\alpha) & 1-e^{-i\zeta kL}\alpha \\ \zeta(1-e^{-i\zeta^2 kL}\alpha) & \zeta^2(1-e^{-i\zeta^2 kL}\alpha) & 1-e^{-i\zeta^2 kL}\alpha \end{pmatrix} \begin{pmatrix} k^2\tilde{f}_0(t,k) \\ -ik\tilde{f}_1(t,k) \\ -\tilde{f}_2(t,k) \end{pmatrix}$$

$$= \begin{pmatrix} \hat{q}_0(k) \\ \hat{q}_0(\zeta k) \\ \hat{q}_0(\zeta^2 k) \end{pmatrix} - \begin{pmatrix} e^{-ik^3t}\hat{q}(t,k) \\ e^{-ik^3t}\hat{q}(t,\zeta k) \\ e^{-ik^3t}\hat{q}(t,\zeta^2 k) \end{pmatrix}.$$

The determinant function $\Delta(k)$ is given by

$$\begin{aligned} \Delta(k) &= 3\left(\zeta-\zeta^2\right)\left(1-e^{-ikL}\alpha\right)\left(1-e^{-i\zeta kL}\alpha\right)\left(1-e^{-i\zeta^2 kL}\alpha\right) \\ &= 3\left(\zeta-\zeta^2\right)\Big\{1-\alpha^3-\alpha\Big(e^{-ikL}+e^{-i\zeta kL}+e^{-i\zeta^2 kL}\Big) \\ &\qquad +\alpha^2\left(e^{ikL}+e^{i\zeta kL}+e^{i\zeta^2 kL}\right)\Big\}, \end{aligned}$$

and has zeros at the points k_n, ζk_n and $\zeta^2 k_n$ where

$$k_n = -\frac{i\ln\alpha \pm 2n\pi}{L}, \quad n \in \mathbb{Z}.$$

Remark 3.2.2. We observe that the presence of the constant term $1-\alpha^3$, in the expression for the determinant function $\Delta(k)$, prevents the use of Levin's Theorem (Theorem 1.3.22) for locating the zeros in the complex k-plane. Indeed, $1 = e^{0\times k}$, but α^3 cannot be written as an exponential of k.

The zeros k_n depend upon the value of α that is prescribed and in general, there are three cases to consider:

- If $0 < \alpha < 1$ then the zeros lie on the three straight-line rays that run parallel to the boundary rays ∂D, such that infinitely many zeros lie outside of the domain D (Figure 3.6).
- If $\alpha = 1$ then the boundary conditions are periodic and the zeros lie on the boundary rays of D (Section 3.2.2, Figure 3.5(b)).
- If $\alpha > 1$ then the zeros lie on the three straight-line rays that run parallel to the boundary rays ∂D, such that infinitely many zeros lie inside of the domain D (Figure 3.7).

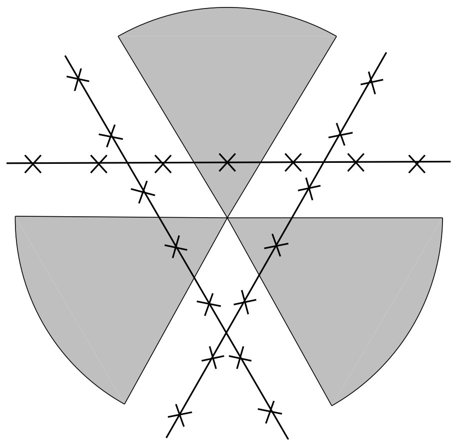


Figure 3.6: The regions $D^{\pm}$ for the third order problem $q_t(x,t) + q_{xxx}(x,t) = 0$ with quasi-periodic boundary conditions $q_x^j(L,t) = \alpha q_x^j(0,t)$ for $j = 0,1,2$ and $0 < \alpha < 1$, and the location of the zeros of the determinant function $\Delta(k) = 3\left(\zeta - \zeta^2\right)\left(1 - e^{-ikL}\alpha\right)\left(1 - e^{-i\zeta kL}\alpha\right)\left(1 - e^{-i\zeta^2 kL}\alpha\right)$.

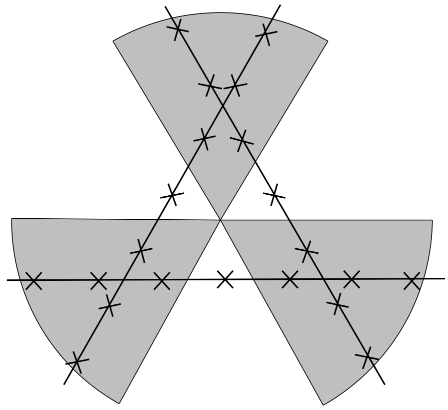


Figure 3.7: The regions $D^{\pm}$ for the third order problem $q_t(x,t) + q_{xxx}(x,t) = 0$ with quasi-periodic boundary conditions $q_x^j(L,t) = \alpha q_x^j(0,t)$ for $j = 0,1,2$ and $\alpha > 1$, and the location of the zeros of the determinant function $\Delta(k) = 3\left(\zeta - \zeta^2\right)\left(1 - e^{-ikL}\alpha\right)\left(1 - e^{-i\zeta kL}\alpha\right)\left(1 - e^{-i\zeta^2 kL}\alpha\right)$.

The Series Representation of the Solution

The series representation of the solution $q(x,t)$ can be obtained directly from any of the three global relations. Solving equation (3.34) at $k = k_n$, or equation (3.35) at $k = \zeta^2 k_n$, or equation (3.36) at $k = \zeta k_n$, for which the left-hand side vanishes, we obtain

$$\hat{q}(t,k_n) = e^{ik_n^3 t}\hat{q}_0(k_n)\,. \tag{3.37}$$

This is the same as expression (3.29), except here $k_n = -\frac{i\ln\alpha \pm 2n\pi}{L}$, $n \in \mathbb{Z}$. Therefore, following the periodic example on Page 93, we suppose that $q(x,t)$ can be expanded in terms of the eigenfunctions $e^{ik_n x}$, and by (3.30) and (3.31), conclude that the solution takes the form of an infinite discrete series, written in terms of *all* of the zeros k_n, in the form

$$q(x,t) = \frac{1}{L} \sum_{n=1}^{\infty} e^{ik_n x + ik_n^3 t} \hat{q}_0(k_n) \,, \tag{3.38}$$

for all $\alpha > 0$.

In what follows, we derive the integral representations of the solution, for the cases $0 < \alpha < 1$ and $\alpha > 1$ separately, and prove their equivalence to (3.38), and show that in both cases, the contours of integration of the integral representation of the solution can be deformed, and the solution expressed entirely in terms of the residue contributions at the zeros k_n of the determinant function $\Delta(k)$.

The Integral Representation of the Solution

The integral representation of the solution is given by (2.56) where the domains $D^{\pm}$ in the complex k-plane are defined by (3.28). Hence

$$\begin{aligned} q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ & \int_{-\infty}^{\infty} e^{ikx+ik^3 t}\hat{q}_0(k)\,\mathrm{d}k \\ & - \int_{\partial D^+} e^{ikx+ik^3 t}\left(k^2 \tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right)\mathrm{d}k \\ & - \int_{\partial D^-} e^{ik(x-L)+ik^3 t}\alpha\left(k^2 \tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right)\mathrm{d}k \Bigg\}\,, \end{aligned}$$

and expression (3.34), yields the following:

$$k^2 \tilde{f}_0(t,k) - ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k) = \frac{\hat{q}_0(k)}{1-e^{-ikL}\alpha} - e^{-ik^3 t}\frac{\hat{q}(t,k)}{1-e^{-ikL}\alpha}\,.$$

Hence the integral representation of the solution is given by

$$\begin{aligned} q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ & \int_{-\infty}^{\infty} e^{ikx+ik^3 t}\hat{q}_0(k)\,\mathrm{d}k \\ & - \int_{\partial D^+} e^{ikx+ik^3 t}\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}\alpha} - e^{-ik^3 t}\frac{\hat{q}(t,k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k \\ & - \int_{\partial D^-} e^{ik(x-L)+ik^3 t}\alpha\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}\alpha} - e^{-ik^3 t}\frac{\hat{q}(t,k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k \Bigg\}\,. \end{aligned} \tag{3.39}$$

Case 1: $0 < \alpha < 1$

Proposition 3.2.3. *The terms involving the unknown function $\hat{q}(t,k)$ do not contribute to the integral representation of the solution, given by (3.39).*

Proof. Consider the integral around ∂D^+ of the term involving the unknown function $\hat{q}(t,k)$, given explicitly by

$$\int_{\partial D^+} e^{ikx}\left(\frac{\hat{q}(t,k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k\,,$$

and let $k \in D^+$. It follows that e^{ikL} is bounded and e^{-ikL} is unbounded. Therefore, the asymptotic behaviour of the integrand is given by

$$e^{ikx}e^{ikL}\hat{q}(t,k) = e^{ikx}\int_0^L e^{ik(L-x')}q(x',t)\,\mathrm{d}x'\,.$$

All of the exponential terms in this expression are bounded as $k \to \infty$ and the proof is complete. $\square$

Remark 3.2.4. A similar argument can be used for the proof around ∂D^-.

Therefore, the integral representation of the solution is given by

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\Bigg\{\int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx+ik^3t}\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k \\ &\quad - \int_{\partial D^-} e^{ik(x-L)+ik^3t}\alpha\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k\Bigg\}. \end{aligned} \tag{3.40}$$

Equivalence of the Series and Integral Representations

The proof that the integral representation of the solution, given by (3.40), and the series solution, given by (3.38), are equivalent, is straightforward, and offers an alternative derivation of the series solution.

Proposition 3.2.5. *The integrands of the integrals around ∂D^+ and ∂D^-, in expression (3.40) are analytic and bounded in the domains D_c^+ and D_c^- respectively, implying that the contours of integration around ∂D^+ and ∂D^- can be deformed to the real line.*

Proof. i.) Consider the integral around ∂D^+, given by

$$\int_{\partial D^+} e^{ikx+ik^3t}\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k\,,$$

and let k be such that $0 < \arg(k) < \frac{\pi}{3}$. Then for k in this wedge, ζk will be such that $\frac{2\pi}{3} < \arg(\zeta k) < \pi$ and $\zeta^2 k$ will be such that $\frac{4\pi}{3} < \arg(\zeta^2 k) < \frac{5\pi}{3}$. Hence

- e^{ikx+ik^3t} is bounded,
- e^{ikL}, $e^{i\zeta kL}$ and $e^{-i\zeta^2 kL}$ are bounded,
- e^{-ikL}, $e^{-i\zeta kL}$ and $e^{i\zeta^2 kL}$ are unbounded.

Since e^{-ikL} is unbounded, the asymptotic behaviour of the integrand is given by

$$e^{ikx+ik^3t}e^{ikL}\hat{q}_0(k) = e^{ikx+ik^3t}\int_0^L e^{ik(L-x')}q_0(x')\,\mathrm{d}x' .$$

All of the exponential terms in this expression are bounded as $k \to \infty$, implying that the contour of integration around ∂D^+ can be deformed to the real line.

Remark 3.2.6. We remark that a similar argument can be used for the proof that the integrand is analytic and bounded for k such that $\frac{2\pi}{3} < \arg(k) < \pi$.

ii.) Consider the integral around ∂D^-, given by

$$\int_{\partial D^-} e^{ik(x-L)+ik^3t}\alpha\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k ,$$

and let $k \in D_c^-$. Then ζk is such that $0 < \arg(\zeta k) < \frac{\pi}{3}$ and $\zeta^2 k$ is such that $\frac{2\pi}{3} < \arg(\zeta^2 k) < \pi$. Hence

- $e^{ik(x-L)+ik^3t}$ is bounded,
- e^{-ikL}, $e^{i\zeta kL}$ and $e^{i\zeta^2 kL}$ are bounded,
- e^{ikL}, $e^{-i\zeta kL}$ and $e^{-i\zeta^2 kL}$ are unbounded.

Since e^{-ikL} is bounded, the asymptotic behaviour of the integrand is given by

$$e^{ik(x-L)+ik^3t}\hat{q}_0(k) = e^{ik(x-L)+ik^3t}\int_0^L e^{-ikx'}q_0(x')\,\mathrm{d}x' .$$

All of the exponential terms in this expression are bounded as $k \to \infty$ and the proof is complete.

□

Deforming the contours of integration in (3.40) to the real line gives the expression

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\Bigg\{\int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{-\infty}^{\infty} e^{ikx+ik^3t}\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k \\ &\quad + \int_{-\infty}^{\infty} e^{ik(x-L)+ik^3t}\alpha\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k\Bigg\} \\ &\quad + \sum_{\substack{k_n: \\ \Delta(k_n)=0}} \textit{residue contributions} . \end{aligned}$$

The integral terms trivially sum to zero, proving that the only contribution to the solution is the series term due to the explicit computation of the principal value contributions at the zeros k_n of the determinant function $\Delta(k)$. We let

$$\begin{aligned} p^+(k) &= e^{ikx+ik^3t}\hat{q}_0(k)\,, \\ p^-(k) &= e^{ik(x-L)+ik^3t}\alpha\hat{q}_0(k)\,, \\ r(k) &= 1-e^{-ikL}\alpha\,. \end{aligned}$$

Therefore the series solution, computed over all of the zeros k_n, is given by

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\left\{\pi i\sum_{n=1}^{\infty}\frac{p^+(k_n)}{r'(k_n)}+\pi i\sum_{n=1}^{\infty}\frac{p^-(k_n)}{r'(k_n)}\right\} \\ &= \frac{1}{2\pi}\left\{\pi i\sum_{n=1}^{\infty}\frac{e^{ik_nx+ik_n^3t}\left(1+e^{-ik_nL}\alpha\right)\hat{q}_0(k_n)}{iL\alpha e^{-ik_nL}}\right\} \\ &= \frac{1}{L}\sum_{n=1}^{\infty}e^{ik_nx+ik_n^3t}\hat{q}_0(k_n)\,, \end{aligned}$$

which coincides with (3.38) and the proof is complete.

Case 2: $\alpha > 1$

The Integral Representation of the Solution

It follows from Proposition 3.2.5 that the contours of integration around $\partial D^{\pm}$, whose integrands involve the term $\hat{q}_0(k)$, can be deformed to the real line, and because in the case $\alpha > 1$, infinitely many zeros lie inside of the domain D, it follows that no residue contributions arise from this deformation. Hence, the integral representation of the solution is given by

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\Bigg\{ &\int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{-\infty}^{\infty} e^{ikx+ik^3t}\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k \\ &+\int_{-\infty}^{\infty} e^{ik(x-L)+ik^3t}\alpha\left(\frac{\hat{q}_0(k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k + \int_{\partial D^+} e^{ikx}\left(\frac{\hat{q}(t,k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k \\ &+\int_{\partial D^-} e^{ik(x-L)}\alpha\left(\frac{\hat{q}(t,k)}{1-e^{-ikL}\alpha}\right)\mathrm{d}k\Bigg\}. \end{aligned}$$

The integrals along $\mathbb{R}$ sum to zero, and hence

$$q(x,t) = \frac{1}{2\pi}\left\{ \int_{\partial D^+} e^{ikx}\left(\frac{\hat{q}(t,k)}{1-e^{-ikL}\alpha}\right) \mathrm{d}k + \int_{\partial D^-} e^{ik(x-L)}\alpha\left(\frac{\hat{q}(t,k)}{1-e^{-ikL}\alpha}\right) \mathrm{d}k \right\}. \tag{3.41}$$

This representation of the solution involves $q(x,t)$ on the RHS, however, we now show how to overcome this, and obtain the solution as an infinite discrete series.

The integrands of the integrals around ∂D^+ and ∂D^-, in the integral representation of the solution given by (3.41), are analytic and bounded in D^+ and D^- respectively. The denominators however, have zeros in D^+ and D^- respectively and it is these that give rise to a residue contribution, which yields the series representation, given by (3.38). Let

$$\begin{aligned} p^+(k) &= e^{ikx}\hat{q}(t,k)\,, \\ p^-(k) &= e^{ik(x-L)}\alpha\hat{q}(t,k)\,, \\ r(k) &= 1-e^{-ikL}\alpha\,. \end{aligned}$$

Therefore, the series solution, computed over all of the zeros k_n, is given by

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\left\{\pi i\sum_{n=1}^{\infty}\frac{p^+(k_n)}{r'(k_n)} + \pi i\sum_{n=1}^{\infty}\frac{p^-(k_n)}{r'(k_n)}\right\} \\ &= \frac{1}{2\pi}\left\{\pi i\sum_{n=1}^{\infty}\frac{e^{ik_nx}(1+\alpha e^{-ik_nL})\hat{q}(t,k_n)}{iL\alpha e^{-ik_nL}}\right\} \\ &= \frac{1}{L}\sum_{n=1}^{\infty} e^{ik_nx}\hat{q}(t,k_n)\,. \end{aligned} \tag{3.42}$$

Via expression (3.37), we substitute $\hat{q}(t,k_n) = e^{ik_n^3t}\hat{q}_0(k_n)$, and conclude that (3.42) is equivalent to (3.38).

3.2.4 An Example of More General Coupled Boundary Conditions

We consider now the third order linear evolution PDE given by (3.10) with the coupled boundary conditions

$$q(0,t) = 0\,, \quad q(L,t) = 0\,, \quad q_x(L,t) = \alpha q_x(0,t)\,, \tag{3.43}$$

for some real $\alpha \neq 0$. These boundary conditions were considered in [53], and it was shown that for $|\alpha| < 1$, one obtains a well-posed problem. Indeed, in this case the x-differential operator is dissipative in nature, hence the energy (i.e., the L_2 norm of the solution) decreases with time. To show this we multiply (3.10a) throughout by $q(x,t)$ and integrate with respect to x over the domain, to obtain

$$\int_0^L \left(q(x,t)q_t(x,t) + q(x,t)q_{xxx}(x,t)\right) \mathrm{d}x = 0\,.$$

Integration by parts, along with the imposition of the boundary conditions, yields

$$\frac{\mathrm{d}}{\mathrm{d}t}||q(x,t)||_2^2 = q_x^2(L,t) - q_x^2(0,t) = \left(\alpha^2 - 1\right) q_x^2(0,t)\,. \tag{3.44}$$

Hence provided $|\alpha| < 1$ and $q_x(0,t)$ does not vanish, the energy will decrease.

This problem has been studied in detail by Zhang and Russell, [53, 42], and we include the following proposition which proves that the solution is expressible in terms of the appropriate eigenfunctions basis. The proof is given in [53].

Proposition 3.2.7. *Assume that $\alpha \neq 0$. Then the operator $L = -\frac{\partial^3}{\partial x^3}$ is a discrete spectral operator. The operator L and its adjoint $L^* = \frac{\partial^3}{\partial x^3}$, on the bounded domain $[0, L]$ have complete sets of eigenvectors, given by*

$$\{\phi_k(x) : -\infty < k < \infty\}\,, \quad \{\psi_k(x) : -\infty < k < \infty\}\,,$$

respectively, which normalised to satisfy the bi-orthogonality condition

$$\langle \phi_k(x), \psi_j(x)\rangle_{L^2([0,L])} = \int_0^L \psi_j^*(x)\,\phi_k(x)\,\mathrm{d}x = \delta_{k,j}\,,$$

where $\delta_{k,j}$ is the Kronecker delta, form the Riesz bases $\{\phi_k(x)\}$ and $\{\psi_j(x)\}$ for $L^2([0,L])$. The corresponding eigenvalues of L have the asymptotic form

$$\lambda_k = \left(8\pi^3k^3 + O(k^2)\right) i - 12\pi^2k^2r + O(k), \quad k \to \infty\,,$$

where

$$r = -\log\left|\frac{1+2\alpha}{2+\alpha}\right| > 0\,.$$

We remark that the crucial property is the completeness of the families of eigenfunctions. In this case, this follows from rather complex general results in functional analysis [9, 10, 11].

It follows from Proposition 3.2.7, that the system given by (3.10) and (3.43) has a unique solution which can be represented as an infinite discrete series in terms of the eigenfunction basis, and in what follows, we derive the series representation of the solution using the transform method. For the analysis that is to follow we do not need to assume that $|\alpha| < 1$.

We use the Fokas transform method and derive an integral representation of the solution involving complex contours. We then show that this alternative form is equivalent to a discrete series representation. Moreover we show that when coupled boundary conditions are prescribed, the zeros of the determinant function $\Delta(k)$ always lie on the six boundary rays of D, and this implies the existence of a series term in the representation of the solution.

Recall the system of global relations given by (3.18). The imposition of the boundary conditions implies that $\tilde{f}_0(t,k) = 0$, $\tilde{g}_0(t,k) = 0$ and $\tilde{g}_1(t,k) = \alpha \tilde{f}_1(t,k)$. Hence the global relations form the following system, solvable for the three unknowns $\tilde{f}_1(t,k)$, $\tilde{f}_2(t,k)$ and $\tilde{g}_2(t,k)$:

$$\begin{pmatrix} 1 - e^{-ikL}\alpha & 1 & e^{-ikL} \\ \zeta\left(1 - e^{-i\zeta kL}\alpha\right) & 1 & e^{-i\zeta kL} \\ \zeta^2\left(1 - e^{-i\zeta^2 kL}\alpha\right) & 1 & e^{-i\zeta^2 kL} \end{pmatrix} \begin{pmatrix} -ik\tilde{f}_1(t,k) \\ -\tilde{f}_2(t,k) \\ \tilde{g}_2(t,k) \end{pmatrix} = \begin{pmatrix} \hat{q}_0(k) \\ \hat{q}_0(\zeta k) \\ \hat{q}_0(\zeta^2 k) \end{pmatrix} - \begin{pmatrix} e^{-ik^3 t}\hat{q}(t,k) \\ e^{-ik^3 t}\hat{q}(t,\zeta k) \\ e^{-ik^3 t}\hat{q}(t,\zeta^2 k) \end{pmatrix}. \tag{3.45}$$

The determinant function $\Delta(k)$ is given by

$$\Delta(k) = \left(\zeta - \zeta^2\right)\left\{e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL} + \alpha\left(e^{ikL} + \zeta e^{i\zeta kL} + \zeta^2 e^{i\zeta^2 kL}\right)\right\},$$

which is a sum of the same six exponential terms as that of the determinant function, given by (3.27), for the periodic case. Therefore, the zeros $k = k_n$, such that $\Delta(k_n) = 0$, are located according to Levin's Theorem (Theorem 1.3.22) and found to cluster asymptotically along the boundary rays of D such that $\arg(k) = \frac{n\pi}{3}$, $n = 0, 1, ..., 5$. The location of the zeros is given in Figure 3.5(b).

The Integral Representation of the Solution

The integral representation of the solution is given by (2.56) where the domains $D^{\pm}$ in the complex k-plane are defined by (3.28). Hence

$$
\begin{aligned}
q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ &\int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx+ik^3t}\left(-ik\tilde{f}_1(t,k)-\tilde{f}_2(t,k)\right)\mathrm{d}k \\
&- \int_{\partial D^-} e^{ik(x-L)+ik^3t}\left(-ik\alpha\tilde{f}_1(t,k)-\tilde{g}_2(t,k)\right)\mathrm{d}k\Bigg\}. \qquad (3.46)
\end{aligned}
$$

The system of global relations, given by (3.45), is solved for the three unknown boundary values. This yields the following expressions in terms of the known functions $\hat{q}_0(k)$, $\hat{q}_0(\zeta k)$ and $\hat{q}_0(\zeta^2 k)$:

$$
\begin{aligned}
-ik\tilde{f}_1(t,k)-\tilde{f}_2(t,k) \;=\; \frac{(1-\zeta)}{\Delta(k)}\Bigg\{ &\hat{q}_0(k)\left(\zeta\alpha e^{ikL}+\zeta^2 e^{-i\zeta kL}+e^{-i\zeta^2 kL}\right) \\
&-\zeta^2\hat{q}_0(\zeta k)\left(e^{-ikL}-\alpha e^{i\zeta kL}\right) \\
&-\hat{q}_0(\zeta^2 k)\left(e^{-ikL}-\alpha e^{i\zeta^2 kL}\right)\Bigg\} \qquad (3.47)
\end{aligned}
$$

and

$$
\begin{aligned}
-ik\alpha\tilde{f}_1(t,k)-\tilde{g}_2(t,k) \;=\; \frac{(1-\zeta)}{\Delta(k)}\Bigg\{ &\hat{q}_0(k)\left(-\zeta-\alpha e^{-i\zeta kL}-\zeta^2\alpha e^{-i\zeta^2 kL}\right) \\
&-\zeta^2\hat{q}_0(\zeta k)\left(1-\alpha e^{-i\zeta^2 kL}\right) \\
&-\hat{q}_0(\zeta^2 k)\left(1-\alpha e^{-i\zeta kL}\right)\Bigg\}. \qquad (3.48)
\end{aligned}
$$

Substituting these expressions into (3.46) achieves the explicit integral representation of the solution:

$$
\begin{aligned}
q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ &\int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k \\
&-\int_{\partial D^+} e^{ikx+ik^3t}\frac{(1-\zeta)}{\Delta(k)}\Big\{\hat{q}_0(k)\Big(\zeta\alpha e^{ikL}+\zeta^2 e^{-i\zeta kL}+e^{-i\zeta^2 kL}\Big) \\
&\qquad -\zeta^2\hat{q}_0(\zeta k)\Big(e^{-ikL}-\alpha e^{i\zeta kL}\Big)-\hat{q}_0(\zeta^2 k)\Big(e^{-ikL}-\alpha e^{i\zeta^2 kL}\Big)\Big\}\,\mathrm{d}k \\
&-\int_{\partial D^-} e^{ik(x-L)+ik^3t}\frac{(1-\zeta)}{\Delta(k)}\Big\{\hat{q}_0(k)\Big(-\zeta-\alpha e^{-i\zeta kL}-\zeta^2\alpha e^{-i\zeta^2 kL}\Big) \\
&\qquad -\zeta^2\hat{q}_0(\zeta k)\Big(1-\alpha e^{-i\zeta^2 kL}\Big)-\hat{q}_0(\zeta^2 k)\Big(1-\alpha e^{-i\zeta kL}\Big)\Big\}\,\mathrm{d}k\Bigg\}. \qquad (3.49)
\end{aligned}
$$

We observe that, according to Proposition 2.4.3(ii), the unknown terms do not contribute to the integral representation of the solution.

Equivalence of the Series and Integral Representations

In this section we show that we can deform the contour ∂D to the real axis and realise the integral representation, given by (3.49), entirely on $\mathbb{R}$. In the process we achieve a series term due to the explicit computation of the principal value contributions at the zeros of the determinant function $\Delta(k)$.

Proposition 3.2.8. *The integrands, of the integrals around ∂D^+ and ∂D^-, are analytic and bounded, as $k \to \infty$, for $k \in D_c^+$ and $k \in D_c^-$ respectively.*

Proof. We prove the case only for the integral around ∂D^+, (the proof for the integral around ∂D^- follows analogously), and consider the integrand given by

$$e^{ikx+ik^3t}\frac{(1-\zeta)}{\Delta(k)}\Big\{\hat{q}_0(k)\Big(\zeta\alpha e^{ikL}+\zeta^2e^{-i\zeta kL}+e^{-i\zeta^2kL}\Big)-\zeta^2\hat{q}_0(\zeta k)\Big(e^{-ikL}-\alpha e^{i\zeta kL}\Big)$$
$$-\hat{q}_0(\zeta^2k)\Big(e^{-ikL}-\alpha e^{i\zeta^2kL}\Big)\Big\},$$

where $\Delta(k)=(\zeta-\zeta^2)\Big\{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2e^{-i\zeta^2kL}+\alpha\Big(e^{ikL}+\zeta e^{i\zeta kL}+\zeta^2e^{i\zeta^2kL}\Big)\Big\}$.

We divide the proof into two parts. In the first we prove that the integrand around ∂D^+, in (3.49), is analytic and bounded as $k\to\infty$, for k such that $\frac{\pi}{6}<\arg(k)<\frac{\pi}{3}$, and in the second part we prove that the integrand is analytic and bounded as $k\to\infty$, for k such that $0<\arg(k)<\frac{\pi}{6}$. (A similar argument can be used for the proof that the integrand is analytic and bounded as $k\to\infty$ for k such that $\frac{2\pi}{3}<\arg(k)<\pi$).

i.) Consider the wedge such that $\frac{\pi}{6}<\arg(k)<\frac{\pi}{3}$. Then for k in this wedge

- e^{ikx+ik^3t} is bounded,
- e^{ikL}, $e^{i\zeta kL}$ and $e^{-i\zeta^2kL}$ are bounded,
- e^{-ikL}, $e^{-i\zeta kL}$ and $e^{i\zeta^2kL}$ are unbounded.

Therefore asymptotically, the denominator behaves like $(\zeta-\zeta^2)\Big(e^{-ikL}+\zeta e^{-i\zeta kL}+\alpha\zeta^2e^{i\zeta^2kL}\Big)$. To establish which of these terms is dominant we substitute $k=k_R+ik_I$. This implies that

$$|e^{-ikL}|=e^{k_IL},\qquad |e^{-i\zeta kL}|=e^{\frac{1}{2}\left(\sqrt{3}k_R-k_I\right)L},\qquad |e^{i\zeta^2kL}|=e^{\frac{1}{2}\left(\sqrt{3}k_R+k_I\right)L}.\quad(3.50)$$

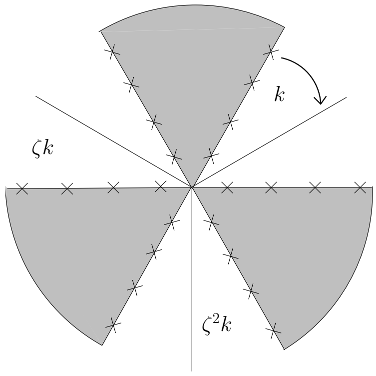


Figure 3.8: The deformation of the contour ∂D^+ for k such that $\frac{\pi}{6} < \arg(k) < \frac{\pi}{3}$.

Now, $\frac{\pi}{6} < \arg(k) < \frac{\pi}{3}$ implies that $k_R < \sqrt{3}k_I < 3k_R$. Therefore

$$\frac{1}{2}\left(\sqrt{3}k_R - k_I\right) < k_I < \frac{1}{2}\left(\sqrt{3}k_R + k_I\right) ,$$

and we conclude that the dominant term in the denominator is given by $(\zeta - \zeta^2)\, \alpha\zeta^2 e^{i\zeta^2 kL}$. Therefore, the asymptotic behaviour of the integrand is given by

$$\frac{e^{ikx+ik^3t}}{\alpha} e^{-i\zeta^2 kL}\Big\{\hat{q}_0(k)\Big(\zeta\alpha e^{ikL} + \zeta^2 e^{-i\zeta kL} + e^{-i\zeta^2 kL}\Big) - \zeta^2\hat{q}_0(\zeta k)\Big(e^{-ikL} - \alpha e^{i\zeta kL}\Big)$$
$$- \hat{q}_0(\zeta^2 k)\Big(e^{-ikL} - \alpha e^{i\zeta^2 kL}\Big)\Big\} .$$

By definition (2.22) for D, it follows immediately that e^{ik^3t} is bounded in D_c. The behaviour of the x-exponential terms are given as follows:

(a) $e^{ikx}e^{-i\zeta^2 kL}e^{-ikx}e^{ikL} + e^{ikx}e^{-i\zeta^2 kL}e^{-ikx}e^{-i\zeta kL} + e^{ikx}e^{-i\zeta^2 kL}e^{-ikx}e^{-i\zeta^2 kL}$
$= e^{-i\zeta^2 kL}e^{ikL} + e^{ikL} + e^{-i\zeta^2 kL}e^{-i\zeta^2 kL}$,

(b) $e^{ikx}e^{-i\zeta^2 kL}e^{-i\zeta kx}e^{-ikL} + e^{ikx}e^{-i\zeta^2 kL}e^{-i\zeta kx}e^{i\zeta kL}$
$= e^{ikx}e^{i\zeta k(L-x)} + e^{ikx}e^{-i\zeta^2 kL}e^{i\zeta k(L-x)}$,

(c) $e^{ikx}e^{-i\zeta^2 kL}e^{-i\zeta^2 kx}e^{-ikL} + e^{ikx}e^{-i\zeta^2 kL}e^{i\zeta^2 kL}e^{-i\zeta^2 kx}$
$= e^{ikx}e^{-i\zeta^2 kx}e^{i\zeta kL} + e^{ikx}e^{-i\zeta^2 kx}$.

All of the terms in these expressions are analytic and bounded as $k \to \infty$, for k such that $\frac{\pi}{6} < \arg(k) < \frac{\pi}{3}$ and the proof is complete.

ii.) Consider the wedge such that $0 < \arg(k) < \frac{\pi}{6}$. Then for k in this wedge

- e^{ikx+ik^3t} is bounded,

- e^{ikL}, $e^{i\zeta kL}$ and $e^{-i\zeta^2 kL}$ are bounded,
- e^{-ikL}, $e^{-i\zeta kL}$ and $e^{i\zeta^2 kL}$ are unbounded.

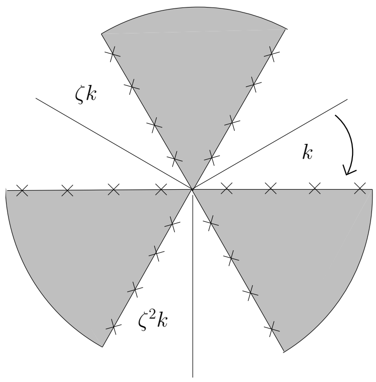


Figure 3.9: The deformation of the contour ∂D^+ for k such that $0 < \arg(k) < \frac{\pi}{6}$.

Therefore, as in case (i), the denominator behaves like $(\zeta - \zeta^2)\left(e^{-ikL} + \zeta e^{-i\zeta kL} + \alpha\zeta^2 e^{i\zeta^2 kL}\right)$ and expressions (3.50) apply. Now, $0 < \arg(k) < \frac{\pi}{6}$ implies that $0 < \sqrt{3}k_I < k_R$. Therefore $k_I < \sqrt{3}k_R - k_I < \sqrt{3}k_R + k_I$, and it follows that the dominant term in the denominator is given by $(\zeta - \zeta^2)\,\alpha\zeta^2 e^{i\zeta^2 kL}$.

Since the dominant term in the denominator is the same as for case (i), the remainder of the proof follows identically.

$\square$

Corollary 3.2.9. *The contours of integration, ∂D^+ and ∂D^-, in expression (3.49) can be deformed to the real axis. Hence representation (3.49) can be rewritten as*

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\left\{ \int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{-\infty}^{\infty} e^{ikx+ik^3t}\left(-ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right)\mathrm{d}k \right. \\ &\qquad \left. + \int_{-\infty}^{\infty} e^{ik(x-L)+ik^3t}\left(-ik\alpha\tilde{f}_1(t,k) - \tilde{g}_2(t,k)\right)\mathrm{d}k \right\} \\ &\quad + \sum_{\substack{k_n:\\ \Delta(k_n)=0}} \textit{residue contributions}. \end{aligned} \tag{3.51}$$

Proof. The expressions for $-ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)$ and $-ik\alpha\tilde{f}_1(t,k) - \tilde{g}_2(t,k)$ are given by (3.47) and (3.48) respectively. A simple computation shows that the all the integral terms sum to zero, proving that the only contribution left in the solution is due to the series term. The integrand in (3.51) is given explicitly as

$$
\begin{aligned}
e^{ikx+ik^3t}\Bigg\{ \hat{q}_0(k) - \frac{(1-\zeta)}{\Delta(k)}\Big[\hat{q}_0(k)\left(\zeta\alpha e^{ikL}+\zeta^2 e^{-i\zeta kL}+e^{-i\zeta^2 kL}\right) \\
-\zeta^2\hat{q}_0(\zeta k)\left(e^{-ikL}-\alpha e^{i\zeta kL}\right)-\hat{q}_0(\zeta^2 k)\left(e^{-ikL}-\alpha e^{i\zeta^2 kL}\right)\Big] \\
+e^{-ikL}\frac{(1-\zeta)}{\Delta(k)}\Big[\hat{q}_0(k)\left(-\zeta-\alpha e^{-i\zeta kL}-\zeta^2\alpha e^{-i\zeta^2 kL}\right) \\
-\zeta^2\hat{q}_0(\zeta k)\left(1-\alpha e^{-i\zeta^2 kL}\right)-\hat{q}_0(\zeta^2 k)\left(1-\alpha e^{-i\zeta kL}\right)\Big]\Bigg\},
\end{aligned}
$$

and the terms can be grouped together to give

$$
\begin{aligned}
\frac{e^{ikx+ik^3t}}{\Delta(k)}\Bigg\{\hat{q}_0(k)\Big(\Delta(k)-(\zeta-\zeta^2)\Big\{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL} \\
+\alpha\left(e^{ikL}+\zeta e^{i\zeta kL}+\zeta^2 e^{i\zeta^2 kL}\right)\Big\}\Big) \\
+\hat{q}_0(\zeta k)(\zeta^2-1)\left(e^{-ikL}-\alpha e^{i\zeta kL}-e^{-ikL}+\alpha e^{i\zeta kL}\right) \\
+\hat{q}_0(\zeta^2 k)(1-\zeta)\left(e^{-ikL}-\alpha e^{i\zeta^2 kL}-e^{-ikL}+\alpha e^{i\zeta^2 kL}\right)\Bigg\}.
\end{aligned}
$$

These trivially sum to zero, and the proof is complete. We remark that the proof that the integral terms sum to zero is sufficient for the proof that the solution can be expressed entirely as an infinite discrete series. □

We now derive the series solution as the explicit computation of the residue contributions at the zeros k_n of the determinant function $\Delta(k)$.

$$
\begin{aligned}
\int_{\partial D^+}\frac{p^+(k)}{r(k)}\,\mathrm{d}k &= \int_{-\infty}^{\infty}\frac{p^+(k)}{r(k)}\,\mathrm{d}k+\pi i\sum_{\substack{k_n:\\ \Delta(k_n)=0}}\frac{p^+(k_n)}{r'(k_n)}, \\
\int_{\partial D^-}\frac{p^-(k)}{r(k)}\,\mathrm{d}k &= -\int_{-\infty}^{\infty}\frac{p^-(k)}{r(k)}\,\mathrm{d}k+\pi i\sum_{\substack{k_n:\\ \Delta(k_n)=0}}\frac{p^-(k_n)}{r'(k_n)},
\end{aligned}
$$

where

$$
\begin{aligned}
p^+(k) &= e^{ikx+ik^3t}(1-\zeta)\Big\{\hat{q}_0(k)\left(\zeta\alpha e^{ikL}+\zeta^2 e^{-i\zeta kL}+e^{-i\zeta^2 kL}\right) \\
&\quad -\zeta^2\hat{q}_0(\zeta k)\left(e^{-ikL}-\alpha e^{i\zeta kL}\right)-\hat{q}_0(\zeta^2 k)\left(e^{-ikL}-\alpha e^{i\zeta^2 kL}\right)\Big\},
\end{aligned}
\tag{3.52}
$$

$$p^-(k) \;=\; e^{ik(x-L)+ik^3t}(1-\zeta)\Big\{\hat{q}_0(k)\left(-\zeta-\alpha e^{-i\zeta kL}-\zeta^2\alpha e^{-i\zeta^2kL}\right)$$
$$-\zeta^2\hat{q}_0(\zeta k)\left(1-\alpha e^{-i\zeta^2kL}\right)-\hat{q}_0(\zeta^2 k)\left(1-\alpha e^{-i\zeta kL}\right)\Big\}\,, \tag{3.53}$$

$$r(k)=\Delta(k)=\left(\zeta-\zeta^2\right)\left\{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2kL}+\alpha\left(e^{ikL}+\zeta e^{i\zeta kL}+\zeta^2 e^{i\zeta^2kL}\right)\right\}.$$

It has already been proved that all of the integral terms sum to zero, and therefore all that remains are the summation terms which yield the series solution of the form

$$q(x,t) \;=\; \frac{i}{2}\sum_{\substack{k_n:\\ \Delta(k_n)=0}}\frac{p^+(k_n)+p^-(k_n)}{r'(k_n)}\,.$$

We observe that, (3.52) and (3.53) imply that

$$p^+(k)-p^-(k)=e^{ikx+ik^3t}(\zeta-\zeta^2)\hat{q}_0(k)\Delta(k)\,,$$

and therefore $p^+(k_n)=p^-(k_n)$. It follows that the infinite series representation of the solution, computed over all the zeros k_n, is given by

$$q(x,t)=\frac{1}{L}\sum_{\substack{k_n:\\ \Delta(k_n)=0}} e^{ik_nx+ik_n^3t}\left(\frac{\begin{array}{r}\hat{q}_0(k_n)\left(\alpha e^{ik_nL}+\zeta e^{-i\zeta k_nL}+\zeta^2 e^{-i\zeta^2k_nL}\right)\\ -\zeta\hat{q}_0(\zeta k_n)\left(e^{-ik_nL}-\alpha e^{i\zeta k_nL}\right)-\zeta^2\hat{q}_0(\zeta^2k_n)\left(e^{-ik_nL}-\alpha e^{i\zeta^2k_nL}\right)\end{array}}{-e^{-ik_nL}-\zeta^2e^{-i\zeta k_nL}-\zeta e^{-i\zeta^2k_nL}+\alpha\left(e^{ik_nL}+\zeta^2e^{i\zeta k_nL}+\zeta e^{i\zeta^2k_nL}\right)}\right).$$

Remark 3.2.10. We remark that the procedure for deriving the explicit series representation of the solution is constructive, and the existence of the solution, in this form, is known to exist by general theorems [35, 36].

Remark 3.2.11. In order to derive the series representation of the solution directly from the system of global relations, for the problem with coupled boundary conditions and $\alpha \neq 1$, it is necessary to analyse the adjoint problem. We stress that this analysis was not required for the derivation of the series solution via the contour deformation approach.

3.2.5 Uncoupled Boundary Conditions

In this section we consider the imposition of uncoupled boundary conditions and illustrate the method for the boundary conditions of the form

$$q(0,t)=f_0(t)\,,\quad q(L,t)=g_0(t)\,,\quad q_x(L,t)=g_1(t)\,,$$

where $f_0(t)$, $g_0(t)$ and $g_1(t)$ are given functions. It is shown that the solution cannot be represented by a discrete series but the integral representation of the solution, which always exists, can be written alternatively as the sum of an infinite series and an integral term involving complex contours of integration.

The boundary conditions imply that the functions $\tilde{f}_0(t,k)$, $\tilde{g}_0(t,k)$ and $\tilde{g}_1(t,k)$ are known, and therefore the three global relations given by (3.15), (3.16) and (3.17) reduce to the system given by

$$\begin{pmatrix} 1 & 1 & e^{-ikL} \\ \zeta & 1 & e^{-i\zeta kL} \\ \zeta^2 & 1 & e^{-i\zeta^2 kL} \end{pmatrix} \begin{pmatrix} -ik\tilde{f}_1(t,k) \\ -\tilde{f}_2(t,k) \\ \tilde{g}_2(t,k) \end{pmatrix} = \begin{pmatrix} N(t,k) \\ N(t,\zeta k) \\ N(t,\zeta^2 k) \end{pmatrix} - \begin{pmatrix} e^{-ik^3t}\hat{q}(t,k) \\ e^{-ik^3t}\hat{q}(t,\zeta k) \\ e^{-ik^3t}\hat{q}(t,\zeta^2 k) \end{pmatrix}, \tag{3.54}$$

where

$$N(t,k) = \hat{q}_0(k) - k^2\left(\tilde{f}_0(t,k) - e^{-ikL}\tilde{g}_0(t,k)\right) - ike^{-ikL}\tilde{g}_1(t,k)\,,$$

and the determinant function $\Delta(k)$ of the system is given by

$$\Delta(k) = (\zeta - \zeta^2)\left(e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL}\right).$$

This problem was introduced in Section 2.5 where the homogeneous uncoupled boundary conditions of the form $q(0,t) = 0$, $q(L,t) = 0$ and $q_x(L,t) = 0$ were considered. The system that was obtained, and given by (2.57), is of the same form as (3.54) but with $N(t,k) = \hat{q}_0(k)$.

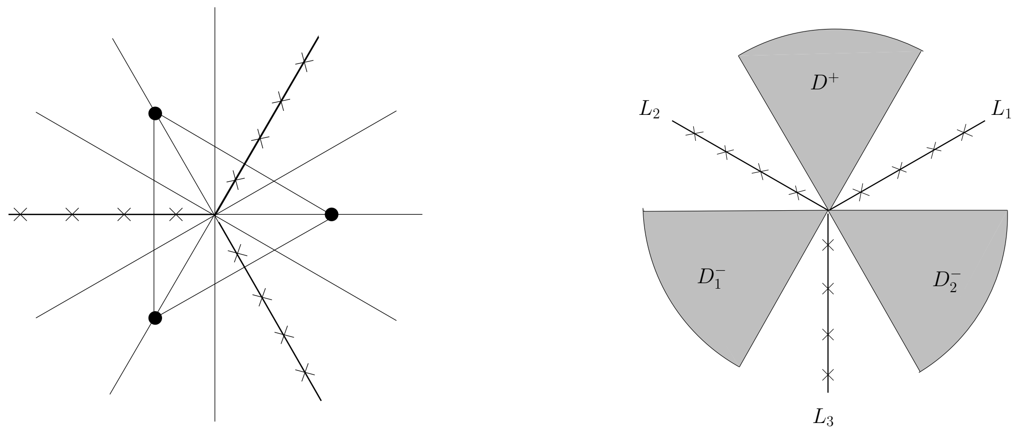
We conclude immediately, from the analysis of Section 2.5, along with the result of Levin given in Section 1.3.7, that there are infinitely many zeros of the determinant function $\Delta(k)$ in the complex k-plane. These zeros accumulate only at infinity and are clustered exactly along the lines

$$L_1 = \left\{k : \arg(k) = \frac{\pi}{6}\right\},\ L_2 = \left\{k : \arg(k) = \frac{5\pi}{6}\right\},\ L_3 = \left\{k : \arg(k) = \frac{3\pi}{2}\right\}.$$

The domain $D = \{k \in \mathbb{C} : \operatorname{Re}\omega(k) \leqslant 0\}$ has three connected components given by

$$\begin{aligned} D^+ &= \left\{k \in \mathbb{C} : \tfrac{\pi}{3} \leqslant \arg(k) \leqslant \tfrac{2\pi}{3}\right\} \\ &\left.\begin{aligned} D_1^- &= \left\{k \in \mathbb{C} : \pi \leqslant \arg(k) \leqslant \tfrac{4\pi}{3}\right\} \\ D_2^- &= \left\{k \in \mathbb{C} : \tfrac{5\pi}{3} \leqslant \arg(k) \leqslant 2\pi\right\} \end{aligned}\right\} \quad D^- = D_1^- \cup D_2^-\,, \end{aligned} \tag{3.55}$$

and it can be seen in Figure 3.10 that the zeros cluster along the three bisecting rays of the complement regions D_c of $D = D^+ \cup D^-$.



(a) z-plane ($z = -ikL$). (b) k-plane.

Figure 3.10: The regions $D^{\pm}$ for the third order problem $q_t(x,t) + q_{xxx}(x,t) = 0$ with the boundary conditions $q(0,t) = f_0(t)$, $q(L,t) = g_0(t)$ and $q_x(L,t) = g_1(t)$ and the location of the zeros of the determinant function $\Delta(k) = (\zeta - \zeta^2)\left(e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL}\right)$ found using Levin's Theorem, (Theorem 1.3.22).

Remark 3.2.12. Having established where the zeros of $\Delta(k)$ lie in the complex plane we might try to establish a series representation of the solution, according to the global relations, as we did for the second order problem. Solving for example with respect to $\tilde{g}_2(t,k)$ yields the following

$$\begin{aligned} \tilde{\Delta}(k)\tilde{g}_2(t,k) &= N(t,k) + \zeta N(t,\zeta k) + \zeta^2 N(t,\zeta^2 k) \\ &\quad - e^{-ik^3 t}\left(\hat{q}(t,k) + \zeta\hat{q}(t,\zeta k) + \zeta^2\hat{q}(t,\zeta^2 k)\right), \end{aligned}$$

where

$$\tilde{\Delta}(k) = e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL}.$$

Evaluating this expression at the zeros of $\Delta(k)$ appears to yield an expression for the unknown functions in terms of the known functions $N(t,k)$, $N(t,\zeta k)$ and $N(t,\zeta^2 k)$. However, if we consider for example the zeros k_n such that $\arg(k_n) = \frac{3\pi}{2}$ then we see that whilst the functions $\hat{q}(t,k)$ and $N(t,k)$ are bounded as $k \to \infty$, the functions $\hat{q}(t,\zeta k)$, $\hat{q}(t,\zeta^2 k)$, $N(t,\zeta k)$ and $N(t,\zeta^2 k)$ are not. Furthermore, no multiple of the identity

$$N(t,k) + \zeta N(t,\zeta k) + \zeta^2 N(t,\zeta^2 k) = e^{-ik^3 t}\left(\hat{q}(t,k) + \zeta\hat{q}(t,\zeta k) + \zeta^2\hat{q}(t,\zeta^2 k)\right)$$

makes all of the terms bounded. Therefore, these expressions have no meaning as $k \to \infty$, and cannot be used to derive directly a series representation.

The Integral Representation

The integral representation of the solution is given by (2.56) where the domains $D^{\pm}$ in the complex k-plane are defined by (3.55), and $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$ are given by (3.13) and (3.14) respectively. Hence,

$$\begin{aligned} q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ & \int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx+ik^3t}k^2\tilde{f}_0(t,k)\,\mathrm{d}k \\ & - \int_{\partial D^+} e^{ikx+ik^3t}\left(-ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k)\right)\mathrm{d}k \\ & - \int_{\partial D^-} e^{ik(x-L)+ik^3t}\left(k^2\tilde{g}_0(t,k) - ik\tilde{g}_1(t,k)\right)\mathrm{d}k \\ & + \int_{\partial D^-} e^{ik(x-L)+ik^3t}\tilde{g}_2(t,k)\,\mathrm{d}k \Bigg\}. \end{aligned} \tag{3.56}$$

The three global relations given by system (3.54) are solved using Cramer's rule to achieve expressions for the three unknown boundary values $\tilde{f}_1(t,k)$, $\tilde{f}_2(t,k)$ and $\tilde{g}_2(t,k)$:

$$\begin{aligned} -ik\tilde{f}_1(t,k) &= \frac{1}{\Delta(k)}\Big\{N(t,k)\left(e^{-i\zeta^2kL} - e^{-i\zeta kL}\right) + N(t,\zeta k)\left(e^{-ikL} - e^{-i\zeta^2kL}\right) \\ &\qquad + N(t,\zeta^2k)\left(e^{-i\zeta kL} - e^{-ikL}\right)\Big\}, \\ -\tilde{f}_2(t,k) &= \frac{1}{\Delta(k)}\Big\{N(t,k)\left(\zeta^2e^{-i\zeta kL} - \zeta e^{-i\zeta^2kL}\right) + N(t,\zeta k)\left(e^{-i\zeta^2kL} - \zeta^2e^{-ikL}\right) \\ &\qquad + N(t,\zeta^2k)\left(\zeta e^{-ikL} - e^{-i\zeta kL}\right)\Big\}, \\ \tilde{g}_2(t,k) &= \frac{1}{\Delta(k)}\Big\{N(t,k)\left(\zeta-\zeta^2\right) + N(t,\zeta k)\left(\zeta^2-1\right) + N(t,\zeta^2k)\left(1-\zeta\right)\Big\}. \end{aligned}$$

Using the identity $\Delta(k) = (\zeta-\zeta^2)\tilde{\Delta}(k)$ where $\tilde{\Delta}(k) = e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2kL}$, along with the equality

$$\begin{aligned} & \frac{N(t,k)\left(\zeta e^{-i\zeta kL} + \zeta^2e^{-i\zeta^2kL}\right) - N(t,\zeta k)\zeta e^{-ikL} - N(t,\zeta^2k)\zeta^2e^{-ikL}}{\tilde{\Delta}(k)} \\ &= -\left(\frac{N(t,k) + \zeta N(t,\zeta k) + \zeta^2N(t,\zeta^2k)}{\tilde{\Delta}(k)}\right)e^{-ikL} + N(t,k)\,, \end{aligned}$$

the expressions for the unknown functions are given by

$$\begin{aligned} -ik\tilde{f}_1(t,k) - \tilde{f}_2(t,k) &= -\left(\frac{N(t,k) + \zeta N(t,\zeta k) + \zeta^2N(t,\zeta^2k)}{\tilde{\Delta}(k)}\right)e^{-ikL} + N(t,k)\,, \\ \tilde{g}_2(t,k) &= \frac{N(t,k) + \zeta N(t,\zeta k) + \zeta^2N(t,\zeta^2k)}{\tilde{\Delta}(k)}\,. \end{aligned}$$

Proposition 3.2.13. *The unknown terms involving $\hat{q}(t,k)$, $\hat{q}(t,\zeta k)$ and $\hat{q}(t,\zeta^2 k)$ do not contribute to the integral representation of the solution, given by (3.56).*

Proof. The proof follows directly from Proposition 2.4.3(i), however, we give the rigorous proof for the integral around ∂D^- of the unknown functions $\hat{q}(t,\lambda_l(k))$, $l = 0,1,2$, given explicitly by

$$\int_{\partial D^-} e^{ik(x-L)} \left(\frac{\hat{q}(t,k) + \zeta\hat{q}(t,\zeta k) + \zeta^2\hat{q}(t,\zeta^2 k)}{e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL}} \right) \mathrm{d}k \,.$$

We show that the integrand is bounded as $k \to \infty$ in D^- and therefore by Jordan's Lemma (Lemma 1.3.3) does not contribute to the integral representation.

Let $k \in D_1^-$. (The proof for $k \in D_2^-$ follows analogously). Then

- e^{-ikL}, $e^{-i\zeta kL}$ and $e^{i\zeta^2 kL}$ are bounded,
- e^{ikL}, $e^{i\zeta kL}$ and $e^{-i\zeta^2 kL}$ are unbounded.

Therefore, asymptotically the denominator behaves like $e^{-i\zeta^2 kL}$ and the exponential terms of the integrand are given by

- $e^{ik(x-L)}e^{i\zeta^2 kL}e^{-ikx} = e^{-ikL}e^{i\zeta^2 kL}$,
- $e^{ik(x-L)}e^{i\zeta^2 kL}e^{-i\zeta kx}$,
- $e^{ik(x-L)}e^{i\zeta^2 kL}e^{-i\zeta^2 kx} = e^{ik(x-L)}e^{i\zeta^2 k(L-x)}$.

All of these terms are bounded and analytic in D^- and the proof is complete. □

Remark 3.2.14. An analogous proof can be used to show that the unknown terms involving $\hat{q}(t,k)$, $\hat{q}(t,\zeta k)$ and $\hat{q}(t,\zeta^2 k)$ do not contribute to the integral representation of the solution around ∂D^+.

Therefore the integral representation of the solution is given by

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\Bigg\{ & \int_{-\infty}^{\infty} e^{ikx+ik^3 t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx+ik^3 t}k^2 \tilde{f}_0(t,k)\,\mathrm{d}k \\ & - \int_{\partial D^-} e^{ik(x-L)+ik^3 t}\left(k^2\tilde{g}_0(t,k) - ik\tilde{g}_1(t,k)\right)\mathrm{d}k \\ & + \int_{\partial D^+} e^{ikx+ik^3 t}\left(\frac{N(t,k)+\zeta N(t,\zeta k)+\zeta^2 N(t,\zeta^2 k)}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL}} e^{-ikL} - N(t,k)\right)\mathrm{d}k \\ & + \int_{\partial D^-} e^{ik(x-L)+ik^3 t}\left(\frac{N(t,k)+\zeta N(t,\zeta k)+\zeta^2 N(t,\zeta^2 k)}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL}}\right)\mathrm{d}k \Bigg\}. \end{aligned} \tag{3.57}$$

Contour Deformation and Alternative Representations

In this section, we show that the contour of integration in expression (3.57) cannot be deformed. As a consequence, we show that the integral representation is not equivalent to a series representation obtained by a residue computation, as in previous sections.

The deformation of the integral around ∂D^+ to L_1

Proposition 3.2.15. *The contour of integration cannot be deformed from ∂D^+, where* $\arg(k) = \frac{\pi}{3}$, *to the ray* $L_1 = \left\{k : \arg(k) = \frac{\pi}{6}\right\}$.

Proof. For the proof, we show that the integrand is not bounded in the closed region $\frac{\pi}{6} < \arg(k) < \frac{\pi}{3}$. Hence we show that the integrand contains either an x-exponential term and/or a t-exponential term which is unbounded in the closed region $\frac{\pi}{6} < \arg(k) < \frac{\pi}{3}$.

It will suffice to consider the integral around ∂D^+ in (3.57), of only the term involving $\tilde{f}_1(t,k)$ given explicitly by

$$\int_{\partial D^+} e^{ikx+ik^3t}\left(\frac{N(t,k)\left(e^{-i\zeta^2kL}-e^{-i\zeta kL}\right)+N(t,\zeta k)\left(e^{-ikL}-e^{-i\zeta^2kL}\right)+N(t,\zeta^2k)\left(e^{-i\zeta kL}-e^{-ikL}\right)}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2kL}}\right)\mathrm{d}k.$$

Consider the wedge such that $\frac{\pi}{6} < \arg(k) < \frac{\pi}{3}$. Then for k in this wedge, ζk will be such that $\frac{5\pi}{6} < \arg(\zeta k) < \pi$ and $\zeta^2 k$ will be such that $\frac{3\pi}{2} < \arg(\zeta^2 k) < \frac{5\pi}{3}$. Hence k and ζk will lie in the upper half of the complex k-plane and $\zeta^2 k$ will lie in the lower half of the complex k-plane (Figure 3.11). Hence, for k in this wedge

- e^{ikx+ik^3t} is bounded,
- e^{ikL}, $e^{i\zeta kL}$ and $e^{-i\zeta^2kL}$ are bounded,
- e^{-ikL}, $e^{-i\zeta kL}$ and $e^{i\zeta^2kL}$ are not bounded.

To determine the asymptotic behaviour of the denominator, we substitute $\zeta = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ into the expressions for $e^{-i\zeta kL}$ and $e^{-i\zeta^2kL}$ which gives

$$e^{-i\zeta kL} = e^{\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)kL}, \qquad e^{-i\zeta^2kL} = e^{\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)kL},$$

and conclude that asymptotically the denominator behaves like $e^{-ikL} + \zeta e^{-i\zeta kL}$. To establish which of these terms is dominant we set $k = k_R + ik_I$. This implies that

$$\mathrm{Im}(k) = k_I\,, \qquad \mathrm{Im}(\zeta k) = \frac{1}{2}\left(\sqrt{3}k_R - k_I\right).$$

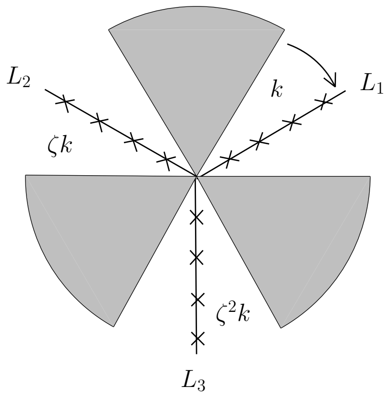


Figure 3.11: The deformation of the contour ∂D^+ for k such that $\frac{\pi}{6} < \arg(k) < \frac{\pi}{3}$.

Since $\mathrm{Im}(k) > \mathrm{Im}(\zeta k)$, we conclude that the dominant term in the denominator is e^{-ikL}. Therefore, the asymptotic behaviour of the integrand is given by

$$e^{ikx+ik^3t}\Big\{N(t,k)\left(e^{(1-\zeta^2)ikL} - e^{(1-\zeta)ikL}\right) + N(t,\zeta k)\left(1 - e^{(1-\zeta^2)ikL}\right)$$
$$+ N(t,\zeta^2 k)\left(e^{(1-\zeta)ikL} - 1\right)\Big\}.$$

Consider the integral around ∂D^+ of the second of the terms involving $N(t,\zeta k)$, and more specifically the term involving $\hat{q}_0(\zeta k)$:

$$\int_{\partial D^+} e^{ikx+ik^3t}\hat{q}_0(\zeta k)\,\mathrm{d}k = \int_{\partial D^+} e^{ikx+ik^3t}\left(\int_0^L e^{-i\zeta kx'}q_0(x')\,\mathrm{d}x'\right)\mathrm{d}k\,.$$

Integration by parts implies that

$$\begin{aligned} e^{ikx}\int_0^L e^{-i\zeta kx'}q_0(x')\,\mathrm{d}x' &= e^{ikx}\left\{\left[\frac{1}{-i\zeta k}e^{-i\zeta kx'}q_0(x')\right]_0^L + \int_0^L \frac{e^{-i\zeta kx'}}{i\zeta k}q_0'(x')\,\mathrm{d}x'\right\} \\ &= \frac{i\zeta^2}{k}e^{ikx-i\zeta kL}q_0(L) - \frac{i\zeta^2}{k}e^{ikx}\int_0^L e^{-i\zeta kx'}q_0'(x')\,\mathrm{d}x'\,. \end{aligned}$$

To analyse the term $e^{ikx-i\zeta kL}$ for any $x \in [0,L]$, we substitute $k = R(\cos\theta + i\sin\theta)$ where $\frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{3}$ and $R > 0$. Therefore $\zeta k = R\left(\cos(\theta + \frac{2\pi}{3}) + i\sin(\theta + \frac{2\pi}{3})\right)$ and hence

$$\mathrm{Re}\left(e^{ikx-i\zeta kL}\right) = e^{-R\sin\theta x}e^{R\sin\left(\theta+\frac{2\pi}{3}\right)L}\,.$$

If $\theta = \frac{\pi}{6}$ then $\sin\theta = \sin\left(\theta + \frac{2\pi}{3}\right) = \frac{1}{2}$ and hence $\mathrm{Re}\left(e^{ikx-i\zeta kL}\right) = e^{-\frac{Rx}{2}}e^{\frac{RL}{2}} = e^{\frac{R}{2}(L-x)}$. Since $L - x > 0$ this expression is unbounded as $R \to \infty$.

This concludes the proof that the contour of integration cannot be deformed through the region $\frac{\pi}{6} < \arg(k) < \frac{\pi}{3}$ to the ray $L_1 = \left\{k : \arg(k) = \frac{\pi}{6}\right\}$. $\square$

Remark 3.2.16. We remark that a similar argument can be used for the proof that the contour of integration cannot be deformed from ∂D^+, where $\arg(k) = \frac{2\pi}{3}$, to the ray $L_2 = \left\{k : \arg(k) = \frac{5\pi}{6}\right\}$.

The deformation of the integral around ∂D^- to L_1

Proposition 3.2.17. *The contour of integration cannot be deformed from the positive real axis to the ray* $L_1 = \left\{k : \arg(k) = \frac{\pi}{6}\right\}$.

Proof. For the proof, we show that integrand is unbounded in the closed region $0 < \arg(k) < \frac{\pi}{6}$.

It will suffice to consider the integral around ∂D^-, in expression (3.57), of only the term involving $\hat{q}_0(k)$ given explicitly by

$$\int_{\partial D^-} e^{ik(x-L)+ik^3t}\left(\frac{\hat{q}_0(k)}{e^{-ikL}+\zeta e^{i\zeta kL}+\zeta^2 e^{i\zeta^2 kL}}\right) \mathrm{d}k\,.$$

Consider the wedge such that $0 < \arg(k) < \frac{\pi}{6}$. Then for k in this wedge, ζk will be such that $\frac{2\pi}{3} < \arg(\zeta k) < \frac{5\pi}{6}$ and $\zeta^2 k$ will be such that $\frac{4\pi}{3} < \arg(\zeta^2 k) < \frac{3\pi}{2}$. Hence k and ζk will lie in the upper half of the complex k-plane and $\zeta^2 k$ will lie in the lower half of the complex k-plane (Figure 3.12). Hence for k in this wedge,

- e^{ikx+ik^3t} is bounded,
- e^{ikL}, $e^{i\zeta kL}$ and $e^{-i\zeta^2 kL}$ are bounded,
- e^{-ikL}, $e^{-i\zeta kL}$ and $e^{i\zeta^2 kL}$ are unbounded.

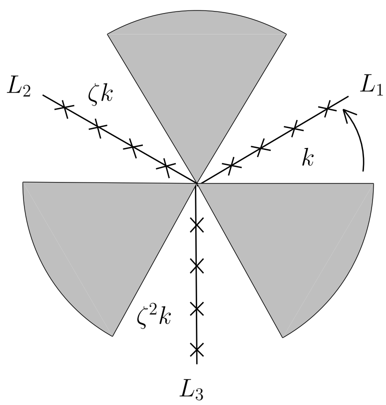


Figure 3.12: The deformation of the contour ∂D^- for k such that $0 < \arg(k) < \frac{\pi}{6}$.

To establish the asymptotic behaviour of the denominator, we substitute $k = k_R + ik_I$ into each of the exponents in the denominator:

$$\mathrm{Re}\left(e^{-ikL}\right) = e^{k_I L}\,, \quad \mathrm{Re}\left(e^{-i\zeta kL}\right) = e^{\frac{1}{2}\left(\sqrt{3}k_R - k_I\right)L}\,, \quad \mathrm{Re}\left(e^{-i\zeta^2 kL}\right) = e^{-\frac{1}{2}\left(\sqrt{3}k_R + k_I\right)L}\,,$$

and conclude immediately that since $k_I > 0$, e^{-ikL} grows as $k \to \infty$. Similarly, $0 < \arg(k) < \frac{\pi}{6}$ implies $0 < \sqrt{3}k_R - k_I$ and therefore $e^{-i\zeta^2 kL}$ also grows as $k \to \infty$. In contrast the term $e^{-i\zeta^2 kL}$ decays as $k \to \infty$ since $\sqrt{3}k_R + k_I > 0$. Therefore asymptotically the denominator behaves like $e^{-ikL} + \zeta e^{-i\zeta kL}$. The dominant term is established using the fact that $\mathrm{Im}(k) < \mathrm{Im}(\zeta k)$. Hence

$$e^{-ikL} + \zeta e^{-i\zeta kL} = e^{-i\mathrm{Re}(k)L} e^{\mathrm{Im}(k)L} + \zeta e^{-i\mathrm{Re}(\zeta k)L} e^{\mathrm{Im}(\zeta k)L}\,,$$

and the dominant contribution as $k \to \infty$ is therefore given by $e^{-i\zeta kL}$. The asymptotic behaviour of the integrand is therefore given by

$$e^{ik(x-L)+ik^3 t} e^{i\zeta kL} \hat{q}_0(k)\,.$$

Explicitly the x-exponential terms are given by

$$\begin{aligned} e^{ik(x-L)} e^{i\zeta kL} \hat{q}_0(k) &= e^{ik(x-L)} e^{i\zeta kL} \int_0^L e^{-ikx'} q_0(x')\,\mathrm{d}x' \\ &= e^{ik(x-L)} e^{i\zeta kL} \left\{ \left[\frac{1}{-ik} e^{-ikx'} q_0(x') \right]_0^L + \int_0^L \frac{1}{ik} e^{-ikx'} q_0'(x')\,\mathrm{d}x' \right\}. \end{aligned}$$

To analyse the term $e^{ik(x-L)} e^{i\zeta kL} e^{-ikL}$, we substitute $k = R(\cos\theta + i\sin\theta)$ where $0 < \arg(k) < \frac{\pi}{6}$ and $R > 0$. Therefore $\zeta k = R\left(\cos(\theta + \frac{2\pi}{3}) + i\sin(\theta + \frac{2\pi}{3})\right)$ and hence

$$\mathrm{Re}\,\left(e^{ik(x-L)} e^{i\zeta kL} e^{-ikL}\right) = e^{-R\sin\theta(x-L)} e^{-R\sin\left(\theta + \frac{2\pi}{3}\right)L} e^{R\sin\theta L}\,.$$

If $\theta = \frac{\pi}{6}$ then $\mathrm{Re}\,\left(e^{ik(x-L)} e^{i\zeta kL} e^{-ikL}\right) = e^{-\frac{R}{2}(x-L)}$ which is unbounded as $R \to \infty$ since $x - L < 0$. This concludes the proof that the contour of integration cannot be deformed through the region $0 < \arg(k) < \frac{\pi}{6}$ to the ray $L_1 = \left\{k : \arg(k) = \frac{\pi}{6}\right\}$. $\square$

Remark 3.2.18. We remark that a similar argument can be used for the proof that the contour of integration cannot be deformed from the negative real axis to the ray $L_2 = \left\{k : \arg(k) = \frac{5\pi}{6}\right\}$.

The deformation of the integral around ∂D^- to L_3

Proposition 3.2.19. *The integrand of the integral around ∂D^- is analytic and bounded in the region such that $\frac{4\pi}{3} < \arg(k) < \frac{3\pi}{2}$, implying that the contour of integration can be deformed through this region to the ray $L_3 = \left\{k : \arg(k) = \frac{3\pi}{2}\right\}$.*

Proof. We remark that the analyticity of the integrand around ∂D^- involving the known boundary values $k^2\tilde{g}_0(t,k) - ik\tilde{g}_1(t,k)$, follows immediately from the definition of D (since the only exponential appearing in the integrand is $e^{ik(x-L)+ik^3t}$ which is necessarily analytic and bounded for $k \in D_c^-$).

Therefore we consider only the integral around ∂D^- in (3.57), of the terms involving $N(t,k)$, $N(t,\zeta k)$ and $N(t,\zeta^2 k)$ given explicitly by

$$\int_{\partial D^-} e^{ik(x-L)+ik^3t}\left(\frac{N(t,k)+\zeta N(t,\zeta k)+\zeta^2 N(t,\zeta^2 k)}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL}}\right)\mathrm{d}k.$$

Consider the wedge such that $\frac{4\pi}{3} < \arg(k) < \frac{3\pi}{2}$. Then for k in this wedge, ζk will be such that $0 < \arg(\zeta k) < \frac{\pi}{6}$ and $\zeta^2 k$ will be such that $\frac{2\pi}{3} < \arg(\zeta^2 k) < \frac{5\pi}{6}$. Hence ζk and $\zeta^2 k$ will lie in the upper half of the complex k-plane and k will lie in the lower half of the complex k-plane (Figure 3.13). Hence

- $e^{ik(x-L)+ik^3t}$ is bounded,
- e^{-ikL}, $e^{i\zeta kL}$ and $e^{i\zeta^2 kL}$ are bounded,
- e^{ikL}, $e^{-i\zeta kL}$ and $e^{-i\zeta^2 kL}$ are unbounded.

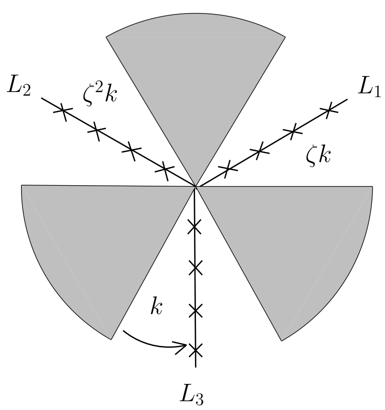


Figure 3.13: The deformation of the contour ∂D^- for k such that $\frac{4\pi}{3} < \arg(k) < \frac{3\pi}{2}$.

We must now establish the asymptotic behaviour of the denominator. Substituting $k = k_R + ik_I$ into each of the exponents in the denominator gives

$$\mathrm{Re}\,(e^{-ikL}) = e^{k_I L}\,, \quad \mathrm{Re}\,(e^{-i\zeta kL}) = e^{\frac{1}{2}\left(\sqrt{3}k_R - k_I\right)L}\,, \quad \mathrm{Re}\,(e^{-i\zeta^2 kL}) = e^{-\frac{1}{2}\left(\sqrt{3}k_R + k_I\right)L}\,.$$

Since $k_I < 0$, the term e^{-ikL} will decay as $k \to \infty$, and since $\sqrt{3}k_R - k_I > 0$, the term $e^{-i\zeta kL}$ will grow as $k \to \infty$. To determine the nature of $e^{-i\zeta^2 kL}$ we need to compare the imaginary part of the term to the imaginary part of $e^{-i\zeta kL}$ and see which is dominant.

$$\mathrm{Im}\,(\zeta k) = \frac{1}{2}\left(\sqrt{3}k_R - k_I\right)\,, \quad \mathrm{Im}\,\left(\zeta^2 k\right) = \frac{1}{2}\left(-\sqrt{3}k_R - k_I\right)\,,$$

therefore $\mathrm{Im}\,(\zeta k) = \mathrm{Im}\,(\zeta^2 k) + \sqrt{3}k_R$ and since $k_R < 0$ this implies that $\mathrm{Im}\,(\zeta^2 k) > \mathrm{Im}\,(\zeta k)$. Hence the dominant contribution in the dominator is given by $e^{-i\zeta^2 kL}$ as $k \to \infty$, and the asymptotic behaviour of the integrand is given by

$$e^{ik(x-L)+ik^3 t} e^{i\zeta^2 kL}\left(N(t,k) + \zeta N(t,\zeta k) + \zeta^2 N(t,\zeta^2 k)\right)\,.$$

The dominant behaviour of the x-exponential terms in the integrand is given as follows:

i.) $e^{ik(x-L)} e^{i\zeta^2 kL} e^{-ikx'} - e^{ik(x-L)} e^{i\zeta^2 kL} e^{-ikL}$,

ii.) $e^{ik(x-L)} e^{i\zeta^2 kL} e^{-i\zeta kx'} - e^{ik(x-L)} e^{i\zeta^2 kL} e^{-i\zeta kL}$,

iii.) $e^{ik(x-L)} e^{i\zeta^2 kL} e^{-i\zeta^2 kx'} - e^{ik(x-L)} e^{i\zeta^2 kL} e^{-i\zeta^2 kL}$,

for any $0 < x, x' < L$. Now, the term $e^{ik(x-L)} = e^{-k_I(x-L)} e^{ik_R(x-L)}$ for $k = k_R + ik_I$, and since $k_I < 0$ and $x - L < 0$ we conclude that $e^{ik(x-L)}$ is bounded. We now analyse the remaining terms by considering each of the expressions above, separately.

i.) Each of the exponentials $e^{ik(x-L)}$, $e^{i\zeta^2 kL}$, $e^{-ikx'}$ and e^{-ikL} are bounded and hence the expression $e^{ik(x-L)} e^{i\zeta^2 kL} e^{-ikx'} - e^{ik(x-L)} e^{i\zeta^2 kL} e^{-ikL}$ is a bounded analytic function.

ii.) In this region $\mathrm{Im}\,(\zeta^2 k) > \mathrm{Im}\,(\zeta k)$ and hence $e^{-i\zeta^2 kx'}$ dominates over $e^{-i\zeta kx'}$. Therefore

$$e^{ik(x-L)} e^{i\zeta^2 kL} e^{-i\zeta kx'} < e^{ik(x-L)} e^{i\zeta^2 kL} e^{-i\zeta^2 kx'} = e^{ik(x-L)} e^{i\zeta^2 k(L-x')}\,,$$

and since $L - x' > 0$ we conclude that $e^{ik(x-L)} e^{i\zeta^2 kL} e^{-i\zeta kx'}$ is a bounded analytic function. Similarly, $e^{-i\zeta^2 kL}$ dominates over $e^{-i\zeta kL}$, therefore

$$e^{ik(x-L)} e^{i\zeta^2 kL} e^{-i\zeta kL} < e^{ik(x-L)} e^{i\zeta^2 kL} e^{-i\zeta^2 kL} = e^{ik(x-L)}\,,$$

which is a bounded analytic function.

iii.) To show that the final two terms are bounded we group the terms as follows:

$$e^{ik(x-L)}e^{i\zeta^2 kL}e^{-i\zeta^2 kx'} - e^{ik(x-L)}e^{i\zeta^2 kL}e^{-i\zeta^2 kL} = e^{ik(x-L)}e^{i\zeta^2 k(L-x')} - e^{ik(x-L)}\,,$$

and then conclude immediately that the expression is a bounded analytic function.

This concludes the proof that the contour of integration can be deformed through the region $\frac{4\pi}{3} < \arg(k) < \frac{3\pi}{2}$ to the ray $L_3 = \left\{k : \arg(k) = \frac{3\pi}{2}\right\}$. □

Remark 3.2.20. We remark that a similar argument can be used for the proof that the contour of integration can be deformed from where $\arg(k) < \frac{5\pi}{3}$ to the ray $L_3 = \left\{k : \arg(k) = \frac{3\pi}{2}\right\}$.

The integral along the wedge $\partial D_c^- = \left\{k \in \mathbb{C} : \arg(k) = \frac{4\pi}{3}\right\} \cup \left\{k \in \mathbb{C} : \arg(k) = \frac{5\pi}{3}\right\}$, orientated such that the interior of the domain is always to the right, can be computed by Cauchy's Theorem (Theorem 1.3.1), and is equal to the sum of the residues at the poles of the determinant function $\tilde{\Delta}(k) = e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL}$ inside the integration contour. However, we now show that this function does not have any poles in this region, and therefore the integral along the wedge ∂D_c^- vanishes. Recall that

$$\int_{\partial D^-} = -\int_{-\infty}^{\infty} + \int_{\partial D_c^-}\,,$$

and therefore the integral representation of the solution, given by (3.57), can be written in the form

$$\begin{aligned}
q(x,t) = \frac{1}{2\pi}\Bigg\{ &\int_{-\infty}^{\infty} e^{ikx+ik^3t}\left(\hat{q}_0(k) + e^{-ikL}(k^2\tilde{g}_0(t,k) - ik\tilde{g}_1(t,k))\right)\,\mathrm{d}k \\
&- \int_{-\infty}^{\infty} e^{ik(x-L)+ik^3t}\left(\frac{N(t,k) + \zeta N(t,\zeta k) + \zeta^2 N(t,\zeta^2 k)}{e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL}}\right)\,\mathrm{d}k \\
&- \int_{\partial D^+} e^{ikx+ik^3t}k^2\tilde{f}_0(t,k)\,\mathrm{d}k \\
&+ \int_{\partial D^+} e^{ikx+ik^3t}\left(\frac{N(t,k) + \zeta N(t,\zeta k) + \zeta^2 N(t,\zeta^2 k)}{e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL}}e^{-ikL} - N(t,k)\right)\,\mathrm{d}k \\
&- \int_{\partial D_c^-} e^{ik(x-L)+ik^3t}\left(k^2\tilde{g}_0(t,k) - ik\tilde{g}_1(t,k)\right)\,\mathrm{d}k \\
&+ \int_{\partial D_c^-} e^{ik(x-L)+ik^3t}\left(\frac{N(t,k) + \zeta N(t,\zeta k) + \zeta^2 N(t,\zeta^2 k)}{e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL}}\right)\,\mathrm{d}k \Bigg\}.
\end{aligned}$$

Writing the third integral around ∂D^+ as an integral along $\mathbb{R}$, and combining with the terms of the first integral, the integral representation can be written as

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\Bigg\{ \int_{-\mathbb{R}\cup\partial D^+} e^{ikx+ik^3t}\left(\frac{N(t,k)+\zeta N(t,\zeta k)+\zeta^2 N(t,\zeta^2 k)}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL}}e^{-ikL} - N(t,k)\right)\mathrm{d}k \\ &\quad + \int_{\partial D_c^-} e^{ik(x-L)+ik^3t}\left(\frac{N(t,k)+\zeta N(t,\zeta k)+\zeta^2 N(t,\zeta^2 k)}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL}}\right)\mathrm{d}k \\ &\quad - \int_{\partial D_c^-} e^{ik(x-L)+ik^3t}\left(k^2\tilde{g}_0(t,k) - ik\tilde{g}_1(t,k)\right)\mathrm{d}k \Bigg\}. \end{aligned}$$

The remainder of the example will be on the integrals around ∂D_c^- and the zeros of the denominator $\tilde{\Delta}(k) = e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL}$ which lie on the negative imaginary axis in the complex k-plane.

Consider the integrand around ∂D_c^-, involving the boundary data, given by

$$e^{ik(x-L)+ik^3t}\left(k^2\tilde{g}_0(t,k) - ik\tilde{g}_1(t,k)\right).$$

This expression is analytic and bounded for k such that $\frac{4\pi}{3} < \arg(k) < \frac{5\pi}{3}$, and therefore, via Cauchy's Theorem (Theorem 1.3.1), the integral around ∂D_c^- of the integrand involving the boundary data vanishes. The integral representation of the solution, is therefore given by

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\Bigg\{ \int_{-\mathbb{R}\cup\partial D^+} e^{ikx+ik^3t}\left(\frac{N(t,k)+\zeta N(t,\zeta k)+\zeta^2 N(t,\zeta^2 k)}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL}}e^{-ikL} - N(t,k)\right)\mathrm{d}k \\ &\quad + \int_{\partial D_c^-} e^{ik(x-L)+ik^3t}\left(\frac{N(t,k)+\zeta N(t,\zeta k)+\zeta^2 N(t,\zeta^2 k)}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL}}\right)\mathrm{d}k \Bigg\}. \end{aligned} \tag{3.58}$$

Proposition 3.2.19, and a similar argument for the region such that $\frac{4\pi}{3} < \arg(k) < \frac{3\pi}{2}$, proves the analyticity and boundedness of the integrand, of the integral around ∂D_c^-, in the region D_c^-. Therefore, the integral around ∂D_c^-, in expression (3.58), can alternatively be written as an infinite series in terms of the explicit computation, at the zeros of the determinant function $\Delta(k)$ situated on the negative imaginary axis, of the principal value contributions. Therefore

$$\int_{\partial D_c^-} \frac{p(k)}{r(k)}\,\mathrm{d}k = 2\pi i \sum_{\substack{k_I: \\ \Delta(k_I)=0}} \frac{p(k_I)}{r'(k_I)},$$

where $\{k_I\}$ represents the set of zeros of $\Delta(k)$, situated on the ray $L_3 = \left\{k : \arg(k) = \frac{3\pi}{2}\right\}$, and

$$\begin{aligned} p(k) &= e^{ik(x-L)+ik^3t}\left(N(t,k) + \zeta N(t,\zeta k) + \zeta^2 N(t,\zeta^2 k)\right), \\ r(k) &= \tilde{\Delta}(k) = e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL}. \end{aligned}$$

Hence expression (3.58) can alternatively be written in the form

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\int_{-\mathbb{R}\cup\partial D^+} e^{ikx+ik^3t}\left(\frac{N(t,k)+\zeta N(t,\zeta k)+\zeta^2 N(t,\zeta^2 k)}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL}} e^{-ikL} - N(t,k)\right) \mathrm{d}k \\ &\quad - \frac{1}{L}\sum_{\substack{k_I: \\ \Delta(k_I)=0}} e^{ik_I(x-L)+ik_I^3 t}\left(\frac{N(t,k_I)+\zeta N(t,\zeta k_I)+\zeta^2 N(t,\zeta^2 k_I)}{e^{-ik_I L}+\zeta^2 e^{-i\zeta k_I L}+\zeta e^{-i\zeta^2 k_I L}}\right). \end{aligned}$$

Homogeneous Boundary Conditions

For the remainder of the example, we consider the case where $\tilde{f}_0(t,k) = 0$, $\tilde{g}_0(t,k) = 0$ and $\tilde{g}_1(t,k) = 0$.

Proposition 3.2.21. *Consider the homogeneous boundary conditions $q(0,t) = 0$, $q(L,t) = 0$ and $q_x(L,t) = 0$ such that $N(t,k) = q_0(k)$. The integrand, of the integral around ∂D_c^-, in the representation of the solution given by (3.58), is analytic for k such that $\left\{\frac{4\pi}{3} < \arg(k) < \frac{5\pi}{3}\right\}$.*

Proof. The integral around ∂D_c^-, in (3.58) is given explicitly in terms of the given initial data as

$$\int_{\partial D_c^-} e^{ik(x-L)+ik^3t}\left(\frac{\hat{q}_0(k)+\zeta\hat{q}_0(\zeta k)+\zeta^2\hat{q}_0(\zeta^2 k)}{e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL}}\right) \mathrm{d}k. \tag{3.59}$$

Using the formula for the Fourier transform (2.26) and integration by parts, the numerator of the integrand involving the initial data, can be expressed as follows:

$$\hat{q}_0(k)+\zeta\hat{q}_0(\zeta k)+\zeta^2\hat{q}_0(\zeta^2 k) = \int_0^L \left(e^{-ikx}+\zeta e^{-i\zeta kx}+\zeta^2 e^{-i\zeta^2 kx}\right) q_0(x)\,\mathrm{d}x$$

$$
\begin{aligned}
&= \left(e^{-ikL}+e^{-i\zeta kL}+e^{-i\zeta^2 kL}\right)\left(-\frac{1}{k^4}q_0'''(L)\right)+\frac{3}{k^4}q_0'''(0)\\
&\quad+\left(e^{-ikL}+\zeta^2 e^{-i\zeta kL}+\zeta e^{-i\zeta^2 kL}\right)\left(\frac{i}{k^5}q_0^{(iv)}(L)\right)\\
&\quad+\left(e^{-ikL}+\zeta e^{-i\zeta kL}+\zeta^2 e^{-i\zeta^2 kL}\right)\left(-\frac{i}{k^3}q_0''(L)+\frac{1}{k^6}q_0^{(v)}(L)\right)\\
&\quad-\frac{1}{k^6}\int_0^L\left(e^{-ikx}+\zeta e^{-i\zeta kx}+\zeta^2 e^{-i\zeta^2 kx}\right)q_0^{(vi)}(x)\,\mathrm{d}x\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\hat{q}_0(k)+\zeta\hat{q}_0(\zeta k)+\zeta^2\hat{q}_0(\zeta^2 k) \;=\; &\frac{1}{k^4L^2}\tilde{\Delta}''(k)q_0'''(L)+\frac{3}{k^4}q_0'''(0)-\frac{1}{k^5L}\tilde{\Delta}'(k)q_0^{(iv)}(L)\\
&+\tilde{\Delta}(k)\left(-\frac{i}{k^3}q_0''(L)+\frac{1}{k^6}q_0^{(v)}(L)\right)\\
&-\frac{1}{k^6}\int_0^L\left(e^{-ikx}+\zeta e^{-i\zeta kx}+\zeta^2 e^{-i\zeta^2 kx}\right)q_0^{(vi)}(x)\,\mathrm{d}x\,.
\end{aligned}
$$

This expression can be further simplified using the relations

$$
q_0'''(0)=-\frac{\partial}{\partial t}q_0(0)=0\,,\quad q_0'''(L)=-\frac{\partial}{\partial t}q_0(L)=0\,,\quad q_0^{(iv)}(L)=-\frac{\partial}{\partial t}q_0'(L)=0\,,
$$

implied by the PDE itself, when the boundary conditions are homogeneous. Hence

$$
\begin{aligned}
\hat{q}_0(k)+\zeta\hat{q}_0(\zeta k)+\zeta^2\hat{q}_0(\zeta^2 k) \;=\; &\tilde{\Delta}(k)\left(-\frac{i}{k^3}q_0''(L)+\frac{1}{k^6}q_0^{(v)}(L)\right)\\
&-\frac{1}{k^6}\int_0^L\left(e^{-ikx}+\zeta e^{-i\zeta kx}+\zeta^2 e^{-i\zeta^2 kx}\right)q_0^{(vi)}(x)\,\mathrm{d}x\,.
\end{aligned}
$$

Further integration by parts achieves terms of the form $\tilde{\Delta}(k)q_0^{(p)}(L)$ where p is an arbitrary positive integer congruent to 2 modulo 3. Hence,

$$
\hat{q}_0(k)+\zeta\hat{q}_0(\zeta k)+\zeta^2\hat{q}_0(\zeta^2 k)=\tilde{\Delta}(k)\left(-\frac{i}{k^3}q_0''(L)+\frac{1}{k^6}q_0^{(v)}(L)+\cdots\right)\,.
$$

Therefore the integral around $\partial D_c^- = \left\{k : \arg(k)=\frac{4\pi}{3}\right\}\cup\left\{k : \arg(k)=\frac{5\pi}{3}\right\}$ involving the initial data, given by (3.59), can be written as

$$
\int_{\partial D_c^-} e^{ik(x-L)+ik^3t}\left(-\frac{i}{k^3}q_0''(L)+\frac{1}{k^6}q_0^{(v)}(L)+\cdots\right)\mathrm{d}k\,.
$$

The integrand is an analytic function, and it follows via Cauchy's Theorem (Theorem 1.3.1) that the integral vanishes. □

Therefore, when the boundary conditions are homogeneous, $N(t,k) = \hat{q}_0(k)$ and the integral representation of the solution is given by

$$q(x,t) = \frac{1}{2\pi} \int_{-\mathbb{R}\cup\partial D^+} e^{ikx+ik^3t} \left(\frac{\hat{q}_0(k) + \zeta\hat{q}_0(\zeta k) + \zeta^2\hat{q}_0(\zeta^2 k)}{e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL}} e^{-ikL} - \hat{q}_0(k) \right) \mathrm{d}k \,.$$

3.2.6 General Uncoupled Boundary Conditions

We have analysed in detail the case when the given boundary conditions are $q(0,t) = f_0(t)$, $q(L,t) = g_0(t)$ and $q_x(L,t) = g_1(t)$ and have shown that the corresponding determinant function of the algebraic system of global relations, is given by

$$\Delta(k) \sim \left(e^{-ikL} + \zeta e^{-i\zeta^2 kL} + \zeta^2 e^{-i\zeta^2 kL} \right) \,,$$

with zeros that cluster asymptotically along the three semilines L_1, L_2 and L_3 given by

$$L_1 = \left\{ k : \arg(k) = \frac{\pi}{6} \right\} \,, \quad L_2 = \left\{ k : \arg(k) = \frac{5\pi}{6} \right\} \,, \quad L_3 = \left\{ k : \arg(k) = \frac{2\pi}{3} \right\} \,. \tag{3.60}$$

In general, the following lemma is valid and the proof is a simple computation.

Lemma 3.2.22. *The determinant function $\Delta(k)$ of the system of global relations, obtained from the imposition of one of the functions $\tilde{f}_j(t,k)$ and two of the functions $\tilde{g}_j(t,k)$, $j = 0,1,2$, has (up to linear terms in ζ and ζ^2) one of the following three forms:*

i.) $\Delta(k) \sim e^{ikL} \left(e^{-ikL} + \zeta e^{-i\zeta kL} + \zeta^2 e^{-i\zeta^2 kL} \right)$,

ii.) $\Delta(k) \sim e^{ikL} \left(e^{-ikL} + e^{-i\zeta kL} + e^{-i\zeta^2 kL} \right)$,

iii.) $\Delta(k) \sim e^{ikL} \left(e^{-ikL} + \zeta^2 e^{-i\zeta kL} + \zeta e^{-i\zeta^2 kL} \right)$.

Hence, if the boundary conditions are uncoupled, an application of the general result given in [33], summarised by Levin's Theorem (Theorem 1.3.22), implies that the argument of the zeros depends only on the exponents in the exponential terms, which in all three cases is the same. Therefore, for all cases of uncoupled boundary conditions, the determinant function $\Delta(k)$ always has infinitely many zeros accumulating at infinity and clustering asymptotically along the lines L_1, L_2 and L_3, given by (3.60). These zeros are given pictorially in Figure 3.14 for the third order problem $q_t(x,t) + q_{xxx}(x,t) = 0$. It

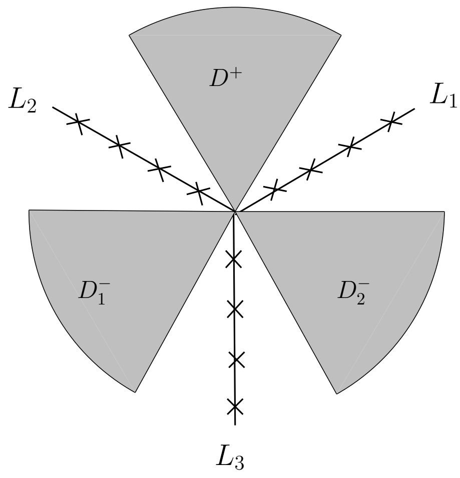


Figure 3.14: The regions $D^{\pm}$ for the third order problem $q_t(x,t) + q_{xxx}(x,t) = 0$ with uncoupled boundary conditions, and the location of the zeros of the determinant function $\Delta(k)$ of the form given in Lemma 3.2.22.

follows that in all cases the zeros of the determinant function lie outside of the domain D. Hence $\Delta(k) \neq 0$ for $k \in D$ and it is therefore necessary to deform the contours of integration to the rays upon which the zeros lie to explicitly compute the principal value contributions at the zeros and realise the series representation of the solution. However, analyticity arguments prove that this deformation is not possible throughout D_c, and it is therefore never possible to write the integral representation of the solution as an equivalent discrete series representation.

It has been shown however that the deformation of the contours is possible in the domain D_c^-, and the terms integrated around this domain give rise to a series contribution. It is therefore possible to express the solution as the sum of an integral around a complex contour and an infinite discrete series.

Chapter 4

Higher Order Problems

In this chapter, we consider some extensions and generalisations to higher order of the results discussed in the previous sections for the second and third order problems. In the first part, we use the Fokas transform method to solve some illustrative examples of fourth order on the bounded domain $[0, L]$, and in the second part we study the eigenvalues of two-point boundary value problems and prove, for problems of order $n \leqslant 4$, that what we defined as the spectrum of the boundary value problem, coincides with the classical discrete spectrum of the associated differential operator in x. We also prove a general result regarding the location in the complex k-plane of the zeros of the determinant function $\Delta_n(k)$, for the general n^{th} order problem of the form (2.1).

It is well known that the fourth order linear differential operator $D_4 = \frac{\partial^4}{\partial x^4}$ is formally self-adjoint [7], and that on $[0, L]$ the adjointness properties of this operator depend on the particular boundary conditions imposed at the endpoints of the interval. The solution of such a boundary value problem admits always a series representation, but to find it one must in general consider not only the given problem but also its adjoint.

We begin by recalling the characterisation of the boundary conditions for which the operator is self-adjoint, and in general, given such a problem, which corresponding problem is its adjoint. We also recall how one can find the series representation of the solution using the separation of variables approach. We then apply the method to solve a self-adjoint example, and two examples of problems that are not self-adjoint but are adjoint of each other. In both cases, our transform method can be used to derive the series representation of the solution without making any assumption on the existence of a bi-orthogonal basis, or any knowledge of the eigenvalues and eigenfunctions of the associated differential operator.

4.1 Fourth Order Problems

4.1.1 The Characterisation of Boundary Conditions

In this section, we study uncoupled boundary conditions for a fourth order two-point boundary value problem. We only look at conditions of the form (4.2) below. It is not difficult to generalise our results to any form of uncoupled boundary conditions.

We analyse those among such conditions that yield a self-adjoint operator, and discuss the boundary conditions for which the operator fails to satisfy the requirements for self-adjointness, and for these cases indicate what the adjoint problem is.

For the remainder of this chapter, we shall consider a specific fourth order PDE, given by

$$q_t(x,t) + q_{xxxx}(x,t) = 0\,, \quad t > 0\,, \quad x \in [0, L]\,, \tag{4.1a}$$

$$q(x,0) = q_0(x)\,, \quad x \in [0, L]\,, \tag{4.1b}$$

where $q_0(x)$ is a given function. For the problem to be well-posed, two boundary conditions must be prescribed at $x = 0$ and two boundary conditions must be prescribed at $x = L$, according to Theorem 2.1.1, and we shall consider the imposition of uncoupled boundary conditions of the form

$$q_x^{(i)}(0,t) = f_i(t)\,, \quad q_x^{(j)}(L,t) = g_j(t)\,, \quad i,j \in \{0,1,2,3\}\ , \tag{4.2}$$

for some given functions $f_i(t)$ and $g_j(t)$. It follows, from (1.25), that for the fourth order problem to be self-adjoint, the boundary conditions that are imposed must be such that the boundary terms that arise during the integration by parts process are eliminated. For any functions $q(x), r(x) \in L^2\left([0,L]\right)$, integration by parts yields

$$\begin{aligned}
\langle D_4 q(x), r(x)\rangle &= \int_0^L D_4 q(x)\,\overline{r}(x)\,\mathrm{d}x \\
&= \Big[D_3 q(x)\,\overline{r}(x) - D_2 q(x)\,D\overline{r}(x) + Dq(x)\,D_2\overline{r}(x) \\
&\qquad - q(x)\,D_3\overline{r}(x) \Big]_0^L + \langle q(x), D_4 r(x)\rangle\,.
\end{aligned} \tag{4.3}$$

If the boundary conditions are such that not all of the boundary terms in the brackets in (4.3) can be eliminated, then the resulting operator is not self-adjoint. There are four cases for which the fourth order operator $D_4 = \frac{\partial^4}{\partial x^4}$ is not self-adjoint:

i.) $q(0,t) = f_0(t)$, $q_{xxx}(0,t) = f_3(t)$, $q(L,t) = g_0(t)$ and $q_{xxx}(L,t) = g_3(t)$,

ii.) $q(0,t) = f_0(t)$, $q_{xxx}(0,t) = f_3(t)$, $q_x(L,t) = g_1(t)$ and $q_{xx}(L,t) = g_2(t)$,

iii.) $q_x(0,t) = f_1(t)$, $q_{xx}(0,t) = f_2(t)$, $q(L,t) = g_0(t)$ and $q_{xxx}(L,t) = g_3(t)$,

iv.) $q_x(0,t) = f_1(t)$, $q_{xx}(0,t) = f_2(t)$, $q_x(L,t) = g_1(t)$ and $q_{xx}(L,t) = g_2(t)$.

The boundary conditions posed by (i) and (iv) are adjoint to each other, and similarly the boundary conditions posed by (ii) and (iii) are adjoint to each other. This implies that the imposition of the boundary conditions given by (i) yield eigenfunctions $e_n(x)$ satisfying the boundary conditions $e_n(0) = 0$, $e_n''(0) = 0$ and $e_n(L) = 0$, $e_n''(L) = 0$, defining problem (iv). Similarly, the imposition of the boundary conditions given by (ii), yield eigenfunctions $e_n(x)$ satisfying the boundary conditions $e_n'(0) = 0$, $e_n''(0) = 0$ and $e_n(L) = 0$, $e_n'''(L) = 0$, defining problem (iii). Analogously, the imposition of the boundary conditions given by (iii) and (iv) yield eigenfunctions satisfying the boundary conditions defining problems (ii) and (i) respectively.

When the boundary conditions yield a self-adjoint operator, the method of separation of variables can be used to obtain the discrete series representation of the solution. However, for the fourth order problem, this approach becomes cumbersome when the boundary conditions are non-homogeneous. When the operator is not self-adjoint, the method of separation of variables can be used to derive the series representation, but to find it one must also consider the adjoint problem. In both cases, the results rely on the specific knowledge of the relevant eigenvalues.

In the next section, we consider the spectral representation of the solution of two-point boundary value problems for fourth order linear evolution PDEs. We shall begin with an example for which the boundary conditions yield a self-adjoint operator. Thereafter we consider boundary value problems such that the operator fails to satisfy the conditions for self-adjointness. In all cases the Fokas transform method yields an integral representation of the solution, which is shown to be equivalent to the series representation of the solution.

4.1.2 The Solution Representation

In this section we consider the fourth order PDE, given by (4.1), with the imposition of the boundary conditions of the form (4.2).

For every case of boundary conditions we derive the integral representation. We aim then to show that the integral representation of the solution can always be written as an infinite discrete series. To do this, we apply the Fokas transform method to (4.1) for a variety of boundary conditions and show that, although the zeros of the determinant function $\Delta(k)$ are always such that $\Delta(k) \neq 0$ for $k \in D$, the integration contours appearing in the representation can be deformed to include them. More specifically, we prove that, as in the case of the heat equation, the zeros always lie on the rays that bisect the domain D_c.

In Section 4.1.4 we consider the imposition of the boundary conditions $q(0,t) = f_0(t)$, $q(L,t) = g_0(t)$, $q_{xx}(0,t) = f_2(t)$ and $q_{xx}(L,t) = g_2(t)$ for some given smooth functions $f_0(t)$, $f_2(t)$, $g_0(t)$ and $g_2(t)$, and show that in this case the operator is self-adjoint and the appropriate basis of eigenfunctions is algorithmically constructed. Hence the solution is easily expressible as an infinite discrete series. In Section 4.1.5 we consider the imposition of the boundary conditions $q(0,t) = f_0(t)$, $q(L,t) = g_0(t)$, $q_{xxx}(0,t) = f_3(t)$ and $q_{xxx}(L,t) = g_3(t)$, for some given smooth functions $f_0(t)$, $g_0(t)$, $f_3(t)$ and $g_3(t)$, and show that the method does not directly yield the appropriate basis of eigenfunctions, but rather the adjoint basis.

The Main Elements of the Solution Method

Following the steps of the method outlined in Section 2.1.1, we derive the Lax pair, given by (2.11), where

$$\omega(k) = k^4 ,$$

$$X(x,t,k) = ik^3 q(x,t) + k^2 q_x(x,t) - ikq_{xx}(x,t) - q_{xxx}(x,t) .$$

The dispersion relation implies that the domain D comprises the four connected components given by

$$\left.\begin{aligned} D_1^+ &= \left\{k \in \mathbb{C} : \tfrac{\pi}{8} \leqslant \arg(k) \leqslant \tfrac{3\pi}{8}\right\} \\ D_2^+ &= \left\{k \in \mathbb{C} : \tfrac{5\pi}{8} \leqslant \arg(k) \leqslant \tfrac{7\pi}{8}\right\} \end{aligned}\right\} \quad D^+ = D_1^+ \cup D_2^+ , \tag{4.4}$$

$$\left.\begin{aligned} D_1^- &= \left\{k \in \mathbb{C} : \tfrac{9\pi}{8} \leqslant \arg(k) \leqslant \tfrac{11\pi}{8}\right\} \\ D_2^- &= \left\{k \in \mathbb{C} : \tfrac{13\pi}{8} \leqslant \arg(k) \leqslant \tfrac{15\pi}{8}\right\} \end{aligned}\right\} \quad D^- = D_1^- \cup D_2^- \,, \tag{4.5}$$

given in Figure 4.1.

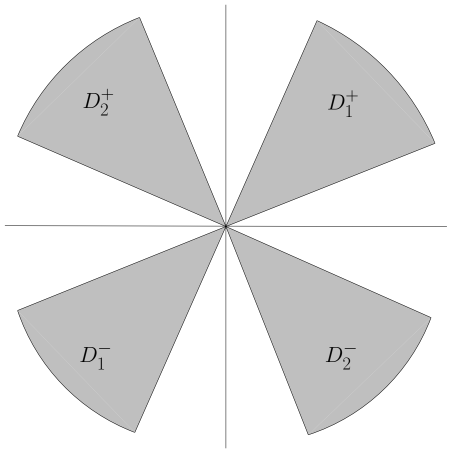


Figure 4.1: The regions $D_1^+ = \left\{k \in \mathbb{C} : \frac{\pi}{8} \leqslant \arg(k) \leqslant \frac{3\pi}{8}\right\}$, $D_2^+ = \left\{k \in \mathbb{C} : \frac{5\pi}{8} \leqslant \arg(k) \leqslant \frac{7\pi}{8}\right\}$, $D_1^- = \left\{k \in \mathbb{C} : \frac{9\pi}{8} \leqslant \arg(k) \leqslant \frac{11\pi}{8}\right\}$ and $D_2^- = \left\{k \in \mathbb{C} : \frac{13\pi}{8} \leqslant \arg(k) \leqslant \frac{15\pi}{8}\right\}$ for the fourth order problem $q_t(x,t) + q_{xxxx}(x,t) = 0$.

The functions $c_0 = ik^3$, $c_1 = k^2$, $c_2 = -ik$ and $c_3 = -1$ imply that the functions $\tilde{f}(t,k)$ and $\tilde{g}(t,k)$, according to (2.23) and (2.24) respectively, are given by

$$\begin{aligned} \tilde{f}(t,k) &= ik^3 \tilde{f}_0(t,k) + k^2 \tilde{f}_1(t,k) - ik\tilde{f}_2(t,k) - \tilde{f}_3(t,k)\,, \\ \tilde{g}(t,k) &= ik^3 \tilde{g}_0(t,k) + k^2 \tilde{g}_1(t,k) - ik\tilde{g}_2(t,k) - \tilde{g}_3(t,k)\,, \end{aligned}$$

where $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$, $j = 0,1,2,3$, represent the t-transforms of the boundary functions. The integral representation of the solution is given by (2.56) where the domains D^+ and D^- in the complex k-plane are defined by (4.4) and (4.5) respectively. Hence

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\Bigg\{ &\int_{-\infty}^{\infty} e^{ikx-k^4t} \hat{q}_0(k)\,\mathrm{d}k \\ &- \int_{\partial D^+} e^{ikx-k^4t} \left(ik^3 \tilde{f}_0(t,k) + k^2 \tilde{f}_1(t,k) - ik\tilde{f}_2(t,k) - \tilde{f}_3(t,k)\right) \mathrm{d}k \\ &- \int_{\partial D^-} e^{ik(x-L)-k^4t} \left(ik^3 \tilde{g}_0(t,k) + k^2 \tilde{g}_1(t,k) - ik\tilde{g}_2(t,k) - \tilde{g}_3(t,k)\right) \mathrm{d}k \Bigg\}. \end{aligned} \tag{4.6}$$

In order to rewrite the unknown boundary functions in terms of the known functions, it is necessary to exploit the invariance properties of the functions $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$, $j=0,1,2,3$. The transformations that leave $\omega(k)$ invariant are determined by the roots of the equation $\omega(k)=\omega(\lambda)$ and are given explicitly by

$$\lambda_0(k)=k\,,\quad \lambda_1(k)=ik\,,\quad \lambda_2(k)=-k\,,\quad \lambda_3(k)=-ik\,. \tag{4.7}$$

Therefore, the system of global relations, given by (2.30), relating the eight unknown boundary functions $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$, $j=0,1,2,3$, is given by

$$\begin{aligned}
&ik^3\left(\tilde{f}_0(t,k)-e^{-ikL}\tilde{g}_0(t,k)\right)+k^2\left(\tilde{f}_1(t,k)-e^{-ikL}\tilde{g}_1(t,k)\right)\\
&\quad -ik\left(\tilde{f}_2(t,k)-e^{-ikL}\tilde{g}_2(t,k)\right)-\left(\tilde{f}_3(t,k)-e^{-ikL}\tilde{g}_3(t,k)\right) = \hat{q}_0(k)-e^{k^4t}\hat{q}(t,k)\,,\\
&k^3\left(\tilde{f}_0(t,k)-e^{kL}\tilde{g}_0(t,k)\right)-k^2\left(\tilde{f}_1(t,k)-e^{kL}\tilde{g}_1(t,k)\right)\\
&\quad +k\left(\tilde{f}_2(t,k)-e^{kL}\tilde{g}_2(t,k)\right)-\left(\tilde{f}_3(t,k)-e^{kL}\tilde{g}_3(t,k)\right) = \hat{q}_0(ik)-e^{k^4t}\hat{q}(t,ik)\,,\\
&-ik^3\left(\tilde{f}_0(t,k)-e^{ikL}\tilde{g}_0(t,k)\right)+k^2\left(\tilde{f}_1(t,k)-e^{ikL}\tilde{g}_1(t,k)\right)\\
&\quad +ik\left(\tilde{f}_2(t,k)-e^{ikL}\tilde{g}_2(t,k)\right)-\left(\tilde{f}_3(t,k)-e^{ikL}\tilde{g}_3(t,k)\right) = \hat{q}_0(-k)-e^{k^4t}\hat{q}(t,-k)\,,\\
&-k^3\left(\tilde{f}_0(t,k)-e^{-kL}\tilde{g}_0(t,k)\right)-k^2\left(\tilde{f}_1(t,k)-e^{-kL}\tilde{g}_1(t,k)\right)\\
&\quad -k\left(\tilde{f}_2(t,k)-e^{-kL}\tilde{g}_2(t,k)\right)-\left(\tilde{f}_3(t,k)-e^{-kL}\tilde{g}_3(t,k)\right) = \hat{q}_0(-ik)-e^{k^4t}\hat{q}(t,-ik)\,.
\end{aligned} \tag{4.8}$$

After the imposition of four well-posed boundary conditions, the four global relations that result, form a system that is solvable for the remaining four unknown boundary functions.

4.1.3 The Determinant Function

In this section we study the determinant function that arises from the system of global relations, given by (4.8), after the imposition of four well-posed boundary conditions. In general the following proposition is valid:

Proposition 4.1.1. *The determinant function $\Delta(k)$ of the system of global relations, given by (4.8), obtained from the imposition of two of the $\tilde{f}_j(t,k)$ and two of the $\tilde{g}_j(t,k)$,*

$j = 0, 1, 2, 3$, *is always of the form*

$$\Delta(k) \sim \alpha_1 + \alpha_2 e^{(1+i)kL} + \alpha_3 e^{(-1+i)kL} + \alpha_4 e^{(1-i)kL} + \alpha_5 e^{(-1-i)kL} \,,$$

where $\alpha_i \neq 0$, $i = 0, 1, \dots 5$, *are constant coefficients and the zeros* k_n, *such that* $\Delta(k_n) = 0$, *lie on the real and imaginary axes in the complex* k*-plane.*

Proof. The matrix corresponding to the system of global relations, written in terms of the eight unknown boundary functions $\tilde{f}_j(t,k)$ and $\tilde{g}_j(t,k)$, $j = 0, 1, 2, 3$, is of the form

$$\left(A1 \middle| A2 \right) = \left(\begin{array}{cccc|cccc} i & 1 & -i & 1 & -ie^{-ikL} & -e^{-ikL} & ie^{-ikL} & e^{-ikL} \\ 1 & -1 & 1 & 1 & -e^{kL} & e^{kL} & -e^{kL} & e^{kL} \\ -i & 1 & i & 1 & ie^{ikL} & -e^{ikL} & -ie^{ikL} & e^{ikL} \\ -1 & -1 & -1 & 1 & e^{-kL} & e^{-kL} & e^{-kL} & e^{-kL} \end{array} \right) .$$

The imposition of four boundary conditions (two at $x = 0$ and two at $x = L$) corresponds to selecting two columns from each of the matrices $A1$ and $A2$.

The determinant function $\Delta(k)$ of the resulting 4×4 matrix, will comprise a linear combination of exponential terms whose exponents correspond to summing two of the λ_j's, $j = 0, 1, 2, 3$, defined by (4.7). These terms are given explicitly as

$$e^{(\lambda_0+\lambda_1)L}, \quad e^{(\lambda_0+\lambda_2)L}, \quad e^{(\lambda_0+\lambda_3)L}, \quad e^{(\lambda_1+\lambda_2)L}, \quad e^{(\lambda_1+\lambda_3)L}, \quad e^{(\lambda_2+\lambda_3)L} .$$

Taking into account expression (4.7) for the λ_j's and that $\lambda_0 + \lambda_2 = \lambda_1 + \lambda_3 = 0$, it follows that the determinant function is of the form

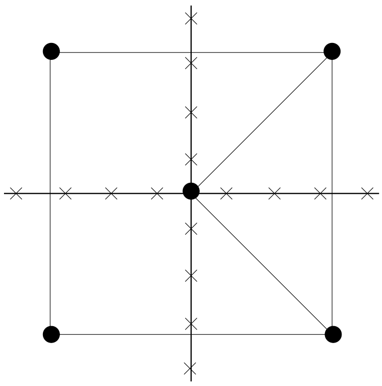
$$\Delta(k) \sim \alpha_1 + \alpha_2 e^{(1+i)kL} + \alpha_3 e^{(-1+i)kL} + \alpha_4 e^{(1-i)kL} + \alpha_5 e^{(-1-i)kL} \,, \tag{4.9}$$

for arbitrary constants $\alpha_1, \dots, \alpha_5$, and the first part of the proof is complete.

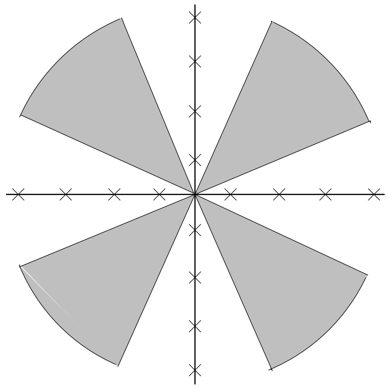
The proof that the zeros lie on the real and imaginary axes in the complex k-plane, follows immediately from an application of Levin's Theorem (Theorem 1.3.22). The transformation $z = -ikL$ is applied to the determinant function, and the five exponents

$$0 \,, \quad (-1+i)z \,, \quad (-1-i)z \,, \quad (1+i)z \,, \quad (1-i)z \,,$$

indicate the four rays emanating from the origin, passing through the points $1+i$, $-1+i$, $-1-i$ and $1-i$ in the complex z-plane, (Figure 4.2(a)). These points are joined to form a convex hull (remarking that the point at the origin is contained within this hull).



(a) z-plane ($z = -ikL$).



(b) k-plane.

Figure 4.2: The regions $D^{\pm}$ for the fourth order problem $q_t(x,t) + q_{xxxx}(x,t) = 0$ with the boundary conditions $q(0,t) = f_0(t)$, $q(L,t) = g_0(t)$, $q_x(0,t) = f_1(t)$ and $q_x(L,t) = g_1(t)$ and the location of the zeros of the determinant function $\Delta(k) = -8i + 2i\left(e^{(1+i)kL} + e^{(-1+i)kL} + e^{(1-i)kL} + e^{(-1-i)kL}\right)$.

The zeros cluster along the rays that perpendicularly bisect the sides of the polygon that is formed. In the complex k-plane, these zeros accumulate only at infinity and are clustered exactly along the real and imaginary axes, which bisect the domain D_c (Figure 4.2(b)). $\square$

The location of the zeros in the complex k-plane, of the determinant function $\Delta(k)$, can be found graphically and the following proposition (Proposition 4.1.2) gives the alternative forms, in terms of trigonometric functions, that the determinant function can take. The proof is less concise than the the proof of Proposition 4.1.1, but makes no reference to general results.

Proposition 4.1.2. *The deteminant function $\Delta(k)$ of the system of global relations, given by (4.8), obtained from the imposition of two of the $\tilde{f}_j(t,k)$ and two of the $\tilde{g}_j(t,k)$, $j = 0, 1, 2, 3$, always has one of the following forms:*

i.) $\Delta(k) \sim \cos(kL)\cosh(kL) \pm 1$,

ii.) $\Delta(k) \sim \sin(kL)\sinh(kL)$,

iii.) $\Delta(k) \sim \cos(kL)\cosh(kL)$,

iv.) $\Delta(k) \sim \cosh(kL)\sin(kL) \pm \cos(kL)\sinh(kL)$,

and the zeros of the determinant function always lie on the real and imaginary axes in the complex k-plane.

Proof. A simple computation proves that (4.9) can be written in one of the forms above. All of the proofs for locating the zeros of the determinant function, use geometric constructions and trigonometric identities:

i.) The zeros of $\cos(kL)\cosh(kL) \pm 1 = 0$ are found to be the intercept points of the functions $\cosh(kL)$ and $\pm\sec(kL)$ in the complex k-plane. We give the proof for the case $\cos(kL)\cosh(kL) - 1 = 0$ and remark that the proof for $\cos(kL)\cosh(kL) + 1 = 0$ follows analogously.

Plotting $\cosh(kL)$ against $\sec(kL)$ in the complex k-plane, implies two sets, $\{r_n\}$ and $\{s_n\}$, of real zeros (Figure 4.3). Since $|\cos(kL)| \leqslant 1\ \forall k$, it follows that

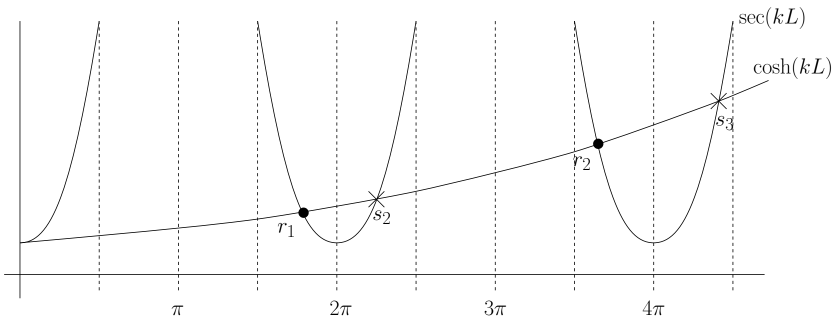


Figure 4.3: The real roots $\{r_n\}$ and $\{s_n\}$ of the equation $\cos(kL)\cosh(kL) = 1$ associated with the determinant function $\Delta(k) = 8i\,(-1 + \cos(kL)\cosh(kL))$ of the fourth order problem $q_t(x,t) + q_{xxxx}(x,t) = 0$ with the boundary conditions $q(0,t) = f_0(t)$, $q(L,t) = g_0(t)$, $q_x(0,t) = f_1(t)$ and $q_x(L,t) = g_1(t)$.

$|\sec(kL)| \geqslant 1\ \forall k$. In the interval $[0, \frac{\pi}{2}]$ the graphs of $\cosh(kL)$ and $\sec(kL)$ intersect at $k = 0$. Since $\cosh(kL)$ is an increasing function there must be a point of intersection in every interval of the form $[\frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2}]$ for $n = 0, 1, 2, \ldots$, and given there are an infinite number of intervals there must be an infinite number of intersection points. Therefore, there are an infinite number of real roots. These take the form

$$r_n \approx \frac{(4n-1)\pi}{2}, \qquad s_n \approx \frac{(4n-3)\pi}{2}, \qquad n = 1, 2, \ldots,$$

and an application of Newton's method achieves the improved approximations

$$r_n \approx \frac{(4n-1)\pi}{2} + \operatorname{sech}\frac{(4n-1)\pi}{2}, \qquad s_n \approx \frac{(4n-3)\pi}{2} - \operatorname{sech}\frac{(4n-3)\pi}{2}.$$

The existence of infinitely many purely imaginary roots in the complex k-plane is an immediate consequence of the identities

$$\cos(ikL) = \cosh(kL)\,, \quad \cosh(ikL) = \cos(kL)\,. \tag{4.10}$$

It follows that if $\Delta(k_n) = 0$ then $\Delta(ik_n) = 0$, and the proof is complete.

ii.) The zeros k_n of $\sin(kL)\,\sinh(kL) = 0$ are found trivially. The zeros of $\sin(kL)$ are given by $k_n = \frac{n\pi}{L}$, $n \in \mathbb{Z}$ and imply an infinite number of zeros lying on the real axis. Using the identities

$$\sin(ik_nL) = i\sinh(k_nL)\,, \quad i\sin(k_nL) = \sinh(ik_nL)\,,$$

it follows that if $\Delta(k_n) = 0$ then $\Delta(ik_n) = 0$.

iii.) The zeros k_n of $\cos(kL)\,\cosh(kL) = 0$ are found analogously to (ii). The zeros of $\cos(kL)$ are given by $k_n = \frac{(2n+1)\pi}{2L}$, $n \in \mathbb{Z}$ and imply an infinite number of zeros lying on the real axis. Using the identities, given by (4.10), it follows that if $\Delta(k_n) = 0$ then $\Delta(ik_n) = 0$.

iv.) We give the proof for the function $\Delta(k) \sim \cosh(kL)\,\sin(kL) - \cos(kL)\,\sinh(kL)$ and remark that the proof for the other case follows analogously. The positive real roots of $\cosh(kL)\,\sin(kL) - \cos(kL)\,\sinh(kL) = 0$ are given by the intercept points of the functions $\tan(kL)$ and $\tanh(kL)$, (Figure 4.4). In the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the

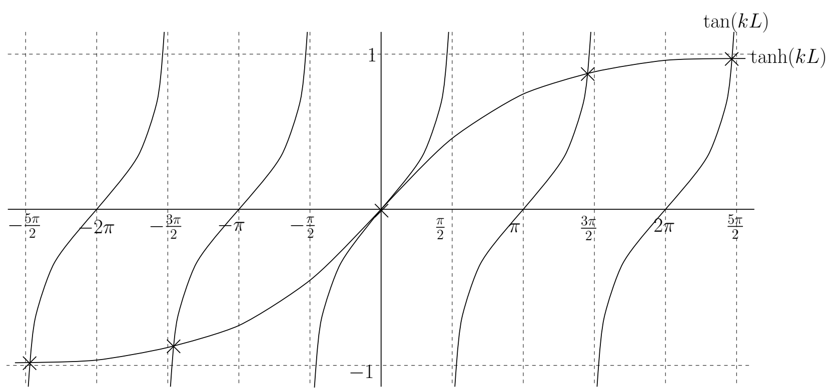


Figure 4.4: The real roots of the equation $\cosh(kL)\,\sin(kL) - \cos(kL)\,\sinh(kL) = 0$ associated with the determinant function $\Delta(k)$ of the fourth order problem $q_t(x,t) + q_{xxxx}(x,t) = 0$.

graphs intersect at $k = 0$. Since $\tanh(kL)$ increases asymptotically to 1, and

$\tan(kL)$ has period π, there must be a point of intersection in every interval of the form

$$\left[\frac{(2n+1)\pi}{2}, \frac{(2n+3)\pi}{2}\right], \quad n = 0, 1, 2, \ldots .$$

Therefore, since there are an infinite number of intervals, there must be an infinite number of intersection points and hence an infinite number of real roots of the equation $\cosh(kL)\,\sin(kL) - \cos(kL)\,\sinh(kL) = 0$. A similar argument holds for the proof that there are an infinite number of negative real roots of the equation $\cosh(kL)\,\sin(kL) - \cos(kL)\,\sinh(kL) = 0$.

The existence of infinitely many purely imaginary roots is proven as follows. Suppose that ikL is an imaginary root. Then

$$\cosh(ikL)\,\sin(ikL) - \cos(ikL)\,\sinh(ikL) = 0$$

$$\Rightarrow \cos(kL)\,\sinh(kL) - \cosh(kL)\,\sin(kL) = 0\,,$$

which is true, and the proof is complete.

$\square$

Remark 4.1.3. We give a simple argument ruling out the existence of any other complex roots. Consider the function

$$\Delta(z) = \cosh(z)\,\sin(z) - \cos(z)\,\sinh(z)\,, \quad z \in \mathbb{C}\,.$$

Writing $z = x + iy$, $x, y, \in \mathbb{R}$, we obtain

$$\begin{aligned}\Delta(x+iy) &= -\frac{1}{4}\Big\{ \left(e^{ix-y} + e^{-ix+y}\right)\left(e^{x+iy} - e^{-x-iy}\right) \\ &\qquad - i\left(e^{x+iy} + e^{-x-iy}\right)\left(e^{ix-y} - e^{-ix+y}\right) \Big\}\,.\end{aligned}$$

Assume, without loss of generality, that $x, y > 0$. Then

$$\lim_{x\to\infty} e^{-x-iy} = 0\,, \qquad \lim_{y\to\infty} e^{ix-y} = 0\,,$$

and therefore

$$\begin{aligned}
\lim_{\substack{x,y,\to\infty\\ x,y,>0}} |\Delta(z)|^2 &= \frac{1}{16} \lim_{\substack{x,y,\to\infty\\ x,y,>0}} \left|(1-i)e^{x+y}e^{i(y-x)}\right|^2 \\
&= \frac{1}{8} \lim_{\substack{x,y,\to\infty\\ x,y,>0}} \left|e^{x+y}e^{i(y-x)}\right|^2 \\
&= \frac{1}{8} \lim_{\substack{x,y,\to\infty\\ x,y,>0}} \left|e^{x+y}\right|^2 \neq 0\,.
\end{aligned}$$

4.1.4 The Case of a Self-Adjoint Operator

In this section we consider the imposition of boundary conditions of the form (4.2), for which the operator is self-adjoint. As an illustrative example, we consider the imposition of the boundary conditions

$$q(0,t) = f_0(t)\,, \quad q(L,t) = g_0(t)\,, \quad q_{xx}(0,t) = f_2(t)\,, \quad q_{xx}(L,t) = g_2(t)\,,$$

for some given smooth functions $f_0(t)$, $g_0(t)$, $f_2(t)$ and $g_2(t)$. The system of global relations, given by (4.8), simplifies to yield

$$\begin{aligned}
k^2\left(\tilde{f}_1(t,k) - e^{-ikL}\tilde{g}_1(t,k)\right) - \left(\tilde{f}_3(t,k) - e^{-ikL}\tilde{g}_3(t,k)\right) &= N(t,k) - e^{k^4t}\hat{q}(t,k)\,, \\
-k^2\left(\tilde{f}_1(t,k) - e^{kL}\tilde{g}_1(t,k)\right) - \left(\tilde{f}_3(t,k) - e^{kL}\tilde{g}_3(t,k)\right) &= N(t,ik) - e^{k^4t}\hat{q}(t,ik)\,, \\
k^2\left(\tilde{f}_1(t,k) - e^{ikL}\tilde{g}_1(t,k)\right) - \left(\tilde{f}_3(t,k) - e^{ikL}\tilde{g}_3(t,k)\right) &= N(t,-k) - e^{k^4t}\hat{q}(t,-k)\,, \\
-k^2\left(\tilde{f}_1(t,k) - e^{-kL}\tilde{g}_1(t,k)\right) - \left(\tilde{f}_3(t,k) - e^{-kL}\tilde{g}_3(t,k)\right) &= N(t,-ik) - e^{k^4t}\hat{q}(t,-ik)\,,
\end{aligned} \tag{4.11}$$

where

$$N(t,k) = \hat{q}_0(k) - ik^3\left(\tilde{f}_0(t,k) - e^{-ikL}\tilde{g}_0(t,k)\right) + ik\left(\tilde{f}_2(t,k) - e^{-ikL}\tilde{g}_2(t,k)\right)\,,$$

is a known function in terms of the t-transforms of the given initial and boundary data. In matrix form we write the system as

$$\begin{pmatrix} 1 & 1 & -e^{-ikL} & e^{-ikL} \\ -1 & 1 & e^{kL} & e^{kL} \\ 1 & 1 & -e^{ikL} & e^{ikL} \\ -1 & 1 & e^{-kL} & e^{-kL} \end{pmatrix} \begin{pmatrix} k^2\tilde{f}_1(t,k) \\ -\tilde{f}_3(t,k) \\ k^2\tilde{g}_1(t,k) \\ \tilde{g}_3(t,k) \end{pmatrix} = \begin{pmatrix} N(t,k) \\ N(t,ik) \\ N(t,-k) \\ N(t,-ik) \end{pmatrix} - \begin{pmatrix} e^{k^4t}\hat{q}(t,k) \\ e^{k^4t}\hat{q}(t,ik) \\ e^{k^4t}\hat{q}(t,-k) \\ e^{k^4t}\hat{q}(t,-ik) \end{pmatrix},$$

where the determinant function $\Delta(k)$ of the system is given by

$$\Delta(k) = 16i\sin(kL)\,\sinh(kL)\,.$$

The Integral Representation

We now start from the integral representation of the solution, given by (4.6). The functions $\tilde{f}_1(t,k)$, $\tilde{f}_3(t,k)$, $\tilde{g}_1(t,k)$ and $\tilde{g}_3(t,k)$ are determined from solving the system of global relations, given by (4.11). This yields the following expressions, in terms of the known functions $N(t,\lambda_l(k))$, $l = 0,1,2,3$:

$$k^2\tilde{f}_1(t,k) - \tilde{f}_3(t,k) \;=\; \frac{4}{\Delta(k)}\left(e^{(1+i)kL} - e^{(-1+i)kL}\right)\left(N(t,k) - e^{-2ikL}N(t,-k)\right), \tag{4.12}$$

$$k^2\tilde{g}_1(t,k) - \tilde{g}_3(t,k) \;=\; \frac{4}{\Delta(k)}\left(e^{kL} - e^{-kL}\right)\left(N(t,k) - N(t,-k)\right). \tag{4.13}$$

An application of Proposition 2.4.3(i) proves that the unknown terms do not contribute to the integral representation of the solution. Expressions (4.12) and (4.13) are substituted into the integral representation of the solution, given by (4.6), to achieve the following solution:

$$\begin{aligned} q(x,t) \;=\; \frac{1}{2\pi}\Bigg\{ & \int_{-\infty}^{\infty} e^{ikx-k^4t}\hat{q}_0(k)\,\mathrm{d}k - \int_{\partial D^+} e^{ikx-k^4t}\left(ik^3\tilde{f}_0(t,k) - ik\tilde{f}_2(t,k)\right)\mathrm{d}k \\ & - \int_{\partial D^+} \frac{4e^{ikx-k^4t}}{\Delta(k)}\left(e^{(1+i)kL} - e^{(-1+i)kL}\right)\left(N(t,k) - e^{-2ikL}N(t,-k)\right)\mathrm{d}k \\ & - \int_{\partial D^-} e^{ik(x-L)-k^4t}\left(ik^3\tilde{g}_0(t,k) - ik\tilde{g}_2(t,k)\right)\mathrm{d}k \\ & - \int_{\partial D^-} \frac{4e^{ik(x-L)-k^4t}}{\Delta(k)}\left(e^{kL} - e^{-kL}\right)\left(N(t,k) - N(t,-k)\right)\mathrm{d}k \Bigg\}. \end{aligned} \tag{4.14}$$

The Direct Derivation of the Series Representation of the Solution

The series representation of the solution can easily be obtained from the system of global relations. Subtracting the third global relation from the first yields the following:

$$\left(k^2\tilde{g}_1(t,k) - \tilde{g}_3(t,k)\right)\left(e^{ikL} - e^{-ikL}\right) = N(t,k) - N(t,-k) - e^{k^4t}\left(\hat{q}(t,k) - \hat{q}(t,-k)\right).$$

Evaluating this expression at an arbitrary positive real zero $k = k_n = \frac{n\pi}{L}$, $n \in \mathbb{Z}$ we obtain

$$e^{-k_n^4 t} \left(N(t, k_n) - N(t, -k_n) \right) = \hat{q}(t, k_n) - \hat{q}(t, -k_n) .$$

Therefore

$$\int_0^L \left(e^{-ik_n x} - e^{ik_n x} \right) q(x,t) \, \mathrm{d}x = e^{-k_n^4 t} \left(N(t, k_n) - N(t, -k_n) \right) .$$

Since the family $e_n(x) = e^{ik_n x} - e^{-ik_n x} = 2i \sin(k_n x)$ is a complete orthogonal basis in $L^2([0, L])$, it follows that there exists functions $c_n(t)$ such that

$$q(x,t) = \sum_{n=1}^{\infty} c_n(t) \, e_n(x) .$$

The coefficients $c_n(t)$ can be found using the orthogonality of the basis functions, and are given by

$$c_n(t) = \frac{1}{2L} e^{-k_n^4 t} \left(N(t, k_n) - N(t, -k_n) \right) .$$

Therefore, the infinite discrete series representation of the solution is given by the sine series

$$\begin{aligned} q(x,t) &= \frac{1}{2L} \sum_{n=1}^{\infty} e^{-k_n^4 t} \left(N(t, k_n) - N(t, -k_n) \right) \left(e^{ik_n x} - e^{-ik_n x} \right) \\ &= \frac{i}{L} \sum_{n=1}^{\infty} e^{-k_n^4 t} \left(N(t, k_n) - N(t, -k_n) \right) \sin(k_n x) , \end{aligned} \tag{4.15}$$

where the index indicates that only the positive real zeros contribute to the series representation.

An Alternative Derivation of the Series Representation of the Solution

In this section we present an alternative derivation for the series representation of the solution, given by (4.15). For this approach we begin by proving that the contours of integration in (4.14) can be deformed throughout D_c. We then show that the only contribution to the solution is from the explicit computation at the zeros of the determinant function $\Delta(k)$, of the principal value contributions in the integral representation of the solution.

The derivation of the series representation of the solution via the global relation, relies on the assumption that $q(x,t)$ can be expressed as a generalised Fourier series, in terms of the eigenfunctions $e_n(x)$. We remark that no such assumptions, regarding any knowledge of bases, orthogonality or eigenfunctions etc., are required for the derivation based on the explicit computation of the residues.

Proposition 4.1.4. *The integrands of the integrals around ∂D^+ and ∂D^-, in (4.14), are analytic and bounded for $k \in D_c^+$ and $k \in D_c^-$ respectively.*

Proof. We prove the case only for the integrand, of the integral around ∂D^+, for k such that $0 < \arg(k) < \frac{\pi}{8}$, and remark that a similar argument can be used for the other regions in D_c.

The integrand, of the integral around ∂D^+, in (4.14), is given explicitly by

$$e^{ikx-k^4t}\Bigg\{\left(ik^3\tilde{f}_0(t,k) - ik\tilde{f}_2(t,k)\right) \\ + \frac{4}{\Delta(k)}\left(e^{(1+i)kL} - e^{(-1+i)kL}\right)\left(N(t,k) - e^{-2ikL}N(t,-k)\right)\Bigg\}.$$

Consider the wedge such that $0 < \arg(k) < \frac{\pi}{8}$. Then for k in this wedge ik will be such that $\frac{\pi}{2} < \arg(ik) < \frac{5\pi}{8}$, $-k$ will be such that $\pi < \arg(-k) < \frac{9\pi}{8}$ and $-ik$ will be such that $\frac{3\pi}{2} < \arg(-ik) < \frac{13\pi}{8}$ (Figure 4.5). Hence in this wedge

- e^{ikx-k^4t} is bounded,
- e^{ikL} and e^{-kL} are bounded,
- e^{-ikL} and e^{kL} are unbounded.

Now, the denominator, $\Delta(k) \sim e^{(1+i)kL} - e^{(-1+i)kL} - e^{(1-i)kL} + e^{(-1-i)kL}$, behaves asymptotically like $e^{(1+i)kL} + e^{(1-i)kL}$, and the terms that matter for the asymptotic behaviour are given by the real part of the exponents. Therefore, if we set $k = k_R + ik_I$ then

$$\mathrm{Re}\left(e^{(1+i)kL}\right) = e^{(k_R-k_I)L}, \quad \mathrm{Re}\left(e^{(1-i)kL}\right) = e^{(k_R+k_I)L}.$$

For k such that $0 < \arg(k) < \frac{\pi}{8}$, if $\theta = \arg(k)$ then $0 < k_I < k_R \tan\left(\frac{\pi}{8}\right) < k_R$. Therefore

$$e^{(k_R-k_I)L} + e^{(k_R+k_I)L} = e^{k_RL}(e^{-k_IL} + e^{k_IL}) < e^{k_RL}(1 + e^{k_IL}) < e^{k_RL}(1 + e^{k_RL}),$$

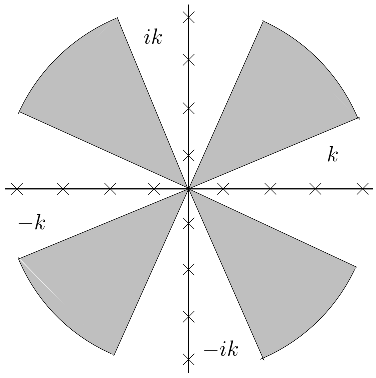


Figure 4.5: The location of k, ik, $-k$ and $-ik$ such that $0 < \arg(k) < \frac{\pi}{8}$.

and we conclude that the dominant term in the denominator is given by e^{2kL}. Therefore, the asymptotic behaviour of the integrand is given by

$$e^{ikx-k^4t}\Big\{ \left(ik^3\tilde{f}_0(t,k) - ik\tilde{f}_2(t,k)\right) \\ + 4\left(e^{(-1+i)kL} - e^{(-3+i)kL}\right)\left(N(t,k) - e^{-2ikL}N(t,-k)\right)\Big\},$$

where $N(t,k)$ is given explicitly by

$$N(t,k) \; = \; \int_0^L e^{-ikx'} q_0(x')\,\mathrm{d}x' - ik^3\left(\tilde{f}_0(t,k) - e^{-ikL}\tilde{g}_0(t,k)\right) \\ + ik\left(\tilde{f}_2(t,k) - e^{-ikL}\tilde{g}_2(t,k)\right).$$

Therefore, in terms of x-exponentials, the integrand is given in terms of the following

i.) $e^{ikx}e^{-ikx'}e^{-kL}e^{ikL} = e^{ikx}e^{ik(L-x')}e^{-kL}$,

$e^{ikx}e^{-ikx'}e^{-3kL}e^{ikL} = e^{ikx}e^{ik(L-x')}e^{-3kL}$,

$e^{ikx}e^{-ikL}e^{-kL}e^{ikL} = e^{ikx}e^{-kL}$,

$e^{ikx}e^{-ikL}e^{-3kL}e^{ikL} = e^{ikx}e^{-3kL}$,

ii.) $e^{ikx}e^{ikx'}e^{-kL}e^{ikL}e^{-2ikL} = e^{ikx}e^{ikx'}e^{(-1-i)kL}$,

$e^{ikx}e^{ikx'}e^{-3kL}e^{ikL}e^{-2ikL} = e^{ikx}e^{ikx'}e^{-2kL}e^{(-1-i)kL}$,

$e^{ikx}e^{ikL}e^{-kL}e^{ikL}e^{-2ikL} = e^{ikx}e^{-kL}$,

$e^{ikx}e^{ikL}e^{-3kL}e^{ikL}e^{-2ikL} = e^{ikx}e^{-3kL}$.

Since $\mathrm{Re}\left(e^{(-1-i)kL}\right) = e^{(k_I-k_R)L}$ and $k_I - k_R < 0$ for k such that $0 < \arg(k) < \frac{\pi}{8}$, it follows that the exponential $e^{(-1-i)kL}$ is bounded as $k \to \infty$. Hence all of the terms in the integrand of the integral around $\partial D^+$are bounded as $k \to \infty$ and the proof is complete. $\square$

Corollary 4.1.5. *The contours of integration, ∂D^+ and ∂D^-, in (4.14), can be deformed to any contour inside D^+ and D^- respectively. Hence (4.14) can be written in the form*

$$
\begin{aligned}
q(x,t) &= \frac{1}{2\pi}\Bigg\{\int_{-\infty}^{\infty} e^{ikx-k^4t}\hat{q}_0(k)\,\mathrm{d}k - \int_{-\infty}^{\infty} e^{ikx-k^4t}\left(ik^3\tilde{f}_0(t,k) - ik\tilde{f}_2(t,k)\right)\mathrm{d}k \\
&\quad - \int_{-\infty}^{\infty} \frac{4e^{ikx-k^4t}}{\Delta(k)}\left(e^{(1+i)kL} - e^{(-1+i)kL}\right)\left(N(t,k) - e^{-2ikL}N(t,-k)\right)\mathrm{d}k \\
&\quad + \int_{-\infty}^{\infty} e^{ik(x-L)-k^4t}\left(ik^3\tilde{g}_0(t,k) - ik\tilde{g}_2(t,k)\right)\mathrm{d}k \\
&\quad + \int_{-\infty}^{\infty} \frac{4e^{ik(x-L)-k^4t}}{\Delta(k)}\left(e^{ikL} - e^{-ikL}\right)\left(N(t,ik) - N(t,-ik)\right)\mathrm{d}k \\
&\quad + \pi i \sum_{\substack{k_n:\\ \Delta(k_n)=0}} \frac{p^+(k_n) + p^-(k_n)}{r'(k_n)}\Bigg\},
\end{aligned}
\tag{4.16}
$$

where

$$
\begin{aligned}
p^+(k) &= 4e^{ikx-k^4t}\left(e^{(1+i)kL} - e^{(-1+i)kL}\right)\left(N(t,k) - e^{-2ikL}N(t,-k)\right), \\
p^-(k) &= 4e^{ik(x-L)-k^4t}\left(e^{kL} - e^{-kL}\right)\left(N(t,k) - N(t,-k)\right), \\
r(k) &= \Delta(k) = 4\left(e^{(1+i)kL} - e^{(-1+i)kL} - e^{(1-i)kL} + e^{(-1-i)kL}\right).
\end{aligned}
$$

By Corollary 4.1.5, it follows that the contours can be deformed through the region D_c, to the zeros that lie on the real and imaginary axes in the complex k-plane. The residue contributions from *all* of the poles k_n are computed using the formulas

$$
\int_{\partial D^+} \frac{p^+(k)}{r(k)}\,\mathrm{d}k = \int_{-\infty}^{\infty} \frac{p^+(k)}{r(k)}\,\mathrm{d}k + \pi i \sum_{\substack{k_n:\\ \Delta(k_n)=0}} \frac{p^+(k_n)}{r'(k_n)}, \tag{4.17}
$$

$$
\int_{\partial D^-} \frac{p^-(k)}{r(k)}\,\mathrm{d}k = -\int_{-\infty}^{\infty} \frac{p^-(k)}{r(k)}\,\mathrm{d}k + \pi i \sum_{\substack{k_n:\\ \Delta(k_n)=0}} \frac{p^-(k_n)}{r'(k_n)}. \tag{4.18}
$$

Substituting these expressions into the integral representation of the solution, given by (4.14), yields the representation in terms of an integral along the real line and a series contribution due to the residues, given by (4.16). All of the integral terms sum to zero and the solution is given only in terms of an infinite series, due to the explicit computation of the residues.

Consider first, the case where k_n represents the real zeros, i.e., $k_n = \frac{n\pi}{L}$, $n \in \mathbb{Z}$. Then it follows that

$$\begin{aligned}
p^+(k_n) &= 4(-1)^n e^{ik_nx-k_n^4t}\left(e^{k_nL}-e^{-k_nL}\right)\left(N(t,k_n)-N(t,-k_n)\right)\,, \\
p^-(k_n) &= 4(-1)^n e^{ik_nx-k_n^4t}\left(e^{k_nL}-e^{-k_nL}\right)\left(N(t,k_n)-N(t,-k_n)\right)\,, \\
r'(k_n) &= 4L(-1)^n\Big\{(1-i)\left(e^{-k_nL}-e^{k_nL}\right)+(1+i)\left(e^{k_nL}-e^{-k_nL}\right)\Big\} \\
&= 8iL(-1)^n\left(e^{k_nL}-e^{-k_nL}\right)\,.
\end{aligned}$$

Hence

$$\frac{i}{2}\sum_{k_n\text{real}}\frac{p^+(k_n)+p^-(k_n)}{r'(k_n)} = \frac{1}{2L}\sum_{k_n\text{real}} e^{ik_nx-k_n^4t}\left(N(t,k_n)-N(t,-k_n)\right)\,.$$

Now let $k_n = \frac{in\pi}{L}$, $n \in \mathbb{Z}$. It follows that $p^+(k_n) = 0$ and $p^-(k_n) = 0$ and therefore, the series solution is calculated from the residue contributions that arise from the real zeros only, and is given by

$$q(x,t) = \frac{1}{2L}\sum_{k_n\text{real}} e^{ik_nx-k_n^4t}\left(N(t,k_n)-N(t,-k_n)\right)\,.$$

To prove that this is the same as (4.15), we sum over the positive zeros only:

$$\begin{aligned}
q(x,t) &= \frac{1}{2L}\sum_{\substack{k_n\text{real}\\ n<0}} e^{ik_nx-k_n^4t}\left(N(t,k_n)-N(t,-k_n)\right) \\
&\quad+\frac{1}{2L}\sum_{\substack{k_n\text{real}\\ n>0}} e^{ik_nx-k_n^4t}\left(N(t,k_n)-N(t,-k_n)\right) \\
&= \frac{1}{2L}\sum_{\substack{k_n\text{real}\\ n>0}} e^{-k_n^4t}\left(N(t,k_n)-N(t,-k_n)\right)\left(e^{ik_nx}-e^{-ik_nx}\right)\,. \qquad (4.19)
\end{aligned}$$

The index of the summation can now trivially by written in terms of n, i.e., from $n = 1$ to ∞, and the proof that (4.19) is identical to (4.15) is complete.

4.1.5 The Case of a Non Self-Adjoint Operator

In this section we consider the imposition of the boundary conditions for which the operator is not self-adjoint. This implies that the eigenfunctions that are found from the transform approach satisfy the homogeneous conditions characterising the adjoint problem, rather than the given conditions of the problem. Therefore, as it is natural to expect, it is not possible to obtain directly from the global relation the appropriate basis of eigenfunctions for a given problem if the operator is non self-adjoint, and one is forced to consider the adjoint problem at the same time as the given one. In contrast, it is possible to obtain the series representation of the solution considering only the given boundary conditions, by deforming the contours of integration and realising the solution entirely as the residue contribution from the poles.

Example 1

We consider the imposition of the boundary conditions

$$q(0,t) = f_0(t)\,, \quad q(L,t) = g_0(t)\,, \quad q_{xxx}(0,t) = f_3(t)\,, \quad q_{xxx}(L,t) = g_3(t)\,, \tag{4.20}$$

for some given smooth functions $f_0(t)$, $g_0(t)$, $f_3(t)$ and $g_3(t)$. The system of global relations is given in matrix form by

$$\begin{pmatrix} 1 & -i & -e^{-ikL} & ie^{-ikL} \\ -1 & 1 & e^{kL} & -e^{kL} \\ 1 & i & -e^{ikL} & -ie^{ikL} \\ -1 & -1 & e^{-kL} & e^{-kL} \end{pmatrix} \begin{pmatrix} k^2\tilde{f}_1(t,k) \\ k\tilde{f}_2(t,k) \\ k^2\tilde{g}_1(t,k) \\ k\tilde{g}_2(t,k) \end{pmatrix} = \begin{pmatrix} N(t,k) \\ N(t,ik) \\ N(t,-k) \\ N(t,-ik) \end{pmatrix} - \begin{pmatrix} e^{k^4t}\hat{q}(t,k) \\ e^{k^4t}\hat{q}(t,ik) \\ e^{k^4t}\hat{q}(t,-k) \\ e^{k^4t}\hat{q}(t,-ik) \end{pmatrix}, \tag{4.21}$$

where

$$N(t,k) = \hat{q}_0(k) - ik^3\left(\tilde{f}_0(t,k) - e^{-ikL}\tilde{g}_0(t,k)\right) + \left(\tilde{f}_3(t,k) - e^{-ikL}\tilde{g}_3(t,k)\right)\,,$$

and the determinant function $\Delta(k)$ is given by

$$\begin{aligned} \Delta(k) &= -8i + 2i\left(e^{(1+i)kL} + e^{(-1+i)kL} + e^{(1-i)kL} + e^{(-1-i)kL}\right) \\ &= -8i + 2i(e^{ikL} + e^{-ikL})(e^{kL} + e^{-kL}) \\ &= 8i\left(-1 + \cos(kL)\cosh(kL)\right)\,. \end{aligned}$$

The Integral Representation of the Solution

The integral representation of the solution, for the general fourth order linear evolution PDE of the form (4.1), is given by (4.6). The functions $\tilde{f}_1(t,k)$, $\tilde{f}_2(t,k)$, $\tilde{g}_1(t,k)$ and

$\tilde{g}_2(t,k)$ are determined from solving the system of global relations, given by (4.21), to yield the following expressions, in terms of the known functions $N(t,\lambda_l(k))$, $l=0,1,2,3$:

$$\begin{aligned} k^2\tilde{f}_1(t,k)-ik\tilde{f}_2(t,k) &= \frac{1}{\Delta(k)}\Big\{N(t,k)2i\left(-2+e^{(1+i)kL}+e^{(-1+i)kL}\right)\\ &\quad -N(t,ik)(2-2i)\left(1-e^{(-1-i)kL}\right)\\ &\quad +N(t,-k)2\left(e^{(-1-i)kL}-e^{(1-i)kL}\right)\\ &\quad +N(t,-ik)(2+2i)\left(1-e^{(1-i)kL}\right)\Big\}, \end{aligned} \tag{4.22}$$

$$\begin{aligned} k^2\tilde{g}_1(t,k)-ik\tilde{g}_2(t,k) &= \frac{1}{\Delta(k)}\Big\{N(t,k)2\left(-ie^{-kL}+2ie^{ikL}-ie^{kL}\right)\\ &\quad +N(t,ik)(2-2i)\left(e^{-kL}-e^{ikL}\right)\\ &\quad +N(t,-k)2\left(e^{-kL}-e^{kL}\right)\\ &\quad +N(t,-ik)(2+2i)\left(e^{ikL}-e^{kL}\right)\Big\}. \end{aligned} \tag{4.23}$$

An application of Proposition 2.4.3(i) proves that the unknown terms involving $\hat{q}(t,\lambda_l(k))$, $l=0,1,2,3$ do not contribute to the integral representation of the solution. Expressions (4.22) and (4.23) are substituted into the integral representation of the solution, given by (4.6) to yield the following:

$$\begin{aligned} q(x,t) &= \frac{1}{2\pi}\Bigg\{\int_{-\infty}^{\infty} e^{ikx-k^4t}\hat{q}_0(k)\,\mathrm{d}k-\int_{\partial D^+} e^{ikx-k^4t}\left(ik^3\tilde{f}_0(t,k)-\tilde{f}_3(t,k)\right)\,\mathrm{d}k\\ &-\int_{\partial D^+}\frac{e^{ikx-k^4t}}{\Delta(k)}\Big\{N(t,k)2i(-2+e^{(1+i)kL}+e^{(-1+i)kL})-N(t,ik)(2-2i)(1-e^{(-1-i)kL})\\ &\quad +N(t,-k)2(e^{(-1-i)kL}-e^{(1-i)kL})+N(t,-ik)(2+2i)(1-e^{(1-i)kL})\Big\}\,\mathrm{d}k\\ &-\int_{\partial D^-} e^{ik(x-L)-k^4t}\left(ik^3\tilde{g}_0(t,k)-\tilde{g}_3(t,k)\right)\,\mathrm{d}k\\ &-\int_{\partial D^-}\frac{e^{ik(x-L)-k^4t}}{\Delta(k)}\Big\{N(t,k)2(-ie^{-kL}+2ie^{ikL}-ie^{kL})+N(t,ik)(2-2i)(e^{-kL}-e^{ikL})\\ &\quad +N(t,-k)2(e^{-kL}-e^{kL})+N(t,-ik)(2+2i)(e^{ikL}-e^{kL})\Big\}\,\mathrm{d}k\Bigg\}. \end{aligned} \tag{4.24}$$

The Series Representation of the Solution

The series representation of the solution can be obtained by deforming the contours of integration, and realising the solution entirely as the explicit computation of the residue contributions due to the poles.

Proposition 4.1.6. *The integrands of the integrals around ∂D^+ and ∂D^-, in (4.24), are analytic and bounded for $k \in D_c^+$ and $k \in D_c^-$ respectively.*

Proof. We prove the case only for the integrand, of the integral around ∂D^+, for k such that $0 < \arg(k) < \frac{\pi}{8}$, and remark that a similar argument can be used for the other regions in D_c.

The integrand, of the integral around ∂D^+, in (4.24), is given explicitly by

$$e^{ikx-k^4t}\Bigg\{\Big(ik^3\tilde{f}_0(t,k)-\tilde{f}_3(t,k)\Big)+\frac{1}{\Delta(k)}\Big\{N(t,k)2i(-2+e^{(1+i)kL}+e^{(-1+i)kL})$$
$$-N(t,ik)(2-2i)(1-e^{(-1-i)kL})+N(t,-k)2(e^{(-1-i)kL}-e^{(1-i)kL})$$
$$+N(t,-ik)(2+2i)(1-e^{(1-i)kL})\Big\}\Bigg\}.$$

Consider the wedge, such that $0 < \arg(k) < \frac{\pi}{8}$. Then for k in this wedge, ζk will be such that $\frac{\pi}{2} < \arg(\zeta k) < \frac{5\pi}{8}$, $\zeta^2 k$ will be such that $\pi < \arg(\zeta^2 k) < \frac{9\pi}{8}$ and $\zeta^3 k$ will be such that $\frac{3\pi}{2} < \arg(\zeta^3 k) < \frac{13\pi}{8}$ (Figure 4.5). Hence

- e^{ikx-k^4t} is bounded,
- e^{ikL} and e^{-kL} are bounded,
- e^{-ikL} and e^{kL} are unbounded.

Now, the denominator, $\Delta(k) \sim 1+e^{(1+i)kL}+e^{(-1+i)kL}+e^{(-1-i)kL}+e^{(1-i)kL}$, behaves asymptotically like $e^{(1+i)kL}+e^{(1-i)kL}$, and the terms that matter for the asymptotic behaviour are given by the real part of the exponents. Therefore, if we set $k = k_R + ik_I$ then

$$\mathrm{Re}\left(e^{(1+i)kL}\right) = e^{(k_R-k_I)L}, \quad \mathrm{Re}\left(e^{(1-i)kL}\right) = e^{(k_R+k_I)L}.$$

For k such that $0 < \arg(k) < \frac{\pi}{8}$, if $\theta = \arg(k)$ then $0 < k_I < k_R \tan\left(\frac{\pi}{8}\right) < k_R$. Therefore

$$e^{(k_R-k_I)L}+e^{(k_R+k_I)L} = e^{k_RL}(e^{-k_IL}+e^{k_IL}) < e^{k_RL}(1+e^{k_IL}) < e^{k_RL}(1+e^{k_RL}),$$

and we conclude that the dominant term in the denominator is given by e^{2kL}. Therefore, the asymptotic behaviour of the integrand is given by

$$e^{ikx-k^4t}\Bigg\{ \Big(ik^3\tilde{f}_0(t,k)-\tilde{f}_3(t,k)\Big)+\Big\{N(t,k)2i(-2e^{-2kL}+e^{(-1+i)kL}+e^{(-3+i)kL})$$

$$-N(t,ik)(2-2i)(e^{-2kL}-e^{(-3-i)kL})+N(t,-k)2(e^{(-3-i)kL}-e^{(-1-i)kL})$$

$$+N(t,-ik)(2+2i)(e^{-2kL}-e^{(-1-i)kL})\Big\}\Bigg\},$$

where $N(t,k)$ is given explicitly by

$$N(t,k) \;=\; \int_0^L e^{-ikx'}q_0(x')\,\mathrm{d}x' - ik^3\left(\tilde{f}_0(t,k)-e^{-ikL}\tilde{g}_0(t,k)\right)$$

$$+\left(\tilde{f}_3(t,k)-e^{-ikL}\tilde{g}_3(t,k)\right).$$

Therefore, in terms of x-exponentials, the integrand is given in terms of the following

i.) $e^{ikx}e^{-ikx'}e^{-2kL}=e^{ikx}e^{ik(L-x')}e^{(-1-i)kL}e^{-kL}$,

$e^{ikx}e^{-ikx'}e^{-kL}e^{ikL}=e^{ikx}e^{ik(L-x')}e^{-kL}$,

$e^{ikx}e^{-ikx'}e^{-3kL}e^{ikL}=e^{ikx}e^{ik(L-x')}e^{-3kL}$,

$e^{ikx}e^{-ikL}e^{-2kL}=e^{ikx}e^{(-1-i)kL}e^{-kL}$,

$e^{ikx}e^{-ikL}e^{-kL}e^{ikL}=e^{ikx}e^{-kL}$,

$e^{ikx}e^{-ikL}e^{-3kL}e^{ikL}=e^{ikx}e^{-3kL}$,

ii.) $e^{ikx}e^{kx'}e^{-2kL}=e^{ikx}e^{k(x'-L)}e^{-kL}$,

$e^{ikx}e^{kx'}e^{-3kL}e^{-ikL}=e^{ikx}e^{k(x'-L)}e^{(-1-i)kL}e^{-kL}$,

$e^{ikx}e^{kL}e^{-2kL}=e^{ikx}e^{-kL}$,

$e^{ikx}e^{kL}e^{-3kL}e^{-ikL}=e^{ikx}e^{-kL}e^{(-1-i)kL}$,

iii.) $e^{ikx}e^{ikx'}e^{-3kL}e^{-ikL}=e^{ikx}e^{ikx'}e^{-2kL}e^{(-1-i)kL}$,

$e^{ikx}e^{ikx'}e^{-kL}e^{-ikL}=e^{ikx}e^{ikx'}e^{(-1-i)kL}$,

$e^{ikx}e^{ikL}e^{-3kL}e^{-ikL}=e^{ikx}e^{-3kL}$,

$e^{ikx}e^{ikL}e^{-kL}e^{-ikL}=e^{ikx}e^{-kL}$,

iv.) $e^{ikx}e^{-kx'}e^{-2kL}$,

$e^{ikx}e^{-kx'}e^{-kL}e^{-ikL}=e^{ikx}e^{-kx'}e^{(-1-i)kL}$,

$e^{ikx}e^{-kL}e^{-2kL}$,

$e^{ikx}e^{-kL}e^{-kL}e^{-ikL}=e^{ikL}e^{-kL}e^{(-1-i)kL}$.

Since $\mathrm{Re}\left(e^{(-1-i)kL}\right) = e^{(k_I-k_R)L}$ and $k_I - k_R < 0$, for k such that $0 < \arg(k) < \frac{\pi}{8}$, it follows that the exponential $e^{(-1-i)kL}$ is bounded as $k \to \infty$. Hence all of the terms in the integrand of the integral around $\partial D^+$are bounded as $k \to \infty$ and the proof is complete. $\square$

Corollary 4.1.7. *The contours of integration, ∂D^+ and ∂D^-, in (4.24), can be deformed to any contour inside D^+ and D^- respectively. Hence (4.24) can be written in the form*

$$
\begin{aligned}
q(x,t) \;=\; & \frac{1}{2\pi}\Bigg\{ \int_{-\infty}^{\infty} e^{ikx-k^4t}\hat{q}_0(k)\,\mathrm{d}k - \int_{-\infty}^{\infty} e^{ikx-k^4t}\left(ik^3\tilde{f}_0(t,k) - \tilde{f}_3(t,k)\right)\mathrm{d}k \\
& - \int_{-\infty}^{\infty} \frac{e^{ikx-k^4t}}{\Delta(k)}\Big\{N(t,k)2i(-2+e^{(1+i)kL}+e^{(-1+i)kL}) - N(t,ik)(2-2i)(1-e^{(-1-i)kL}) \\
& \qquad + N(t,-k)2(e^{(-1-i)kL}-e^{(1-i)kL}) + N(t,-ik)(2+2i)(1-e^{(1-i)kL})\Big\}\,\mathrm{d}k \\
& + \int_{-\infty}^{\infty} e^{ik(x-L)-k^4t}\left(ik^3\tilde{g}_0(t,k) - \tilde{g}_3(t,k)\right)\mathrm{d}k \\
& + \int_{-\infty}^{\infty} \frac{e^{ik(x-L)-k^4t}}{\Delta(k)}\Big\{N(t,k)2(-ie^{-kL}+2ie^{ikL}-ie^{kL}) + N(t,ik)(2-2i)(e^{-kL}-e^{ikL}) \\
& \qquad + N(t,-k)2(e^{-kL}-e^{kL}) + N(t,-ik)(2+2i)(e^{ikL}-e^{kL})\Big\}\,\mathrm{d}k \\
& + \pi i \sum_{\substack{k_n: \\ \Delta(k_n)=0}} \frac{p^+(k_n)+p^-(k_n)}{r'(k_n)}\Bigg\}.
\end{aligned} \tag{4.25}
$$

where

$$
\begin{aligned}
p^+(k) \;=\; & e^{ikx-k^4t}\Big\{N(t,k)2i(-2+e^{(1+i)kL}+e^{(-1+i)kL}) - N(t,ik)(2-2i)(1-e^{(-1-i)kL}) \\
& \qquad + N(t,-k)2(e^{(-1-i)kL}-e^{(1-i)kL}) + N(t,-ik)(2+2i)(1-e^{(1-i)kL})\Big\}, \\
p^-(k) \;=\; & e^{ik(x-L)-k^4t}\Big\{N(t,k)2(-ie^{-kL}+2ie^{ikL}-ie^{kL}) + N(t,ik)(2-2i)(e^{-kL}-e^{ikL}) \\
& \qquad + N(t,-k)2(e^{-kL}-e^{kL}) + N(t,-ik)(2+2i)(e^{ikL}-e^{kL})\Big\}, \\
r(k) \;=\; & \Delta(k) = -8i + 2i\left(e^{(1+i)kL}+e^{(-1+i)kL}+e^{(1-i)kL}+e^{(-1-i)kL}\right).
\end{aligned}
$$

By Corollary 4.1.7, it follows that the contours can be deformed through the region D_c, to the zeros that lie on the real and imaginary axes in the complex k-plane. The residue contributions from the poles are computed using the formulas given by (4.17) and (4.18), and substituting these expressions into (4.24) yields the representation, given by (4.25).

A simple cancellation of terms shows that all of the integral terms sum to zero, proving that the solution is expressible entirely as an infinite discrete series. Substitution and simplification yields the series solution given by

$$\begin{aligned} q(x,t) &= \frac{1}{2L} \sum_{\substack{k_n: \\ \Delta(k_n)=0}} \frac{e^{ik_nx-k_n^4t}p(k_n)}{(1+i)\left(e^{(1+i)k_nL}-e^{(-1-i)k_nL}\right)+(1-i)\left(e^{(1-i)k_nL}-e^{(-1+i)k_nL}\right)} \\ &= \frac{1}{8L} \sum_{\substack{k_n: \\ \Delta(k_n)=0}} \frac{e^{ik_nx-k_n^4t}p(k_n)}{\tanh(k_nL)-\tan(k_nL)}, \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} p(k_n) &= N(t,k_n)2i\left(2-e^{(1-i)k_nL}-e^{(-1-i)k_nL}\right)-N(t,ik_n)(2-2i)\left(1-e^{(-1-i)k_nL}\right) \\ &\quad +N(t,-k_n)2\left(e^{(-1-i)k_nL}-e^{(1-i)k_nL}\right)+N(t,-ik_n)(2+2i)\left(1-e^{(1-i)k_nL}\right). \end{aligned}$$

We observe that since the denominator is invariant under $k \to \zeta k$, with ζ the fourth root of 1, the series can be written in terms of the positive real zeros only:

$$q(x,t) = \frac{1}{8L}\sum_{n=1}^{\infty} e^{-k_n^4t}\left(\frac{e^{ik_nx}p(k_n)-e^{-ik_nx}p(-k_n)+ie^{-k_nx}p(ik_n)-ie^{k_nx}p(-ik_n)}{\tanh(k_nL)-\tan(k_nL)}\right). \tag{4.27}$$

The Direct Derivation of the Series Representation of the Solution using the Adjoint Problem

In this section we show that the eigenfunctions that are found from the analysis of the global relations, satisfy the homogeneous boundary conditions of the adjoint problem, and therefore, the analysis of the adjoint problem is necessary for the derivation of the series solution. This derivation can be compared to the classical separation of variables approach, in that it relies on the assumptions of orthogonality of eigenfunctions and convergence properties etc.

In this first part, we use the system of global relations, given by (4.21) to derive the appropriate basis of eigenfunctions $e_n(x)$ satisfying the homogeneous boundary conditions of the adjoint problem. Therefore, an analogous repeat of this procedure is required in the second part, where the adjoint problem is analysed and the appropriate basis of eigenfunctions $f_m(x)$ is derived, satisfying the homogeneous boundary conditions of this problem. The functions $e_n(x)$ and $f_m(x)$ are shown to satisfy a bi-orthogonality condition, and the series representation of the solution for example 1, is given in terms of the eigenfunctions $f_m(x)$ of the adjoint problem.

The system of global relations, given by (4.21), can be solved for the unknown boundary functions using Cramer's rule. The expression that results for $k^2\tilde{f}_1(t,k)$ is given by

$$
\begin{aligned}
k^2\tilde{f}_1(t,k) = & \frac{1}{\Delta(k)}(1+i)e^{(-1-i)kL} \\
& \Big\{ \Big(N(t,k) - e^{k^4t}\hat{q}(t,k)\Big)\left(ie^{2ikL} - (1+i)e^{(1+i)kL} + e^{(2+2i)kL}\right) \\
& + \Big(N(t,ik) - e^{k^4t}\hat{q}(t,ik)\Big)\left(-e^{2ikL} - i + (1+i)e^{(1+i)kL}\right) \\
& + \Big(N(t,-k) - e^{k^4t}\hat{q}(t,-k)\Big)\left(-(1+i)e^{(1+i)kL} + 1 + ie^{2kL}\right) \\
& + \Big(N(t,-ik) - e^{k^4t}\hat{q}(t,-ik)\Big)\left(-ie^{(2+2i)kL} + (1+i)e^{(1+i)kL} - e^{2kL}\right)\Big\}.
\end{aligned}
$$

Evaluating this expression at an arbitrary, positive real zero k_n of the determinant function $\Delta(k)$, yields an expression for the unknown functions $\hat{q}(t,\lambda_l(k))$ in terms of the known functions $N(t,\lambda_l(k))$, $l=0,1,2,3$. This is given explicitly by

$$
\begin{aligned}
& N(t,k_n)\left(ie^{2ik_nL} - (1+i)e^{(1+i)k_nL} + e^{(2+2i)k_nL}\right) \\
& + N(t,ik_n)\left(-e^{2ik_nL} - i + (1+i)e^{(1+i)k_nL}\right) \\
& + N(t,-k_n)\left(-(1+i)e^{(1+i)k_nL} + 1 + ie^{2k_nL}\right) \\
& + N(t,-ik_n)\left(-ie^{(2+2i)k_nL} + (1+i)e^{(1+i)k_nL} - e^{2k_nL}\right) \\
= \ & e^{k_n^4t}\Big\{\hat{q}(t,k_n)\left(ie^{2ik_nL} - (1+i)e^{(1+i)k_nL} + e^{(2+2i)k_nL}\right) \\
& + \hat{q}(t,ik_n)\left(-e^{2ik_nL} - i + (1+i)e^{(1+i)k_nL}\right) \\
& + \hat{q}(t,-k_n)\left(-(1+i)e^{(1+i)k_nL} + 1 + ie^{2k_nL}\right) \\
& + \hat{q}(t,-ik_n)\left(-ie^{(2+2i)k_nL} + (1+i)e^{(1+i)k_nL} - e^{2k_nL}\right)\Big\}.
\end{aligned}
$$

Multiplying throughout by e^{-2k_nL} yields the following expression which is bounded, for all positive real k_n, as $k_n \to \infty$:

$$\hat{q}(t,k_n)\,A1 + \hat{q}(t,ik_n)\,A2 + \hat{q}(t,-k_n)\,A3 + \hat{q}(t,-ik_n)\,A4$$

$$= e^{-k_n^4 t}\Big\{N(t,k_n)\,A1 + N(t,ik_n)\,A2 + N(t,-k_n)\,A3 + N(t,-ik_n)\,A4\Big\},$$

where

$$A1 = ie^{(-2+2i)k_nL} - (1+i)e^{(-1+i)k_nL} + e^{2ik_nL}, \tag{4.28a}$$

$$A2 = -e^{(-2+2i)k_nL} - ie^{-2k_nL} + (1+i)e^{(-1+i)k_nL}, \tag{4.28b}$$

$$A3 = -(1+i)e^{(-1+i)k_nL} + e^{-2k_nL} + i, \tag{4.28c}$$

$$A4 = -ie^{2ik_nL} + (1+i)e^{(-1+i)k_nL} - 1. \tag{4.28d}$$

Assuming that the RHS can be written as the L^2 inner product $\langle q_0(x), e_n(x)\rangle$ for some basis functions $e_n(x)$, we find

$$\int_0^L \left(A1\,e^{-ik_nx} + A2\,e^{k_nx} + A3\,e^{ik_nx} + A4\,e^{-k_nx}\right) q(x,t)\,\mathrm{d}x = e^{-k_n^4 t} A5\,,$$

where

$$A5 = N(t,k_n)\,A1 + N(t,ik_n)\,A2 + N(t,-k_n)\,A3 + N(t,-ik_n)\,A4\,, \tag{4.29}$$

is a known function in terms of the t-transforms of the given initial and boundary data.

Proposition 4.1.8. *The functions $e_n(x)$, given by*

$$e_n(x) = \overline{A1}\,e^{ik_nx} + \overline{A2}\,e^{k_nx} + \overline{A3}\,e^{-ik_nx} + \overline{A4}\,e^{-k_nx}\,,$$

satisfy the homogeneous boundary conditions

$$e_n'(0) = e_n'(L) = 0\,, \quad e_n''(0) = e_n''(L) = 0\,. \tag{4.30}$$

Proof. All of the conditions follow by a direct cancellation of terms, except for the condition $e_n''(0) = 0$ that requires the identity $\Delta(k_n) = 0$:

$$\begin{aligned} e_n''(0) &= k_n^2\left(-\overline{A1} + \overline{A2} - \overline{A3} + \overline{A4}\right) \\ &= k_n^2\Big(ie^{(-2-2i)k_nL} + (1-i)e^{(-1-i)k_nL} - e^{-2ik_nL} - e^{(-2-2i)k_nL} + ie^{-2k_nL} \\ &\qquad + (1-i)e^{(-1-i)k_nL} + (1-i)e^{(-1-i)k_nL} - e^{-2k_nL} + i + ie^{-2ik_nL} \\ &\qquad + (1-i)e^{(-1-i)k_nL} - 1\Big) \end{aligned}$$

$$
\begin{aligned}
&= -(1-i)k_n^2 e^{(-1-i)k_nL}\left(-4+e^{(1+i)k_nL}+e^{(-1+i)k_nL}+e^{(1-i)k_nL}+e^{(-1-i)k_nL}\right)\\
&= \left(\frac{1+i}{2}\right)k_n^2 e^{(-1-i)k_nL}\Delta(k_n)\\
&= 0\,.
\end{aligned}
$$

□

The conditions, given by (4.30), do not correspond to the boundary conditions, given by (4.20), imposed on the PDE, proving that the analysis of the system of global relations, given by (4.21), is not sufficient for the derivation of the series solution. It follows that to find the appropriate eigenfunctions $f_m(x)$, satisfying the boundary conditions $f_m(0) = f_m(L) = 0$ and $f_m'''(0) = f_m'''(L) = 0$, the transform method must applied to the adjoint problem, posed by (4.1) with the boundary conditions

$$q_x(0,t) = f_1(t)\,,\quad q_x(L,t) = g_1(t)\,,\quad q_{xx}(0,t) = f_2(t)\,,\quad q_{xx}(L,t) = g_2(t)\,,$$

for some given functions $f_1(t)$, $g_1(t)$, $f_2(t)$ and $g_2(t)$, and this is what we do next.

Example 2

In a way analogous to that of the previous example, we find the system for the unknown boundary conditions. The determinant of this system has the same zeros as the determinant of its adjoint. The system of global relations can be solved to yield expressions for the unknown terms. The expression for $k^3\tilde{f}_0(t,k)$ is given by

$$
\begin{aligned}
k^3\tilde{f}_0(t,k) = {} & \frac{1}{\Delta(k)}(1+i)e^{(-1-i)kL}\\
& \Big\{\left(N(t,k)-e^{k^4t}\hat{q}(t,k)\right)\left(ie^{2ikL}+(1-i)e^{(1+i)kL}-e^{(2+2i)kL}\right)\\
& +\left(N(t,ik)-e^{k^4t}\hat{q}(t,ik)\right)\left(-1-ie^{2ikL}+(1+i)e^{(1+i)kL}\right)\\
& +\left(N(t,-k)-e^{k^4t}\hat{q}(t,-k)\right)\left(1-(1-i)e^{(1+i)kL}-ie^{2kL}\right)\\
& +\left(N(t,-ik)-e^{k^4t}\hat{q}(t,-ik)\right)\left((-1-i)e^{(1+i)kL}+ie^{2kL}+e^{(2+2i)kL}\right)\Big\}\,.
\end{aligned}
$$

Evaluating this expression at a positive real zero k_m of the determinant function $\Delta(k)$, and multiplying throughout by e^{-2k_mL} yields the following expression, which is bounded, for all positive real zeros k_m, as $k_m \to \infty$:

$$\begin{aligned}
&\hat{q}(t,k_m)\,B1 + \hat{q}(t,ik_m)\,B2 + \hat{q}(t,-k_m)\,B3 + \hat{q}(t,-ik_m)\,B4 \\
&\quad = e^{-k_m^4 t}\Big\{N(t,k_m)\,B1 + N(t,ik_m)\,B2 + N(t,-k_m)\,B3 + N(t,-ik_m)\,B4\Big\}\,,
\end{aligned}$$

where

$$B1 = ie^{(-2+2i)k_mL} + (1-i)e^{(-1+i)k_mL} - e^{2ik_mL}\,, \tag{4.31a}$$

$$B2 = -e^{-2k_mL} - ie^{(-2+2i)k_mL} + (1+i)e^{(-1+i)k_mL}\,, \tag{4.31b}$$

$$B3 = e^{-2k_mL} - (1-i)e^{(-1+i)k_mL} - i\,, \tag{4.31c}$$

$$B4 = (-1-i)e^{(-1+i)k_mL} + i + e^{2ik_mL}\,. \tag{4.31d}$$

Assuming that the RHS can be written as the L^2 inner product $\langle q_0(x), f_m(x)\rangle$ for some basis functions $f_m(x)$, we find

$$\int_0^L \left(B1\,e^{-ik_mx} + B2\,e^{k_mx} + B3\,e^{ik_mx} + B4\,e^{-k_mx}\right) q(x,t)\,\mathrm{d}x = e^{-k_m^4 t}B5\,,$$

where

$$B5 = N(t,k_m)\,B1 + N(t,ik_m)\,B2 + N(t,-k_m)\,B3 + N(t,-ik_m)\,B4\,.$$

Proposition 4.1.9. *The functions $f_m(x)$, given by*

$$f_m(x) = \overline{B1}\,e^{ik_mx} + \overline{B2}\,e^{k_mx} + \overline{B3}\,e^{-ik_mx} + \overline{B4}\,e^{-k_mx}\,,$$

satisfy the homogeneous boundary conditions

$$f_m(0) = f_m(L) = 0\,, \quad f_m'''(0) = f_m'''(L) = 0\,.$$

Proof. The proof that $f_m(0)=0$, $f_m(L)=0$ and $f_m'''(L)=0$ follow by a direct cancellation of terms. The proof that $f_m'''(0)=0$ however, requires the identity $\Delta(k_m)=0$:

$$\begin{aligned}
f_m'''(0) &= k_m^3\left(-i\,\overline{B1} + \overline{B2} + i\,\overline{B3} - \overline{B4}\right) \\
&= k_m^3\Big(- e^{(-2-2i)k_mL} + (1-i)e^{(-1-i)k_mL} + ie^{-2ik_mL} - e^{-2k_mL} + ie^{(-2-2i)k_mL} \\
&\qquad + (1-i)e^{(-1-i)k_mL} + ie^{-2k_mL} + (1-i)e^{(-1-i)k_mL} - 1 \\
&\qquad + (1+i)e^{(-1+i)k_mL} - i - e^{2ik_mL}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= -k_m^3(1-i)e^{(-1-i)k_mL}\left(-4+e^{(1+i)k_mL}+e^{(-1+i)k_mL}+e^{(1-i)k_mL}+e^{(-1-i)k_mL}\right)\\
&= \left(\frac{1+i}{2}\right)k_m^3e^{(-1-i)k_mL}\Delta(k_m)\\
&= 0\,.
\end{aligned}
$$

□

The Series Representation of the Solution for Example 1

In this section, we show how to derive the series representation of the solution, from the analysis of the global relation of the adjoint problem, in the example for which the operator is non self-adjoint.

The eigenfunctions $e_n(x)$ that result from the problem posed by (4.1), with the imposition of the boundary conditions

$$q(0,t)=f_0(t)\,,\quad q(L,t)=g_0(t)\,,\quad q_{xxx}(0,t)=f_3(t)\,,\quad q_{xxx}(L,t)=g_3(t)\,,$$

for some given functions $f_0(t)$, $g_0(t)$, $f_3(t)$ and $g_3(t)$, are given by

$$e_n(x)=\overline{A1}\,e^{ik_nx}+\overline{A2}\,e^{k_nx}+\overline{A3}\,e^{-ik_nx}+\overline{A4}\,e^{-k_nx}\,,$$

where $A1$, $A2$, $A3$ and $A4$ are given by (4.28), and satisfy the boundary conditions

$$e_n'(0)=0\,,\quad e_n'(L)=0\,,\quad e_n''(0)=0\,,\quad e_n''(L)=0\,.$$

Similarly, the eigenfunctions $f_m(x)$, that result from the imposition of the boundary conditions

$$q_x(0,t)=f_1(t)\,,\quad q_x(L,t)=g_1(t)\,,\quad q_{xx}(0,t)=f_2(t)\,,\quad q_{xx}(L,t)=g_2(t)\,,$$

for some given functions $f_1(t)$, $g_1(t)$, $f_2(t)$ and $g_2(t)$, are given by

$$f_m(x)=\overline{B1}\,e^{ik_mx}+\overline{B2}\,e^{k_mx}+\overline{B3}\,e^{-ik_mx}+\overline{B4}\,e^{-k_mx}\,,$$

where $B1$, $B2$, $B3$ and $B4$ are given by (4.31), and satisfy the boundary conditions

$$f_m(0)=0\,,\quad f_m(L)=0\,,\quad f_m'''(0)=0\,,\quad f_m'''(L)=0\,.$$

Proposition 4.1.10. *The functions $e_n(x)$ and $f_m(x)$ satisfy the bi-orthogonality condition*

$$\int_0^L f_m(x)\,\overline{e_n}(x)\,\mathrm{d}x = \begin{cases} 0\,, & n \neq m\,, \\ c(k_n, L)\,, & n = m\,, \end{cases}$$

for some function $c(k_n, L)$.

Proof. Integration by parts yields the following:

$$\begin{aligned} \langle D_4 f_m(x), e_n(x)\rangle &= \Big[D_3 f_m(x)\,\overline{e_n}(x) - D_2 f_m(x)\,D\overline{e_n}(x) + D f_m(x)\,D_2\overline{e_n}(x) \\ &\qquad - f_m(x)\,D_3\overline{e_n}(x)\Big]_0^L + \langle f_m(x), D_4 e_n(x)\rangle \\ &= \langle f_m(x), D_4 e_n(x)\rangle\,. \end{aligned}$$

Hence

$$\langle D_4 f_m(x), e_n(x)\rangle - \langle f_m(x), D_4 e_n(x)\rangle = \left(k_m^4 - k_n^4\right)\langle f_m(x), e_n(x)\rangle = 0\,,$$

and the proof is complete. □

To obtain the series representation of the solution for example 1, we assume that the solution $q(x,t)$ can be expressed in terms of the eigenfunctions $f_m(x)$ and written in the form

$$q(x,t) = \sum_{m=1}^{\infty} a_m(t)\,f_m(x)\,,$$

for some functions $a_m(t)$ to be found, and where the series converges in the L^2 norm. The index, from $m = 1$ to ∞ indicates that only the positive real zeros contribute to the series representation.

The coefficients are found by multiplying both sides by $\overline{e_n}(x)$ and integrating from 0 to L:

$$\int_0^L \overline{e_n}(x)\,q(x,t)\,\mathrm{d}x = \sum_{m=1}^{\infty}\left(\int_0^L f_m(x)\,\overline{e_n}(x)\,\mathrm{d}x\right)a_m(t) = c(k_n, L)\,a_n(t)\,.$$

Therefore

$$a_m(t) = \frac{1}{c(k_m, L)}\int_0^L \overline{e_m}(x)\,q(x,t)\,\mathrm{d}x = \frac{1}{c(k_m, L)}e^{-k_m^4 t}A5\,,$$

where $A5$ is given by (4.29) in terms of k_m, and

$$c(k_m, L) = 8L\tan(k_m L)\,\sin(k_m L)\Big\{\sin(k_m L) - \sinh(k_m L)\Big\}e^{-2k_m L}\,.$$

Therefore, the series representation of the solution, for example 1, computed in terms of all of the positive real zeros k_m such that $\Delta(k_m) = 0$, is given by

$$q(x,t) = \sum_{m=1}^{\infty} \frac{1}{c(k_m, L)} e^{-k_m^4 t} A5\left(\overline{B1}\,e^{ik_m x} + \overline{B2}\,e^{k_m x} + \overline{B3}\,e^{-ik_m x} + \overline{B4}\,e^{-k_m x}\right)\,. \quad (4.32)$$

The expression (4.32) is identical to the expression one obtains by separating variables and using the classical approach and bi-orthogonal basis. By algebraic manipulations (which are long and tedious, and are therefore omitted) this expression can be put in the more concise form (4.26), via (4.27). We remark that the latter form is arrived at directly when using the integral representation and contour deformation.

Remark 4.1.11. Suppose that instead of (4.1a) we consider the PDE $q_t(x,t) + iq_{xxxx}(x,t) = 0$. The dispersion relation of this PDE is $\omega(k) = ik^4$ and D is given by

$$\left.\begin{aligned} D_1^+ &= \left\{k \in \mathbb{C} : 0 \leqslant \arg(k) \leqslant \tfrac{\pi}{4}\right\} \\ D_2^+ &= \left\{k \in \mathbb{C} : \tfrac{\pi}{2} \leqslant \arg(k) \leqslant \tfrac{3\pi}{4}\right\} \end{aligned}\right\} \quad D^+ = D_1^+ \cup D_2^+\,,$$

$$\left.\begin{aligned} D_1^- &= \left\{k \in \mathbb{C} : \pi \leqslant \arg(k) \leqslant \tfrac{5\pi}{4}\right\} \\ D_2^- &= \left\{k \in \mathbb{C} : \tfrac{3\pi}{2} \leqslant \arg(k) \leqslant \tfrac{7\pi}{4}\right\} \end{aligned}\right\} \quad D^- = D_1^- \cup D_2^-\,.$$

By following the same steps as for the analysis of (4.1a), we find that the zeros of the determinant of any boundary value problem for uncoupled conditions of the same form as in the previous section has zeros lying on the real and purely imaginary axes. Hence these zeros are on the boundary of D, and the general theory implies that the residue contribution, hence a series term, always exists. Indeed, one can show that the representation can be reduced to a series.

Since the x-differential operator associated to (4.1a) and to this PDE is the same, it follows from this observation, without any reference to classical results, that it is possible to deform the contour and realise the corresponding representation as a series also in the case of equation (4.1a), for which the zeros are *outside* D.

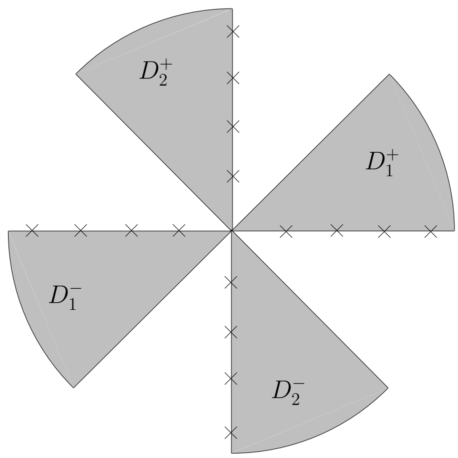


Figure 4.6: The regions $D_1^+ = \left\{k \in \mathbb{C} : 0 \leqslant \arg(k) \leqslant \frac{\pi}{4}\right\}$, $D_2^+ = \left\{k \in \mathbb{C} : \frac{\pi}{2} \leqslant \arg(k) \leqslant \frac{3\pi}{4}\right\}$, $D_1^- = \left\{k \in \mathbb{C} : \pi \leqslant \arg(k) \leqslant \frac{5\pi}{4}\right\}$ and $D_2^- = \left\{k \in \mathbb{C} : \frac{3\pi}{2} \leqslant \arg(k) \leqslant \frac{7\pi}{4}\right\}$ for the fourth order problem $q_t(x,t) + iq_{xxxx}(x,t) = 0$.

4.2 The Eigenvalues of Linear Evolution PDEs

In this section, we return to the notion of spectrum of the boundary value problem. In all examples we have considered, this set indeed coincides with the discrete spectrum of the linear differential operator in x defining the PDE, considered with the given boundary conditions at $x = 0$ and $x = L$. We give here a proof of the fact that this is indeed the general case, and that the notion of effective spectrum of the PDE and discrete spectrum of the differential operator yields the same set, modulo the operation of taking n^{th} roots. Namely, the effective spectrum of the boundary value problem is precisely the set of the n^{th} roots of the discrete spectrum of the differential operator.

We prove this result for the case of second, third and fourth order problems that we have considered in this work.

Notation: In the remaining part of this section we will denote by $S1$ and $S2$ the systems characterising the two different notion of spectra. More precisely, we let D_n denote the n^{th} order linear differential operator $\frac{\partial^n}{\partial x^n}$. We then consider:

Problem 1: The PDE $q_t(x,t) + \alpha D_n q(x,t) = 0$, for some α, with given initial condition $q_0(x)$ and an appropriate number of prescribed boundary conditions. We denote by $f_i(t)$ the given boundary conditions at $x = 0$, and by $g_j(t)$ the prescribed boundary conditions at $x = L$, where $i, j, \in \{0, 1, \dots, n-1\}$.

Problem 2: The eigenvalue problem for the operator D_n, is subject to *homogeneous* boundary conditions corresponding to $f_i(t)$ and $g_j(t)$. This is the problem of determining all values of $\lambda \in \mathbb{C}$ for which the ODE $D_n v(x) = \lambda v(x)$, admits a solution $v(x) \neq 0$, which also satisfies the given homogeneous boundary conditions. The general solution of this ODE is given by

$$v(x) = a_0 e^{\lambda^{\frac{1}{n}} x} + a_1 e^{\alpha \lambda^{\frac{1}{n}} x} + \cdots + a_{n-1} e^{\alpha^{n-1} \lambda^{\frac{1}{n}} x}, \tag{4.33}$$

where α_i, $i = 1, 2, \dots n-1$ is a primitive n^{th} root of unity, chosen as the appropriate n^{th} root of λ. Imposing the boundary conditions at $x = 0$ and $x = L$, which are a total of n homogeneous conditions, then yields a system of n equations for the coefficients $a_0, a_1, \dots, a_{n-1}$.

The System *S1*: We associate with Problem 1 the system *S1*, obtained from the global relation and characterising the unknown boundary conditions.

The System *S2*: We associate with Problem 2 the system *S2*, obtained from the imposition of the homogeneous boundary conditions to determine the solution of the ODE. Because of the presence of n^{th} roots in (4.33), it is convenient to write the ODE in the form $D_n v(x) = (ik)^n v(x)$, and write the system *S2* in terms of k rather than the roots of λ.

Remark 4.2.1. The form $D_n v(x) = (ik)^n v(x)$ is chosen purely for convenience of notation. For example, for the second order problem, it is necessary to choose $\lambda = -k^2$ to be able to satisfy the boundary conditions.

In Proposition 4.2.2, we show that the two determinants arising from systems *S1* and *S2* have the same set of zeros. It will be shown that the reason that these sets are identical is that there is an interplay between the derivatives in the two systems *S1* and *S2*. In one case, the rows that are retained correspond to the boundary conditions that are imposed, and in the other the complementary rows are selected. However, in one case, multiplication by k corresponds to the *order* of the derivative that is imposed, and in the other it corresponds to ($(n-1)$-*order*) of the derivative that is imposed. In the proof that follows, we show that there is a direct balance between the interplay of derivatives and the k multipliers.

Proposition 4.2.2. *The set of zeros of the determinant of the system S1, arising from the analysis of Problem 1, coincides with the set of zeros of the determinant of the system S2, arising from the analysis of Problem 2.*

Proof. To prove this we show that the two determinants involved in the two definitions always have the same set of zeros. We give a proof for each value of n:

i.) $\mathbf{n = 2}$

S1 : Consider the PDE

$$q_t(x,t) - q_{xx}(x,t) = 0\,, \quad t > 0\,, \quad x \in [0, L]\,,$$

$$q(x,0) = q_0(x)\,, \quad x \in [0, L]\,.$$

The analysis of the global relation, using the Fokas transform method, leads to the following pair of equations

$$\begin{aligned} & ik\left(\tilde{f}_0(t,k) - e^{-ikL}\tilde{g}_0(t,k)\right) + \left(\tilde{f}_1(t,k) - e^{-ikL}\tilde{g}_1(t,k)\right) \\ & = \hat{q}_0(k) - e^{k^2t}\hat{q}(t,k)\,, \\ & i\zeta k\left(\tilde{f}_0(t,k) - e^{-i\zeta kL}\tilde{g}_0(t,k)\right) + \left(\tilde{f}_1(t,k) - e^{-i\zeta kL}\tilde{g}_1(t,k)\right) \\ & = \hat{q}_0(\zeta k) - e^{k^2t}\hat{q}(t,\zeta k)\,. \end{aligned}$$

Since one of the $\tilde{f}$'s and one of the $\tilde{g}$'s are known, the matrix of the system for the unknown boundary values has the form

$$\begin{pmatrix} 1 & \zeta^{k_1} \\ e^{ikL} & \zeta^{k_2}e^{i\zeta kL} \end{pmatrix}^T, \qquad k_{1,2} \in \{0, 1\}\;, \tag{4.34}$$

where all the information is in the second column and the factor of ik has been incorporated into the unknown terms. The values of k_1 and k_2 correspond to the boundary functions that are unknown at $x = L$ and $x = 0$ respectively. Hence, the value of k_l, $l = 1, 2$, indicates the order of the derivative of the function sought:

- $k_l = 1 -$ (the order of the derivative that is sought) .

For example, if the boundary conditions at $q(0)$ and $q_x(L)$ are prescribed, then $k_1 = 1$ and $k_2 = 0$.

S2 : Consider the second order eigenvalue problem

$$v_{xx}(x) + k^2 v(x) = 0\,, \quad x \in [0, L]\,,$$

chosen according to Remark 4.2.1. The general solution is given by

$$v(x) = a_0 e^{ikx} + a_1 e^{i\zeta kx}\,, \quad \zeta = e^{\pi i} = -1\,,$$

for constants a_0 and a_1. The imposition of two boundary conditions, one at either end of the interval $[0, L]$, corresponds to choosing two appropriate rows from the following system:

$$\begin{pmatrix} v(0) \\ v_x(0) \\ v(L) \\ v_x(L) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ ik & i\zeta k \\ e^{ikL} & e^{i\zeta kL} \\ ike^{ikL} & i\zeta ke^{i\zeta kL} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}.$$

The coefficient functions of k, in the matrix that results from the imposition of the boundary conditions, appear as a multiplicative constant in the determinant function, and therefore for the purposes of analysing the zeros of the determinant function can be ignored.

Therefore, by eliminating the factors of ik, the boundary condition at $x = 0$ takes the form $a_0 + \zeta^{j_1} a_1$ and the boundary condition at $x = L$ takes the form $a_0 e^{ikL} + \zeta^{j_2} a_1 e^{i\zeta kL}$ where the value of j_l, $l = 1, 2$, indicates the order of the derivative imposed, at $x = 0$ for j_1 and at $x = L$ for j_2. Hence the matrix for the system can always be put in the form

$$\begin{pmatrix} 1 & \zeta^{j_1} \\ e^{ikL} & \zeta^{j_2} e^{i\zeta kL} \end{pmatrix}, \qquad j_{1,2} \in \{0, 1\}\,, \tag{4.35}$$

where all the information about the problem is in the second column of the matrix.

For example, if the boundary conditions at $v(0)$ and $v_x(L)$ are prescribed, then $j_1 = 0$ and $j_2 = 1$.

The structure of the vectors is always

$$\left(\zeta^{j_1},\ \zeta^{j_2}\right)^T, \quad \left(\zeta^{k_1},\ \zeta^{k_2}\right),$$

and it is trivial to prove that these vectors are identical, since the knowledge of j_1 and/or j_2 determines k_1 and/or k_2 respectively:

- $j_1 = 0, 1 \quad \Rightarrow \quad k_2 = 0, 1,$
- $j_2 = 0, 1 \quad \Rightarrow \quad k_1 = 0, 1.$

Therefore the determinants of the matrices, given by (4.34) and (4.35), up to a variant of sign, are identical, and the proof is complete.

ii.) n = 3

S1 : Consider now the third order PDE

$$q_t(x,t) + q_{xxx}(x,t) = 0, \quad t > 0, \quad x \in [0, L],$$

$$q(x,0) = q_0(x), \quad x \in [0, L].$$

The global relation is given by

$$k^2\left(\tilde{f}_0(t,k) - e^{-ikL}\tilde{g}_0(t,k)\right) - ik\left(\tilde{f}_1(t,k) - e^{-ikL}\tilde{g}_1(t,k)\right)$$
$$- \left(\tilde{f}_2(t,k) - e^{-ikL}\tilde{g}_2(t,k)\right) = \hat{q}_0(k) - e^{-ik^3t}\hat{q}(t,k),$$

and the matrix is formed by evaluating this expression at ζk and $\zeta^2 k$ and selecting one of the $\tilde{g}_i$'s and two of the $\tilde{f}_i$'s corresponding to the boundary values that are sought. The coefficient functions of k are eliminated and the resulting matrix is always of the form

$$\begin{pmatrix} 1 & \zeta^{k_1} & \zeta^{3-k_1} \\ e^{ikL} & \zeta^{k_2}e^{i\zeta kL} & \zeta^{3-k_2}e^{i\zeta^2 kL} \\ e^{ikL} & \zeta^{k_3}e^{i\zeta kL} & \zeta^{3-k_3}e^{i\zeta^2 kL} \end{pmatrix}^T, \qquad k_{1,2,3} \in \{0,1,2\}, \tag{4.36}$$

where the first row corresponds to the unknown $\tilde{g}$ which is sought, the second and third rows to the two unknowns $\tilde{f}_i$ and $\tilde{f}_j$ where $i < j$ respectively and all the information is contained in the second column. The value of k_l, $l = 1, 2, 3$, indicates which boundary functions are known. It follows that

- $k_l = 2 -$ (the order of the derivative that is sought),
- $k_2 > k_3$.

For example, if the boundary conditions $q(0), q(L)$ and $q_x(L)$ are prescribed, then $k_1 = 0, k_2 = 1$ and $k_3 = 0$.

S2 : Consider the third order eigenvalue problem

$$v_{xxx}(x) + ik^3 v(x) = 0\,, \quad x \in [0, L]\,.$$

The general solution is given by

$$v(x) = a_0 e^{ikx} + a_1 e^{i\zeta kx} + a_2 e^{i\zeta^2 kx}\,, \quad \zeta = e^{\frac{2\pi i}{3}}\,, \tag{4.37}$$

for constants a_0, a_1 and a_2.

Remark 4.2.3. We remark that the general solution, given by (4.37), has no hope of being bounded for any choice of complex set of k's, and this is the reason that this problem cannot be solved to yield a series solution.

For the problem to be well-posed, one boundary condition must be prescribed at $x = 0$ and two boundary conditions must be prescribed at $x = L$, corresponding to selecting three appropriate rows from the following system:

$$\begin{pmatrix} v(0) \\ v_x(0) \\ v_{xx}(0) \\ v(L) \\ v_x(L) \\ v_{xx}(L) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ ik & i\zeta k & i\zeta^2 k \\ -k^2 & -\zeta^2 k^2 & -\zeta k^2 \\ e^{ikL} & e^{i\zeta kL} & e^{i\zeta^2 kL} \\ ike^{ikL} & i\zeta k e^{i\zeta kL} & i\zeta^2 k e^{i\zeta^2 kL} \\ -k^2 e^{ikL} & -\zeta^2 k^2 e^{i\zeta kL} & -\zeta k^2 e^{i\zeta^2 kL} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}.$$

The boundary condition at $x = 0$ takes the form $a_0 + \zeta^{j_1} a_1 + \zeta^{3-j_1} a_2$ and the two boundary conditions at $x = L$ take the form $a_0 e^{ikL} + \zeta^{j_l} a_1 e^{i\zeta kL} + \zeta^{3-j_l} a_2 e^{i\zeta^2 kL}$ for $l = 2, 3$. The matrix that results is therefore given by

$$\begin{pmatrix} 1 & \zeta^{j_1} & \zeta^{3-j_1} \\ e^{ikL} & \zeta^{j_2} e^{i\zeta kL} & \zeta^{3-j_2} e^{i\zeta^2 kL} \\ e^{ikL} & \zeta^{j_3} e^{i\zeta kL} & \zeta^{3-j_3} e^{i\zeta^2 kL} \end{pmatrix}, \qquad j_{1,2,3} \in \{0, 1, 2\}\,, \tag{4.38}$$

where all the information is in the second column. The values of j_1, j_2 and j_3 represent the order of the derivatives imposed. Hence

- If $\partial_x^i v(0)$ is given then $j_1 = i$ for $i = 0, 1, 2$.
- If $\partial_x^l v(L)$ is given then $j_2, j_3 = l$ for $l = 0, 1, 2$.
- $j_2 < j_3$.

For example if the boundary conditions at $v(0)$, $v(L)$ and $v_x(L)$ are prescribed, then $j_1 = 0$, $j_2 = 0$ and $j_3 = 1$.

The knowledge of the two boundary conditions at $x = L$ determines k_1 and the boundary condition at $x = 0$ determines the values of k_2 and k_3. Since k_1 indicates the order of the unknown function at $x = L$ and the values of j_2 and j_3 are the orders of the derivatives which are known, it follows that $(2-k_1)+j_2+j_3 = 0+1+2$. Hence

$$k_1 = j_2 + j_3 - 1\,.$$

Similarly, since k_2 and k_3 are the orders of the derivatives at $x = 0$ that are unknown, and j_1 is the order of the derivative at $x = 0$ which is known, it follows that $(2 - k_2) + (2 - k_3) + j_1 = 0 + 1 + 2$. Hence

$$k_2 + k_3 = j_1 + 1\,. \tag{4.39}$$

Now, if $j_1 = 0$ or 1 then either $q(0)$ or $q_x(0)$ is the prescribed boundary condition at $x = 0$. In either case it follows that $k_3 = 0$. If however $j_1 = 2$ then it follows that $k_3 = 1$. Therefore, it is simply an interpolation problem to express k_3 in terms of j_1 and then find k_2 from substitution into (4.39). This yields

$$k_3 = \frac{1}{2} j_1(j_1 - 1)\,, \qquad k_2 = 1 + \frac{3}{2} j_1 - \frac{1}{2} j_1^2\,. \tag{4.40}$$

Definition 4.2.4. *Two triples of the form* $\left(\zeta^a, \zeta^b, \zeta^c\right)$ *and* $\left(\zeta^d, \zeta^e, \zeta^f\right)$ *are equivalent if one of the following conditions is satisfied:*

(a) $\zeta\left(\zeta^a, \zeta^b, \zeta^c\right) = \left(\zeta^d, \zeta^e, \zeta^f\right)$,

(b) $\zeta^2\left(\zeta^a, \zeta^b, \zeta^c\right) = \left(\zeta^d, \zeta^e, \zeta^f\right)$,

(c) $a = d$ *and* $b, c = e, f$ *in any order.*

Remark 4.2.5. The matrices corresponding to the triples $\left(\zeta^a, \zeta^b, \zeta^c\right)$ and $\left(\zeta^d, \zeta^e, \zeta^f\right)$, either vary by a constant multiplier in the second column or by the order of the rows. In both cases, the zeros of the determinant functions are unaltered.

Proposition 4.2.6. *The structure of the dominant vectors is always*

$$\left(\zeta^{j_1},\ \zeta^{j_2},\ \zeta^{j_3}\right)^T, \quad \left(\zeta^{k_1},\ \zeta^{k_2},\ \zeta^{k_3}\right),$$

and exactly one of the following conditions is satisfied:

(a) If j_1, j_2, j_3 are distinct, then k_1, k_2, k_3 are distinct. The resulting vectors are identical and given by the triple

- $(1, \zeta, \zeta^2)$.

or one of its equivalents.

(b) If $j_1 = j_2$ then either $k_1 = k_2$ or $k_1 = k_3$. These correspond to the two cases below and necessarily give equivalent triples:

- $(1, 1, \zeta)$,
- $(1, 1, \zeta^2)$.

Proof. It follows immediately from the central columns of the matrices (4.38) and (4.36), that the structure of the dominant vectors is always of the form

$$\left(\zeta^{j_1},\ \zeta^{j_2},\ \zeta^{j_3}\right)^T, \quad \left(\zeta^{k_1},\ \zeta^{k_2},\ \zeta^{k_3}\right) = \left(\zeta^{j_2+j_3-1}, \zeta^{1+\frac{3}{2}j_1-\frac{1}{2}j_1^2}, \zeta^{\frac{1}{2}j_1(j_1-1)}\right),$$

once the common exponential terms to each vector have been omitted.

(a) Suppose j_1, j_2, j_3 are distinct. Then, according to (4.40),

- If $j_1 = 0$ then the vectors are $(1, \zeta^{j_2}, \zeta^{j_3})$ and $(\zeta^{j_2+j_3-1}, \zeta, 1)$ where $j_2, j_3 = 1, 2$.
- If $j_1 = 1$ then the vectors are $(\zeta, \zeta^{j_2}, \zeta^{j_3})$ and $(\zeta^{j_2+j_3-1}, \zeta^2, 1)$ where $j_2, j_3 = 0, 2$.
- If $j_1 = 2$ then the vectors are $(\zeta^2, \zeta^{j_2}, \zeta^{j_3})$ and $(\zeta^{j_2+j_3-1}, \zeta^2, \zeta)$ where $j_2, j_3 = 0, 1$.

In all cases, both vectors are equivalent to $(1, \zeta, \zeta^2)$. Hence $k_1 \neq k_2 \neq k_3$ and the proof is complete.

(b) If $j_1 = j_2$ then there is a pair of known boundary conditions of the same order corresponding to a pair of ζ's of equal power. The two remaining boundary values at $x = 0$ are unknown, and of the two remaining boundary values at

$x = L$, one is known (with a different power of ζ to the pair) and one is sought. Necessarily, one of the unknown boundary values at $x = 0$ must be of the same order as the unknown boundary value at $x = L$, and the proof is complete. In all cases the triples that result take the form $(1, 1, \zeta)$ or $(1, 1, \zeta^2)$ or one of the equivalents, where the pair of known boundary values of the same order, correspond to the first two equal terms of the triple.

$\square$

iii.) n = 4

S1 : Consider now the fourth order PDE

$$q_t(x,t) + q_{xxxx}(x,t) = 0\,, \quad t > 0\,, \quad x \in [0, L]\,,$$

$$q(x,0) = q_0(x)\,, \quad x \in [0, L]\,.$$

The global relation is given by

$$\begin{aligned} &ik^3 \left(\tilde{f}_0(t,k) - e^{-ikL} \tilde{g}_0(t,k) \right) + k^2 \left(\tilde{f}_1(t,k) - e^{-ikL} \tilde{g}_1(t,k) \right) \\ &-ik \left(\tilde{f}_2(t,k) - e^{-ikL} \tilde{g}_2(t,k) \right) - \left(\tilde{f}_3(t,k) - e^{-ikL} \tilde{g}_3(t,k) \right) \\ &= q_0(k) - e^{k^4 t} \hat{q}(t,k)\,, \end{aligned}$$

and the matrix is formed by evaluating this expression at ζk, $\zeta^2 k$ and $\zeta^3 k$ and selecting two of the $\tilde{f}$'s and two of the $\tilde{g}$'s corresponding to the unknown boundary values. The k's are extracted and the matrix that results is always of the form

$$\begin{pmatrix} 1 & \zeta^{k_1} & \zeta^{4-2k_1} & \zeta^{4-k_1} \\ 1 & \zeta^{k_2} & \zeta^{4-2k_2} & \zeta^{4-k_2} \\ e^{ikL} & \zeta^{k_3} e^{i\zeta kL} & \zeta^{4-2k_3} e^{i\zeta^2 kL} & \zeta^{4-k_3} e^{i\zeta^3 kL} \\ e^{ikL} & \zeta^{k_4} e^{i\zeta kL} & \zeta^{4-2k_4} e^{i\zeta^2 kL} & \zeta^{4-k_4} e^{i\zeta^3 kL} \end{pmatrix}^T, \tag{4.41}$$

where $k_{1,2,3,4} \in \{0, 1, 2, 3\}$, and all the information about the problem is in the second column. The first two rows correspond to the two unknowns $\tilde{g}_i$ and $\tilde{g}_j$, where $i < j$, and the last two rows correspond to the two unknowns $\tilde{f}_k$ and $\tilde{f}_l$, where $k < l$. The value of k_l, $l = 1, 2, 3, 4$, indicates the order of the derivative that is sought:

- $k = 3 -$ (the order of the derivative that is sought),
- $k_1 > k_2$ and $k_3 > k_4$.

For example, if the boundary conditions at $q(0)$, $q_x(0)$, $q_{xx}(L)$ and $q_{xxx}(L)$ are prescribed, then $k_1 = 3$, $k_2 = 2$, $k_3 = 1$ and $k_4 = 0$.

S2 : Consider the fourth order eigenvalue problem

$$v_{xxxx}(x) - k^4 v(x) = 0 \,, \quad x \in [0, L] \,.$$

The general solution is given by

$$v(x) = a_0 e^{ikx} + a_1 e^{i\zeta kx} + a_2 e^{i\zeta^2 kx} + a_3 e^{i\zeta^3 kx} \,, \quad \zeta = e^{\frac{\pi i}{2}} = i \,,$$

for constants a_0, a_1, a_2 and a_3. For the problem to be well-posed, two boundary conditions must be prescribed at either end of the interval. Incorporating the coefficient functions of k into the unknown terms, implies that the two boundary conditions at $x = 0$ take the form

$$a_0 + \zeta^{j_1} a_1 + \zeta^{4-2j_1} a_2 + \zeta^{4-j_1} a_3 \,,$$

$$a_0 + \zeta^{j_2} a_1 + \zeta^{4-2j_2} a_2 + \zeta^{4-j_2} a_3 \,,$$

and the two boundary conditions at $x = L$ take the form

$$a_0 e^{ikL} + \zeta^{j_3} a_1 e^{i\zeta kL} + \zeta^{4-2j_3} a_2 e^{i\zeta^2 kL} + \zeta^{4-j_3} a_3 e^{i\zeta^3 kL} \,,$$

$$a_0 e^{ikL} + \zeta^{j_4} a_1 e^{i\zeta kL} + \zeta^{4-2j_4} a_2 e^{i\zeta^2 kL} + \zeta^{4-j_4} a_3 e^{i\zeta^3 kL} \,.$$

The matrix formed from the system is therefore always of the form

$$\begin{pmatrix} 1 & \zeta^{j_1} & \zeta^{4-2j_1} & \zeta^{4-j_1} \\ 1 & \zeta^{j_2} & \zeta^{4-2j_2} & \zeta^{4-j_2} \\ e^{ikL} & \zeta^{j_3} e^{i\zeta kL} & \zeta^{4-2j_3} e^{i\zeta^2 kL} & \zeta^{4-j_3} e^{i\zeta^3 kL} \\ e^{ikL} & \zeta^{j_4} e^{i\zeta kL} & \zeta^{4-2j_4} e^{i\zeta^2 kL} & \zeta^{4-j_4} e^{i\zeta^3 kL} \end{pmatrix} \,, \tag{4.42}$$

where $j_{1,2,3,4} \in \{0, 1, 2, 3\}$, and all the information is in the second column. The value of j_l represents the order of the derivative imposed:

- If $\partial_x^l v(0)$ is given then $j_1, j_2 = l$ for $l = 0, 1, 2, 3$.
- If $\partial_x^j v(L)$ is given then $j_3, j_4 = j$ for $j = 0, 1, 2, 3$.

- $j_1 < j_2$ and $j_3 < j_4$.

For example, if the prescribed boundary conditions are at $v(0)$, $v_x(0)$, $v_{xx}(L)$ and $v_{xxx}(L)$ then $j_1 = 0$, $j_2 = 1$, $j_3 = 2$ and $j_4 = 3$.

Moreover, the values of k_1 and k_2 determine the order of the derivatives at $x = L$ which are known, hence the values of j_3 and j_4. It follows that $(3 - k_1) + (3 - k_2) + j_3 + j_4 = 0 + 1 + 2 + 3$. Hence

$$k_1 + k_2 = j_3 + j_4 \,. \tag{4.43}$$

Similarly the values of k_3 and k_4 determine the order of the derivatives known at $x = 0$, and hence the values of j_1 and j_2. It follows that $(3-k_3)+(3-k_4)+j_1+j_2 = 0 + 1 + 2 + 3$, and therefore

$$k_3 + k_4 = j_1 + j_2 \,. \tag{4.44}$$

Proposition 4.2.7. *The structure of the dominant vectors is always*

$$\left(\zeta^{j_1},\ \zeta^{j_2},\ \zeta^{j_3},\ \zeta^{j_4}\right)^T\ , \quad \left(\zeta^{k_1},\ \zeta^{k_2},\ \zeta^{k_3},\ \zeta^{k_4}\right)\ ,$$

and exactly one of the following cases is satisfied:

(a) If j_1, j_2, j_3, j_4 are distinct then k_1, k_2, k_3, k_4 are distinct. The resulting vectors are identical and given by the quadruple

- $(1, \zeta, \zeta^2, \zeta^3)$,

or one of its equivalents.

(b) If $j_1 = j_3$ and $j_2 = j_4$ then $k_1 = k_3$ and $k_2 = k_4$ corresponding to the quadruple

- $(1, \zeta, 1, \zeta)$,

or one of its equivalents.

(c) The other cases correspond to those for which two of the j's are equal. Either $j_1 = j_3$, $j_1 = j_4$, $j_2 = j_3$ or $j_2 = j_4$ and there is always a pair of equal powers of ζ among the k's. These correspond to the cases below and necessarily give equivalent quadruples:

- $(1, \zeta, 1, \zeta^2)$,
- $(1, \zeta, 1, \zeta^3)$.

Proof. It follows immediately from the second columns of the matrices (4.41) and (4.42), that the structure of the dominant vectors are always of the form

$$\left(\zeta^{j_1},\ \zeta^{j_2},\ \zeta^{j_3},\ \zeta^{j_4}\right)^T, \quad \left(\zeta^{k_1},\ \zeta^{k_2},\ \zeta^{k_3},\ \zeta^{k_4}\right),$$

once the common exponential terms have been omitted.

(a) The proof is an immediate consequence of equalities (4.43) and (4.44).

(b) Suppose $j_1 = j_3$ and $j_2 = j_4$. It follows from equations (4.43) and (4.44) that $k_3 + k_4 = k_1 + k_2$ and the quadruples are of the specified form. Similarly, by construction, it follows trivially that $k_1 = k_3$ and $k_2 = k_4$.

(c) Suppose $j_1 = j_3$. Then there remains three derivatives of $q(x)$ at both $x = 0$ and $x = L$, from which all of the k's have to be selected. Necessarily, either k_1 or k_2 has to be equal to either k_3 or k_4 and the proof is complete. It follows trivially that if only one pair of the j's are equal then the quadruple is necessarily of the required form (or one of the equivalents). The proof follows identically for the other cases.

□

□

Theorem 4.2.8. *The effective discrete spectrum of a PDE boundary value problem, coincides with the classical discrete spectrum of the differential operator D associated with the PDE, equipped with the same boundary conditions.*

Proof. The proof follows immediately, from Proposition 4.2.2. □

4.3 The Location of the Zeros of the Determinant Function

The location of the zeros of the determinant function, for the second, third and fourth order linear evolution PDEs, has been discussed in the previous chapters for a variety of boundary conditions. In this section, we give a general result regarding the location of the zeros of all n^{th} order linear evolution PDEs of the form (2.1), with the boundary conditions chosen according to Theorem 2.1.1.

Theorem 4.3.1. *The zeros of the determinant function $\Delta_n(k)$, of the general n^{th} order linear evolution PDE, of the form (2.1), with the boundary conditions chosen according to Theorem 2.1.1, satisfy one of the following properties:*

- ***i.) Even:*** *If n is even, then the zeros k_n cluster, in the complex k-plane, along the n rays, given by*

$$L_j = \left\{ k_n : \, arg(k_n) = \frac{2(j-1)\pi}{n} \right\} , \quad j = 1, 2, \ldots, n \,. \tag{4.45}$$

- ***ii.) Odd:*** *If n is odd, then the location of the zeros k_n, in the complex k-plane, depends on the boundary conditions that are imposed, and satisfy one of the following properties:*

  - ***(a) Coupled (periodic):*** *If the boundary conditions are periodic, then the zeros cluster along the $2n$ rays, given by*

$$L_j = \left\{ k_n : \, arg(k_n) = \frac{(j-1)\pi}{n} \right\} , \quad j = 1, 2, \ldots, 2n \,, \tag{4.46}$$

  possibly shifted along any fixed axis in the complex k-plane.

  - ***(b) Uncoupled:*** *If the boundary conditions are uncoupled, then the zeros cluster along the n rays, given by*

$$L_j = \left\{ k_n : \, arg(k_n) = \frac{(3n-4+4j)\pi}{2n} \right\} , \quad j = 1, 2, \ldots, n \,, \tag{4.47}$$

  or along the reflection of these rays about the real axis, in the complex k-plane.

Proof. For the proof, we shall use the results given in the proof of Proposition 4.2.2, and analyse the set of zeros of the determinant of the system $S1$. We remark that the results are analogous for all well-posed PDEs of the form (2.1).

- **i.) Even:** We begin with the proof for the second and fourth order PDEs. It follows from the matrices given by (4.34) and (4.41), corresponding to the systems for the unknown boundary values, for the second and fourth order linear evolution PDEs respectively, that the determinant functions $\Delta_2(k)$ and $\Delta_4(k)$, can always be written in the form

$$\begin{aligned} \Delta_2(k) &= F_2\left(e^{-ikL}, e^{ikL}\right) , \\ \Delta_4(k) &= F_4\left(e^0, e^{(1+i)kL}, e^{(-1+i)kL}, e^{(-1-i)kL}, e^{(1-i)kL}\right) , \end{aligned}$$

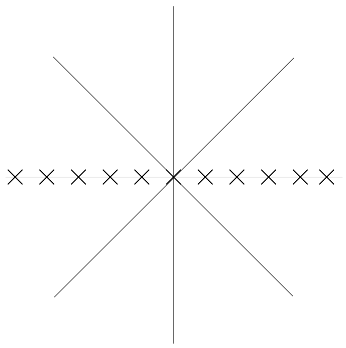
where F_2 and F_4 are linear functions of the indicated exponential terms.

Remark 4.3.2. The coefficients of the exponential terms in the determinant functions are given in terms of powers of ζ. For example, $\Delta_2(k)$ is given explicitly as $\Delta_2(k) = \zeta^{k_2}e^{-ikL} - \zeta^{k_1}e^{ikL}$, where $k_{1,2} \in \{0, 1\}$. However, in order to locate the rays in the complex k-plane upon which the zeros lie, it is sufficient to consider only the exponential terms.

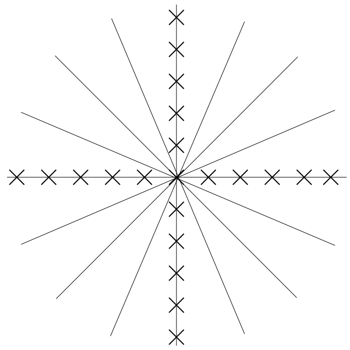
The zeros are located using Levin's Theorem (Theorem 1.3.22). Substituting $z = -ikL$, the determinant functions can be written in the form

$$\begin{aligned} \Delta_2(z) &= F_2\left(e^{z}, e^{\zeta z}\right), \\ \Delta_4(z) &= F_4\left(e^{0}, e^{(-1+i)z}, e^{(-1-i)z}, e^{(1-i)z}, e^{(1+i)z}\right). \end{aligned}$$

In each case, the points, indicated by the exponents, are joined to form a convex hull in the complex z-plane, and the zeros found to cluster asymptotically along the rays that perpendicularly bisect the sides of the polygon that is formed. In the complex k-plane, these rays correspond to the n rays given by (4.45), (Figure 4.7), and the proof for $n = 2$ and $n = 4$ is complete.



(a) $n = 2$.



(b) $n = 4$.

Figure 4.7: The location of the zeros, in the complex k-plane, of the determinant functions $\Delta_2(k)$ and $\Delta_4(k)$, of the general second and fourth order linear evolution PDEs, with the boundary conditions chosen according to Theorem 2.1.1.

To prove the case for the general n^{th} order problem, we follow the approach used for the proof of Proposition 4.2.2. It follows that $\frac{n}{2}$ boundary conditions must be prescribed at either end of the interval, and hence the $n \times n$ matrix corresponding to the system $S1$, comprises exactly $\frac{n}{2}$ rows of complex exponential terms:

$$\begin{array}{c} \frac{n}{2}\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ \frac{n}{2}\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \end{array} \begin{pmatrix} \bullet & \bullet & \cdots\cdots & \bullet \\ \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \cdots\cdots & \bullet \\ e^{ikL} & e^{i\zeta kL} & \cdots\cdots & e^{i\zeta^{n-1}kL} \\ e^{ikL} & e^{i\zeta kL} & \cdots\cdots & e^{i\zeta^{n-1}kL} \\ \vdots & \vdots & \ddots & \vdots \\ e^{ikL} & e^{i\zeta kL} & \cdots\cdots & e^{i\zeta^{n-1}kL} \end{pmatrix}^T,$$

where $\zeta = e^{\frac{2\pi i}{n}}$, and for the purposes of analysing the exponential terms of the determinant function $\Delta_n(k)$, the coefficients of all of the entries have been omitted.

It follows that the determinant function $\Delta_n(k)$, will be a linear function of exponential terms, whose exponents result from summing any $\frac{n}{2}$ of the $\lambda_l(k)$ terms, $k = 0, 1, \ldots n-1$. For example, for the sixth order problem, the determinant function $\Delta_6(k)$ will always take the form

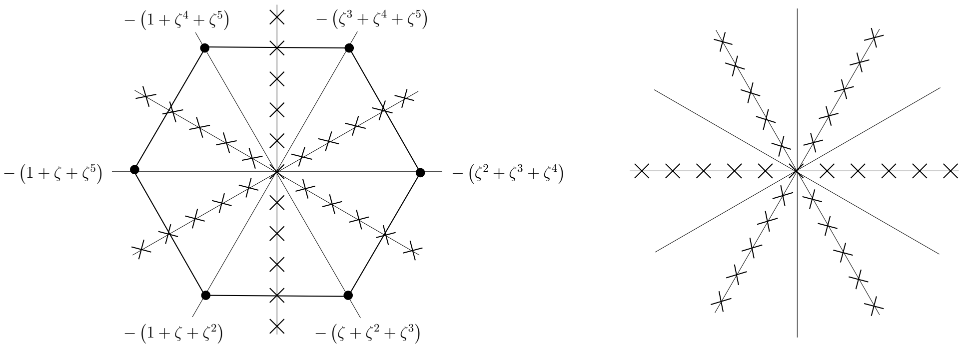
$$\Delta_6(k) = F_6\left(e^{\left(1+\zeta+\zeta^2\right)ikL}, e^{\left(1+\zeta+\zeta^3\right)ikL}, \ldots, e^{\left(\zeta^3+\zeta^4+\zeta^5\right)ikL}\right),$$

and for the general n^{th} order case, $\Delta_n(k)$ can always be written in the form

$$\Delta_n(k) = F_n\left(e^{\left(1+\cdots+\zeta^{\frac{n}{2}-1}\right)ikL}, \ldots, e^{\left(\zeta^{\frac{n}{2}+1}+\cdots+\zeta^{n-1}\right)ikL}\right). \tag{4.48}$$

To locate the zeros in the complex k-plane, we use Levin's Theorem (Theorem 1.3.22), which means constructing a convex hull in the complex plane, of the points indicated by the exponents.

For the sixth order problem, Figure 4.8(a) indicates the 6 exponential terms that form the convex hull. It follows trivially for this case that the rays upon which the zeros lie, satisfy expression (4.45), (Figure 4.8(b)). For the n^{th} order problem, when the exponents in expression (4.48) are analysed, one finds that the exponents that contribute to the convex hull are precisely those whose distinct ζ terms lie on adjacent rays in the complex z-plane. So for example, for the sixth order problem, we see that the exponent given by $(1+\zeta+\zeta^2)ikL$ contributes to the convex hull (Figure 4.8(a)), whereas the exponent given by $(1+\zeta+\zeta^3)ikL$ does not, because ζ does not lie on an adjacent ray to ζ^3 in the complex z-plane, and as a result $-(1+\zeta+\zeta^3)$ lies within the convex hull.



(a) z-plane ($z = -ikL$). (b) k-plane.

Figure 4.8: The location of the zeros of the determinant function $\Delta_6(k)$ of the sixth order linear evolution PDE, of the form (2.1) with the boundary conditions chosen according to Theorem 2.1.1.

It follows that for the n^{th} order problem, there are precisely n exponential terms that contribute to the convex hull, and hence there are n rays in the complex k-plane upon which the zeros of the determinant function $\Delta_n(k)$ lie. The proof that these rays are given by (4.45), follows trivially from the observation that the exponential term, whose exponent comprises the $\frac{n}{2}$ terms, $1, \zeta, \zeta^2, \ldots, \zeta^{\frac{n}{2}-1}$, contributes to the convex hull.

ii.) Odd:

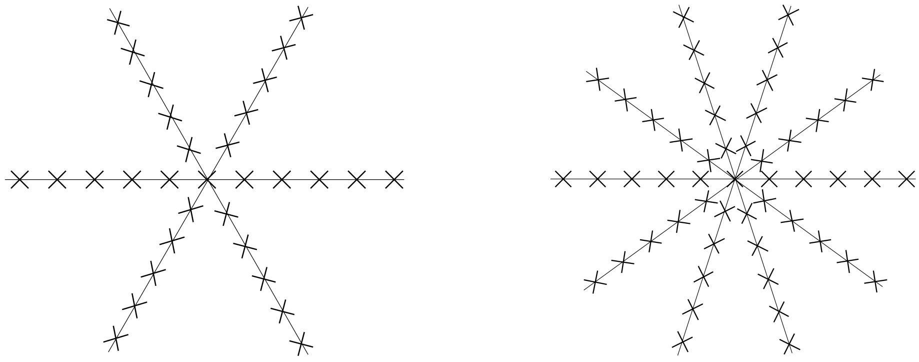
(a) Coupled (periodic): The proof follows trivially from the system of global relations, given by (2.30). The imposition of periodic boundary conditions, yields the system of equations, given by

$$\left(1 - e^{i\lambda_l(k)L}\right)\Big\{c_0(\lambda_l(k))\tilde{f}_0(t,k) + c_1(\lambda_l(k))\tilde{f}_1(t,k) + \cdots$$
$$\cdots + c_{n-1}(\lambda_l(k))\tilde{f}_{n-1}(t,k)\Big\} = \hat{q}_0(\lambda_l(k)) - e^{\omega(k)t}\hat{q}(t,\lambda_l(k)),$$

for $l = 0, 1, \ldots n-1$. It follows that the determinant function is always of the form

$$\Delta_n(k) \sim \left(1 - e^{-ikL}\right)\left(1 - e^{-i\zeta kL}\right)\left(1 - e^{-i\zeta^2 kL}\right)\ldots\left(1 - e^{-i\zeta^{n-1}kL}\right),$$

where $\zeta = e^{\frac{2\pi i}{n}}$. Hence the zeros cluster asymptotically along the rays given by (4.46), and the proof is complete. Figure 4.9(a) and Figure 4.9(b) show the location of the zeros in the complex k-plane, for the cases where $n = 3$ and $n = 5$ respectively.



(a) $n = 3$. (b) $n = 5$.

Figure 4.9: The location of the zeros in the complex k-plane of the determinant functions $\Delta_3(k)$ and $\Delta_5(k)$ of the third and fifth order linear evolution PDEs respectively, with periodic boundary conditions.

Remark 4.3.3. The imposition of coupled boundary conditions, either does not effect the location of the zeros in the complex k-plane, or there is a shift of the axes upon which the zeros lie.

For example, it was shown in Section 3.2.4, that the imposition of coupled boundary conditions yields a determinant function whose zeros lie on the same rays in the complex k-plane as the corresponding problem with periodic boundary conditions, whereas in Section 3.2.3, it was shown that the imposition of quasi-periodic boundary conditions corresponds directly to a shift in the complex k-plane of the rays upon which the zeros lie.

(b) Uncoupled: For the proofs that are to follow, we consider the PDEs of the form $q_t(x,t) + q_x^{(n)}(x,t) = 0$ and prove that the zeros of the determinant function $\Delta_n(k)$ lie on the n rays in the complex k-plane, given by (4.47).

We begin with the proof for the third order problem, and recall the general matrix, given by (4.36), that results from the imposition of one boundary condition at $x = 0$ and two boundary conditions at $x = L$. It follows that the determinant function $\Delta_3(k)$, can always be written in the form

$$\Delta_3(k) = F_3\left(e^{-ikL}, e^{-i\zeta kL}, e^{-i\zeta kL}\right), \quad \zeta = e^{\frac{2\pi i}{3}},$$

where F_3 is a linear function of the indicated exponential terms. Substituting $z = -ikL$, implies that the determinant function can always be written in the form

$$\Delta_3(z) = F_3\left(e^z, e^{\zeta z}, e^{\zeta^2 z}\right).$$

The proof now follows immediately, from Section 2.5.

Let us now consider the following fifth order PDE:

$$q_t(x,t) + q_{xxxxx}(x,t) = 0\,, \quad t > 0\,, \quad x \in [0,L]\,,$$

$$q(x,0) = q_0(x)\,, \quad x \in [0,L]\,.$$

The global relation is given by

$$\begin{aligned}
&-k^4(\tilde{f}_0(t,k) - e^{-ikL}\tilde{g}_0(t,k)) + ik^3(\tilde{f}_1(t,k) - e^{-ikL}\tilde{g}_1(t,k)) \\
&+ k^2(\tilde{f}_2(t,k) - e^{-ikL}\tilde{g}_2(t,k)) - ik(\tilde{f}_3(t,k) - e^{-ikL}\tilde{g}_3(t,k)) \\
&- (\tilde{f}_4(t,k) - e^{-ikL}\tilde{g}_4(t,k)) = \hat{q}_0(k) - e^{ik^5 t}\hat{q}(t,k)\,,
\end{aligned}$$

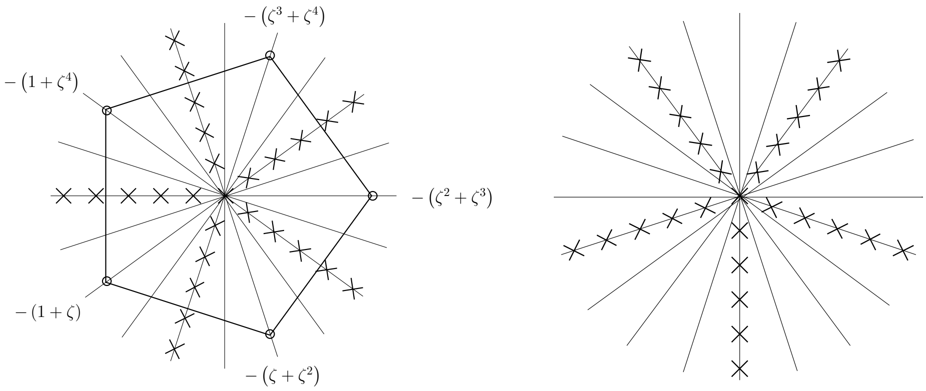
which is supplemented by the four additional expressions, evaluated at ζk, $\zeta^2 k$, $\zeta^3 k$ and $\zeta^4 k$, where $\zeta = e^{\frac{2\pi i}{5}}$. For the problem to be well-posed, three boundary conditions must be prescribed at $x = 0$ and two boundary conditions must be prescribed at $x = L$, corresponding to selecting two of the f's and three of the g's. The resulting matrix is therefore always of the form

$$\begin{pmatrix}
1 & \zeta^{k_1} & \zeta^{5-3k_1} & \zeta^{5-2k_1} & \zeta^{5-k_1} \\
1 & \zeta^{k_2} & \zeta^{5-3k_2} & \zeta^{5-2k_2} & \zeta^{5-k_2} \\
1 & \zeta^{k_3} & \zeta^{5-3k_3} & \zeta^{5-2k_3} & \zeta^{5-k_3} \\
e^{ikL} & \zeta^{k_4}e^{i\zeta kL} & \zeta^{5-3k_4}e^{i\zeta^2 kL} & \zeta^{5-2k_4}e^{i\zeta^3 kL} & \zeta^{5-k_4}e^{i\zeta^4 kL} \\
e^{ikL} & \zeta^{k_5}e^{i\zeta kL} & \zeta^{5-3k_5}e^{i\zeta^2 kL} & \zeta^{5-2k_5}e^{i\zeta^3 kL} & \zeta^{5-k_5}e^{i\zeta^4 kL}
\end{pmatrix}^T.$$

It follows that the determinant function $\Delta_5(k)$ is comprised of a linear combination of the exponential terms, whose exponents result from summing any two of the $\lambda_l(k)$, $l = 0, 1, \ldots, 4$:

$$\begin{aligned}
\Delta_5(k) \;=\; F_5\Big(&e^{(1+\zeta)ikL}\,,\, e^{(1+\zeta^2)ikL}\,,\, e^{(1+\zeta^3)ikL}\,,\, e^{(1+\zeta^4)ikL}\,,\, e^{(\zeta+\zeta^2)ikL}\,, \\
&e^{(\zeta+\zeta^3)ikL}\,,\, e^{(\zeta+\zeta^4)ikL}\,,\, e^{(\zeta^2+\zeta^3)ikL}\,,\, e^{(\zeta^2+\zeta^4)ikL}\,,\, e^{(\zeta^3+\zeta^4)ikL}\Big)\,.
\end{aligned}$$

In the complex z-plane, the points, indicated by the exponents, are joined to form a convex hull (Figure 6.22(a)), and the resulting pentagon indicates that the zeros cluster asymptotically along the five rays in the complex k-plane that satisfy expression (4.47), (Figure 6.22(b)). This completes the proof for



(a) z-plane ($z = -ikL$).

(b) k-plane.

Figure 4.10: The location of the zeros of the determinant function $\Delta_5(k)$ of the fifth order PDE $q_t(x,t) + q_{xxxxx}(x,t) = 0$ with uncoupled boundary conditions.

the fifth order case.

This approach of constructing the determinant function, can be generalised for the n^{th} order problem. If the boundary conditions are uncoupled, then N boundary conditions must be prescribed at $x = 0$ and $n - N$ at $x = L$, where N is determined according to Theorem 2.1.1. It follows that the $n \times n$ matrix corresponding to the system $S1$, comprises exactly $n - N$ rows of complex exponential terms:

$$\begin{array}{r} N \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ n-N \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \end{array} \begin{pmatrix} \bullet & \bullet & \cdots\cdots & \bullet \\ \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \cdots\cdots & \bullet \\ e^{ikL} & e^{i\zeta kL} & \cdots\cdots & e^{i\zeta^{n-1}kL} \\ e^{ikL} & e^{i\zeta kL} & \cdots\cdots & e^{i\zeta^{n-1}kL} \\ \vdots & \vdots & \ddots & \vdots \\ e^{ikL} & e^{i\zeta kL} & \cdots\cdots & e^{i\zeta^{n-1}kL} \end{pmatrix}^T ,$$

where $\zeta = e^{\frac{2\pi i}{n}}$, and for the purposes of analysing the exponential terms of the determinant function $\Delta_n(k)$, the coefficients of all of the entries have been omitted.

Therefore, the determinant function $\Delta_n(k)$ is given by the linear combination of exponential terms, whose exponents result from summing any $n-N$ of the $\lambda_l(k)$ terms, $k = 0, 1, \ldots, n-1$. It follows that $\Delta_n(k)$ will comprise exactly ${}^n\mathrm{C}_{n-N}$ terms:

$$\Delta_n(k) = F_n\left(e^{\left(1+\cdots+\zeta^{n-N-1}\right)ikL}, \ldots, e^{\left(\zeta^{n-N+1}+\cdots+\zeta^{n-1}\right)ikL}\right).$$

The proof that the zeros cluster, in the complex k-plane, along the n rays given by (4.47), follows from the observation that the n exponential terms that contribute to the convex hull are the ones whose exponents comprise powers of ζ that lie on adjacent rays in the complex z-plane. The remainder of the argument follows analogously to the argument used for the general even order problem, and the proof is complete.

Remark 4.3.4. If the boundary conditions are uncoupled and n is odd, then the zeros of the determinant function $\Delta_n(k)$, of the n^{th} order linear evolution PDE of the form $q_t(x,t) - q_x^{(n)}(x,t) = 0$, cluster along the n rays in the complex k-plane, given by

$$L_j = \left\{k_n : \arg(k_n) = \frac{(5n-4+4j)\,\pi}{2n}\right\}, \quad j = 1, 2, \ldots, n,$$

which are precisely the n rays given by (4.47), reflected about the real axis. Therefore the proof follows analogously to the above and is omitted from the work.

□

Chapter 5

Linear Numerical Results

In this chapter we consider the numerical solution of third and fourth order linear differential equations, of the general form

$$u_x^{(n)}(x) + Tu(x) = f(x)\,, \quad x \in [-1, 1]\,, \tag{5.1}$$

where T is a linear x-differential operator, $f(x)$ is a given smooth function, the integer n defines the order of the problem and it is assumed that n boundary conditions are chosen according to Theorem 2.1.1.

In the first section we solve a wide variety of third order problems. The difficulties posed by the third order differential operator are due to the lack of symmetry, characteristic of any odd order problem, and the subsequent non-symmetric nature of the boundary conditions. All of the schemes use Chebyshev interpolation and employ the program *cheb*, which was introduced in Section 1.2.2, to compute the Chebyshev differentiation matrix D_N, defined by Theorem 1.2.1. Of particular interest is the solution of the third order problem with coupled boundary conditions, which is illustrated in detail in the next section.

Our aim is devising numerical schemes for the solution of the boundary value problems under consideration that are simple and user-friendly, and that capture accurately the qualitative behaviour of the solution. Our schemes are based on *Matlab* standard built-in functions and routines, and their novelty is in the ability to model a variety of different boundary conditions. However, they would not be adequate for a detailed study of small scale phenomena in the solution.

Remark 5.0.5. We do not discuss here the direct numerical evaluation of the integral representation formula for the solution of a linear boundary value problem, given by the transform method of Fokas. This approach has recently been considered [24] and appears to yield a very competitive numerical solution technique in a large number of cases. This is due to the property the integral representation sometime possesses, namely that the contour for the computation can be deformed, using analyticity properties, in such a way that all integrands are exponentially decreasing. This is a promising further application to numerical computations of the method of Fokas, and a topic for further investigation.

5.1 Third Order Numerical Results

In this section we solve numerically a variety of third order linear boundary value problems. We begin with a simple second order ODE boundary value problem as an illustrative example before developing the method for a variety of examples that do not currently appear in the litearture, for the numerical imposition of boundary conditions for the case of an odd order differential operator. We conclude this section with a simple example of a third order PDE with non-homogeneous uncoupled boundary conditions to illustrate the technique.

5.1.1 The Imposition of Boundary Conditions

All of the examples included within this section use a matrix-stripping technique of the appropriate rows and columns of the Chebyshev matrix D_N for the explicit imposition of the boundary conditions.

Example 1: Let us begin with one of the simplest possible examples that demonstrates the use of the program *cheb*. Consider the following second order linear ODE boundary value problem, with Dirichlet boundary conditions:

$$u_{xx}(x) = f(x)\,, \quad x \in [-1, 1]\,,$$

$$u(-1) = 0\,, \quad u(1) = 0\,,$$

where $f(x)$ is a prescribed smooth function. To achieve a numerical approximation to the exact solution $u(x)$, we begin by computing the Chebyshev differentiation matrix D_N^2, which is precisely the square of D_N. The imposition of the boundary conditions is straightforward. We begin by taking the interior Chebyshev points $x_1, \ldots, x_{N-1}$ as our computational grid, along with the vector $u = (u_1, \ldots, u_{N-1})^T$ as the corresponding vector of unknowns:

- Let $p(x)$ be the unique polynomial such that $\deg(p(x)) \leqslant N$ with $p(\pm 1) = 0$ and $p(x_j) = u_j$, $1 \leqslant j \leqslant N-1$.
- Set $f_j = p''(x_j)$, $1 \leqslant j \leqslant N-1$.

So, D_N^2 is an $(N+1)\times(N+1)$ matrix, that maps the vector of unknowns $(u_0, \ldots, u_N)^T$ to the vector $(f_0, \ldots, f_N)^T$. Since u_0 and u_N are known to take the values of zero, we can ignore f_0 and f_N, implying that the outer rows and columns of D_N^2 have no effect. This means that all we have to do is invert the reduced $(N-1)\times(N-1)$ matrix, which we shall call $\tilde{D}_N^2$, and multiply by the vector $(f_1, \ldots, f_{N-1})^T$ to achieve an approximation to the interior points $(u_1, \ldots, u_{N-1})^T$.

Example 2: We now examine the following third order linear ODE boundary value problem with homogeneous uncoupled boundary conditions:

$$u_{xxx}(x) = f(x)\,, \quad x \in [-1, 1]\,, \tag{5.2a}$$

$$u(-1) = 0\,, \quad u(1) = 0\,, \quad u_x(1) = 0\,, \tag{5.2b}$$

where $f(x)$ is a prescribed smooth function. The imposition of the boundary conditions is not as straightforward as with the second order problem, because we now have both Dirichlet and Neumann boundary conditions at $x = 1$. However, to solve the problem numerically, we employ a simple trick involving polynomials related as follows:

$$u(x) = (1-x)q(x)\,,$$

for some polynomial $q(x)$. After three differentiations we obtain

$$u_{xxx}(x) = (1-x)q_{xxx}(x) - 3q_{xx}(x)\,.$$

A polynomial $q(x)$ such that $\deg(q(x)) \leqslant N$, with $q(\pm 1) = 0$, corresponds to a polynomial $u(x)$ such that $\deg(u(x)) \leqslant N+1$, with $u(\pm 1) = u_x(1) = 0$. Hence $\deg(u(x)) = \deg(q(x)) + 1$. We take the interior Chebyshev points $x_1, \ldots, x_{N-1}$ as our computational grid with $u = (u_1, \ldots, u_{N-1})^T$ as the corresponding vector of unknowns:

- Let $q(x)$ be the unique polynomial such that $\deg(q(x)) \leqslant N$ with $q(\pm 1) = 0$ and $q(x_j) = \frac{u_j}{1-x_j}$, $j = 1, \ldots, N-1$.

- Set $f_j = (1 - x_j)q_{xxx}(x_j) - 3q_{xx}(x_j)$, $j = 1, \ldots, N-1$.

Hence

$$\left((1 - x_j)D_N^3 - 3D_N^2\right) q(x_j) = f(x_j)\,, \qquad j = 0, \ldots, N\,,$$

where $q(\pm 1) = 0$. Therefore the matrix $[\mathrm{diag}(1 - x)D_N^3 - 3D_N^2] \times \mathrm{diag}\left(\frac{1}{1-x}\right)$ maps a vector $(u_0, \ldots, u_N)^T$ to a vector $(f_0, \ldots, f_N)^T$. The problem has therefore been converted from one in terms of $u(x)$ with three boundary conditions, to one in terms of $q(x)$ with simple Dirichlet boundary conditions, which we know how to solve from the second order problem seen previously. Hence to solve our problem, we define our spectral discrete operator as follows:

$$L = \left[\mathrm{diag}(1 - x_j)\tilde{D}_N^3 - 3\tilde{D}_N^2\right] \times \mathrm{diag}\left(\frac{1}{1 - x_j}\right)\,, \quad j = 1, \ldots, N-1\,,$$

where $\tilde{D}_N^3$ and $\tilde{D}_N^2$ are the matrices obtained by taking the indicated powers of D_N and stripping away the first and last rows and columns. So, solving our original problem spectrally is now equivalent to solving the linear system of equations for $u(x)$:

$$Lu = f\,, \qquad f = (f_1, \ldots, f_{N-1})^T\,.$$

There is an alternative approach to imposing the boundary conditions, which can be adapted to accommodate more complicated boundary conditions, and will prove to be useful later. We begin by writing the problem as follows:

$$D_N^3 \mathbf{u}_j = \mathbf{f}_j\,, \quad 0 \leqslant j \leqslant N\,,$$

where $\mathbf{u}_j = (u(x_0), u(x_1), \ldots, u(x_N))^T$ and $\mathbf{f}_j = (f(x_0), f(x_1), \ldots, f(x_N))^T$. To impose the two Dirichlet boundary conditions, we begin by stripping D_N^3 of its outer rows and columns to produce $\tilde{D}_N^3$:

$$\tilde{D}_N^3 \mathbf{u}_j = \mathbf{f}_j\,, \quad 1 \leqslant j \leqslant N-1\,. \tag{5.3}$$

To impose the Neumann boundary condition at $x = 1$ we replace the first row of $\tilde{D}_N^3$ with the first row of D_N and replace f_1 in (5.3) by 0, since $u_x(1) = 0$. The system can be viewed as follows:

$$\begin{pmatrix} D_N(0,1) & D_N(0,2) & \dots & D_N(0,N-1) \\ \tilde{D}_N^3(2,1) & \tilde{D}_N^3(2,2) & \dots & \tilde{D}_N^3(2,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{D}_N^3(N-1,1) & \tilde{D}_N^3(N-1,2) & \dots & \tilde{D}_N^3(N-1,N-1) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix} .$$

Example 3: The technique of transforming the problem for $u(x_j)$ into one in terms of $q(x_j)$, where just two boundary conditions are imposed on $q(x)$, can be extended to the case where $u(x)$ has non-homogeneous uncoupled boundary conditions. We consider the example

$$u_{xxx}(x) + Au_{xx}(x) + Bu_x(x) + Cu(x) = f(x)\,, \quad x \in [-1,1]\,, \tag{5.4a}$$

$$u(-1) = a\,, \quad u(1) = b\,, \quad u_x(1) = c\,, \tag{5.4b}$$

where A, B, C, a, b and c are constants and $f(x)$ is a prescribed smooth function. So in this case our operator L is given by

$$L = \frac{\mathrm{d}}{\mathrm{d}x^3} + A\frac{\mathrm{d}}{\mathrm{d}x^2} + B\frac{\mathrm{d}}{\mathrm{d}x} + C\,.$$

To impose the boundary conditions, we suppose that $u(x)$ takes the form

$$u(x) = g(x)\,q(x) + h(x)\,, \tag{5.5}$$

for some smooth functions $g(x)$ and $h(x)$ and polynomial $q(x)$ such that $q(\pm 1) = 0$. By inspection we find that

$$g(x) = x - 1\,, \qquad h(x) = \left(\frac{a + 2c - b}{4}\right) x^2 + \left(\frac{b - a}{2}\right) x + \left(\frac{3b + a - 2c}{4}\right) .$$

The original problem can therefore be expressed in terms of $q(x)$ as follows:

$$\begin{aligned}(x-1)q_{xxx}(x) + (3 + A(x-1))\,q_{xx}(x) + (2A + B(x-1))\,q_x(x) + (B + C(x-1))\,q(x) \\ = f(x) - Ah_{xx}(x) - Bh_x(x) - Ch(x)\,,\end{aligned}$$

where $q(\pm 1) = 0$. Hence our spectral discrete operator for the problem is given by

$$\begin{aligned} L \ = \ & \mathrm{diag}(x_j - 1)\tilde{D}_N^3 + 3\tilde{D}_N^2 + A\,\mathrm{diag}(x_j - 1)\tilde{D}_N^2 + 2A\tilde{D}_N \\ & + B\,\mathrm{diag}(x_j - 1)\tilde{D}_N + B\tilde{D}_N^0 + C\,\mathrm{diag}(x_j - 1)\tilde{D}_N^0\,, \end{aligned}$$

for $j = 1, \ldots, N-1$. Thus we solve the linear system for $q(x)$ and compute the solution $u(x)$ via equation (5.5).

Example 4: Consider now the third order linear ODE boundary value problem, with coupled boundary conditions:

$$u_{xxx}(x) = f(x)\,, \quad x \in [-1, 1]\,,$$

$$u(-1) = 0\,, \quad u(1) = 0\,, \quad u_x(1) = \alpha u_x(-1)\,,$$

where $f(x)$ is a prescribed smooth function and α is a given constant. We begin by discretising the problem as follows:

$$D_N^3 \mathbf{u}_j = \mathbf{f}_j\,, \quad 0 \leqslant j \leqslant N\,, \tag{5.6}$$

where $\mathbf{u}_j = (u(x_0), u(x_1), \ldots, u(x_N))^T$ and $\mathbf{f}_j = (f(x_0), f(x_1), \ldots, f(x_N))^T$. The two Dirichlet boundary conditions are imposed by stripping D_N^3 of its outer rows and columns to produce $\tilde{D}_N^3$:

$$\tilde{D}_N^3 \mathbf{u}_j = \mathbf{f}_j, \quad 1 \leqslant j \leqslant N-1.$$

The coupled boundary condition is imposed by replacing the final row of $\tilde{D}_N^3$ by a combination of the first and last rows of D_N. The system of equations that result, can be viewed as follows:

$$\begin{pmatrix} \tilde{D}_N^3(1,1) & \tilde{D}_N^3(1,2) & \ldots & \tilde{D}_N^3(1,N-1) \\ \tilde{D}_N^3(2,1) & \tilde{D}_N^3(2,2) & \ldots & \tilde{D}_N^3(2,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{D}_N^3(N-2,1) & \tilde{D}_N^3(N-2,2) & \ldots & \tilde{D}_N^3(N-2,N-1) \\ d_N^3(N-1,1) & d_N^3(N-1,2) & \ldots & d_N^3(N-1,N-1) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-2} \\ 0 \end{pmatrix},$$

where

$$d_N^3(N-1, i) = D_N(0, i) - \alpha D_N(N, i)\,, \quad i = 1, 2, \ldots N-1\,.$$

The solution on the interior grid points is now easily obtainable by inverting the matrix.

5.1.2 The Time Dependent Boundary Value Problem with Non-Homogeneous Uncoupled Boundary Conditions

We now solve the following time dependent problem

$$u_t(x,t) + A1u_{xxx}(x,t) + A2u_{xx}(x,t) + A3u_x(x,t) + A4u(x,t) + A5 = 0\,,$$

$$u(x,0) = u_0(x)\,, \quad t > 0\,, \quad x \in [-1,1]\,,$$

$$u(-1,t) = a\,, \quad u(1,t) = b\,, \quad u_x(1,t) = c\,,$$

for constants $A1(\neq 0), A2, A3, A4, A5, a, b$ and c.

For the time derivative we use a Backward Euler formula, and we approximate the spatial derivatives via the Chebyshev differentiation matrix D_N. The approximation is given by

$$\begin{aligned}\frac{u(x_j,t+\Delta t) - u(x_j,t)}{\Delta t} &= -A1D_N^3u(x_j,t+\Delta t) - A2D_N^2u(x_j,t+\Delta t) \\ &\quad -A3D_Nu(x_j,t+\Delta t) - A4u(x_j,t+\Delta t) - A5\,,\end{aligned}$$

which we rearrange to give

$$\begin{aligned}&D_N^3u(x_j,t+\Delta t) + \frac{A2}{A1}D_N^2u(x_j,t+\Delta t) + \frac{A3}{A1}D_Nu(x_j,t+\Delta t) \\ &\qquad + \frac{1}{A1}\left(A4 + \frac{1}{\Delta t}\right)u(x_j,t+\Delta t) = \frac{1}{A1}\frac{1}{\Delta t}u(x_j,t) - \frac{A5}{A1}\,.\end{aligned}$$

At each time level, the problem takes the form of (5.4) where

$$A = \frac{A2}{A1}\,, \quad B = \frac{A3}{A1}\,, \quad C = \frac{1}{A1}\left(A4 + \frac{1}{\Delta t}\right)\,, \quad f(x_j) = \frac{1}{A1}\frac{1}{\Delta t}u(x_j,t) - \frac{A5}{A1}\,,$$

which we know how to solve, via Example 3.

5.2 Fourth Order Numerical Results

We now develop the numerical schemes of Section 5.1 for the solution of fourth order linear boundary value problems.

In the next section we present a numerical scheme involving a simple polynomial trick for imposing clamped boundary conditions, which follows closely the analogous third order problem, given by Example 2 of Section 5.1.1. This approach is then contrasted to a new, more adaptable scheme for the explicit imposition of the boundary conditions, involving the manipulation of the rows and columns of the Chebyshev differentiation matrices. This approach, based on the *Matlab* Differentiation Matrix Suite of Weideman [51], has the advantage of being easily adaptable for more complicated boundary conditions, and essentially follows the general approach that was used in Section 5.1.

We conclude the chapter by introducing the concept of implicit transform methods for the imposition of boundary conditions. This is well known for second order problems, where it essentially reduces to the use of the sine and cosine transform. We illustrate here, the use of these transforms to solve a fourth order example, with a view to extending this idea to more general transforms tailored to the specific problem to be solved, and modelled on the solution representation given by the Fokas transform. This approach is not pushed further here, but will be the focus of further work.

5.2.1 Clamped Boundary Conditions

In this section we present an approach, involving a simple polynomial trick, for the imposition of clamped boundary conditions. We consider the fourth order linear ODE boundary value problem, given by

$$u_{xxxx}(x) = f(x)\,, \quad x \in [-1, 1]\,, \tag{5.7a}$$

$$u(\pm 1) = 0\,, \quad u_x(\pm 1) = 0\,, \tag{5.7b}$$

where $f(x)$ is a prescribed smooth function. The spatial domain $[-1, 1]$ is discretised by $N + 1$ unevenly spaced Chebyshev points, defined by (1.12), and the corresponding vector of unknowns is given by $(u(x_0), u(x_1), \ldots , u(x_N))^T$.

To solve the problem numerically, we employ a simple trick involving polynomials related as follows

$$u(x) = (1 - x^2)q(x), \quad q(\pm 1) = 0. \tag{5.8}$$

Differentiating (5.8) four times, we obtain

$$u_{xxxx}(x) = (1 - x^2)q_{xxxx}(x) - 8xq_{xxx}(x) - 12q_{xx}(x)\,.$$

A polynomial $q(x)$ such that $\deg(q(x)) \leqslant N$, with $q(\pm 1) = 0$, corresponds to a polynomial $u(x)$ such that $\deg(u(x)) \leqslant N + 2$, with $u(\pm 1) = u_x(\pm 1) = 0$. Hence $\deg(u(x)) =$

deg$(q(x))+2$. We take the interior Chebyshev points $x_1, \ldots, x_{N-1}$ as our computational grid with $u = (u_1, \ldots, u_{N-1})^T$ as the corresponding vector of unknowns:

- Let $q(x)$ be the unique polynomial such that $\deg(q(x)) \leqslant N$ with $q(\pm 1) = 0$ and $q(x_j) = \frac{u_j}{1-x_j^2}$, $j = 1, \ldots, N-1$.
- Set $f_j = (1-x_j^2)q_{xxxx}(x_j) - 8x_j q_{xxx}(x_j) - 12q_{xx}(x_j)$, $j = 1, \ldots, N-1$.

Hence

$$\left((1-x_j^2)D_N^4 - 8x_j D_N^3 - 12D_N^2\right) q(x_j) = f(x_j)\,, \quad j = 0, \ldots, N\,,$$

where $q(\pm 1) = 0$, and D_N^2, D_N^3 and D_N^4 are defined by taking the appropriate powers of the matrix D_N. Therefore the matrix $\left[\mathrm{diag}(1-x_j^2)D_N^4 - 8\mathrm{diag}(x_j)D_N^3 - 12D_N^2\right] \times \mathrm{diag}\left(\frac{1}{1-x_j^2}\right)$, maps the vector $(u_0, \ldots, u_N)^T$ to the vector $(f_0, \ldots, f_N)^T$. Hence, to solve (5.7), we define our spectral biharmonic operator by

$$L = \left[\mathrm{diag}(1-x_j^2)\tilde{D}_N^4 - 8\mathrm{diag}(x_j)\tilde{D}_N^3 - 12\tilde{D}_N^2\right] \times \mathrm{diag}\left(\frac{1}{1-x_j^2}\right)\,, \quad j = 1, \ldots, N-1\,,$$

where $\tilde{D}_N^2$, $\tilde{D}_N^3$ and $\tilde{D}_N^4$ are the matrices obtained by taking the indicated powers of D_N and stripping away the first and last rows and columns. The solution of the original problem is now obtained from solving the following linear system of equations for $u(x)$ on the interior grid points:

$$Lu = f\,, \qquad f = (f_1, \ldots, f_{N-1})^T\,.$$

Remark 5.2.1. This approach can be compared to the analogous third order case, given by (5.2), where the boundary conditions $u(\pm 1) = 0$ and $u_x(1) = 0$ were imposed by letting $u(x) = (1-x)q(x)$ where $q(\pm 1) = 0$.

5.2.2 Weideman's *Matlab* Differentiation Matrix Suite

The *Matlab* Differentiation Matrix Suite of Weideman and Reddy [51] comprises 17 *Matlab* functions for solving differential equations by the spectral collocation (pseudospectral) method, and combines the concepts of the differentiation matrix, along with the matrix-based approach to the numerical solution of differential equations. The codes presented, enable the user to generate spectral differentiation matrices, plus associated nodes, based on Chebyshev, Fourier, Hermite and other interpolants, and can be used to solve a variety of boundary value problems.

The emphasis of the paper is on the matrix-based implementation of the spectral collocation method. It is recognised that transform methods, such as the fast Fourier transform (FFT), can be computed in $O(N \log N)$ operations rather than $O(N^2)$ operations, required by the direct computation. However, for small values of N, the matrix approach is faster than the FFT implementation. Hence, there are situations when the matrix approach is preferable.

Remark 5.2.2. The general approach used for the numerical schemes in the paper of Weideman [51], can be compared to the approach used for the schemes that have already been presented in Section 5.1 for the third order numerical results. However, the schemes of interest in [51] are presented in the context of solving fourth order problems, and hence are discussed in this section for the first time.

Clamped Boundary Conditions

In this section, we present two more approaches for solving the problem, given by (5.7). The first is an adaptation of the method presented by Weideman [51], for the imposition of the hinged boundary conditions $u(\pm 1) = u_{xx}(\pm 1) = 0$, and the second involves the construction of an interpolating polynomial.

Method One

In this section we explain how the approach of Weideman, used for the imposition of hinged boundary conditions, can be adapted for the imposition of the clamped boundary conditions $u(\pm 1) = u_x(\pm 1) = 0$. This approach can be compared to the analogous third order example of Section 5.1.1. We begin by writing the problem $u_{xxxx}(x) = f(x)$, for some given function $f(x)$, on the $N + 1$ point grid, as the linear system of equations, given by

$$D_N^4 \mathbf{u}_j = \mathbf{f}_j \,, \quad 0 \leqslant j \leqslant N \,,$$

where D_N^4 represents the fourth power of the Chebyshev differentiation matrix D_N, $\mathbf{u}_j = (u(x_0), u(x_1), \ldots, u(x_N))^T$ and $\mathbf{f}_j = (f(x_0), f(x_1), ..., f(x_N))^T$. The imposition of the Dirichlet boundary conditions, corresponds to removing the outer rows and columns of D_N^4 and reducing the problem to a linear system of $N-1$ equations. The interpolating polynomial is given by

$$p_{N-1}(x) = \sum_{j=1}^{N-1} u_j \phi_j(x)\,, \quad p_{N-1}(\pm 1) = 0\,, \tag{5.9}$$

where $\{\phi_j(x)\}$ is the Lagrangian basis set corresponding to the set of Chebyshev nodes $\{x_j\}$ on $[-1, 1]$, and the requirement that the equation $u_{xxxx}(x) = f(x)$ is satisfied on the interior $N-3$ grid points, implies

$$p_{N-1}^{''''}(x_k) = \sum_{j=1}^{N-1} u_j \phi_j^{''''}(x_k) = f(x_k)\,, \quad k = 2, \ldots, N-2\,. \tag{5.10}$$

The Dirichlet boundary conditions have already been considered, but the Neumann boundary conditions, $p_{N-1}^{'}(\pm 1) = 0$, must now be accommodated. These imply

$$p_{N-1}^{'}(1) = \sum_{j=1}^{N-1} u_j \phi_j^{'}(x_0) = 0\,, \qquad p_{N-1}^{'}(-1) = \sum_{j=1}^{N-1} u_j \phi_j^{'}(x_N) = 0\,. \tag{5.11}$$

Hence (5.10) and (5.11) form a linear system of $N-2$ equations, solvable for the unknown interior points $u_1, \ldots, u_{N-1}$. In matrix form we write

$$\tilde{D}_N^4 \mathbf{u}_j = \mathbf{f}_j\,, \quad 1 \leqslant j \leqslant N-1\,,$$

which is given explicitly, in matrix form, as

$$\begin{pmatrix} \phi_1^{'}(x_0) & \phi_2^{'}(x_0) & \ldots & \phi_{N-1}^{'}(x_0) \\ \phi_1^{''''}(x_2) & \phi_2^{''''}(x_2) & \ldots & \phi_{N-1}^{''''}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1^{''''}(x_{N-2}) & \phi_2^{''''}(x_{N-2}) & \ldots & \phi_{N-1}^{''''}(x_{N-2}) \\ \phi_1^{'}(x_N) & \phi_2^{'}(x_N) & \ldots & \phi_{N-1}^{'}(x_N) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ f_2 \\ \vdots \\ f_{N-2} \\ 0 \end{pmatrix}. \tag{5.12}$$

The entries of the matrix are numerically computed using the Chebyshev differentiation matrix D_N.

In summary, the interpolation process outlined above, corresponds to replacing the first and last equations of the linear system $\tilde{D}_N^4 \mathbf{u}_j = \mathbf{f}_j$, $1 \leqslant j \leqslant N-1$, with the interpolants, given by (5.11). In agreement with the notation of Section 5.1, if we let

$$\left.\begin{aligned} \phi_i^{'}(x_j) &= D_N(j,\, i)\,, \\ \phi_i^{''''}(x_k) &= \tilde{D}_N^4(k,\, i)\,, \end{aligned}\right\} \qquad \begin{aligned} &i = 1, 2, \ldots, N-1\,, \;\; j = 0 \text{ or } N\,, \\ &k = 2, 3, \ldots, N-2\,, \end{aligned}$$

then the matrix system, given by (5.12), can alternatively be written in the form

$$\begin{pmatrix} D_N(0,1) & D_N(0,2) & \dots & D_N(0,N-1) \\ \tilde{D}^4_N(2,1) & \tilde{D}^4_N(2,2) & \dots & \tilde{D}^4_N(2,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{D}^4_N(N-2,1) & \tilde{D}^4_N(N-2,2) & \dots & \tilde{D}^4_N(N-2,N-1) \\ D_N(N,1) & D_N(N,2) & \dots & D_N(N,N-1) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ f_2 \\ \vdots \\ f_{N-2} \\ 0 \end{pmatrix}.$$

Method Two

The final approach that we present for the imposition of clamped boundary conditions, is the approach used by Weideman [51]. This involves the construction of an interpolating polynomial $p_{N+2}(x)$ of degree $N+2$, satisfying the $N-1$ interpolating conditions

$$p_{N+2}(x_k) = u_k\,, \quad k = 1, \dots, N-1\,, \tag{5.13}$$

and the four boundary conditions

$$p_{N+2}(\pm 1) = 0\,, \quad p'_{N+2}(\pm 1) = 0\,. \tag{5.14}$$

The nodes $\{x_k\}$ are the interior Chebyshev points, with corresponding Lagrangian interpolating polynomials, given by

$$\phi_j(x) = (-1)^j \frac{1-x_j^2}{(N-1)^2} \frac{T'_N(x)}{x-x_j}\,, \quad j = 1, \dots, N-1\,,$$

where $T_N(x)$ is the Chebyshev polynomial of degree N. We define $\left\{\tilde{\phi}_j(x)\right\}$ by

$$\tilde{\phi}_j(x) = \left(\frac{1-x^2}{1-x_j^2}\right)^2 \phi_j(x)\,, \quad j = 1, \dots, N-1\,,$$

and conclude that the interpolating polynomial $p_{N+2}(x)$, satisfying all of the conditions, posed by (5.13) and (5.14), is given by

$$p_{N+2}(x) = \sum_{j=1}^{N-1} u_j \tilde{\phi}_j(x)\,.$$

The approximation to $u_{xxxx}(x) = f(x)$, is therefore given by

$$p''''_{N+2}(x_k) = \sum_{j=1}^{N-1} u_j \tilde{\phi}''''_j(x_k)\,, \quad k = 1\,, \dots, N-1\,,$$

and hence the approximation can be written as the following matrix $\times$ vector multiplication:

$$\begin{pmatrix} \tilde{\phi}_1''''(x_1) & \tilde{\phi}_2''''(x_1) & \ldots & \tilde{\phi}_{N-1}''''(x_1) \\ \tilde{\phi}_1''''(x_2) & \tilde{\phi}_2''''(x_2) & \ldots & \tilde{\phi}_{N-1}''''(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\phi}_1''''(x_{N-2}) & \tilde{\phi}_2''''(x_{N-2}) & \ldots & \tilde{\phi}_{N-1}''''(x_{N-2}) \\ \tilde{\phi}_1''''(x_{N-1}) & \tilde{\phi}_2''''(x_{N-1}) & \ldots & \tilde{\phi}_{N-1}''''(x_{N-1}) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-2} \\ f_{N-1} \end{pmatrix}.$$

The Imposition of Alternative Boundary Conditions

In this section we generalise the approach used by Weideman [51], and develop a numerical scheme for the solution of the general problem, given by

$$u_{xxxx}(x) = f(x)\,, \quad x \in [-1, 1]\,, \tag{5.15a}$$

$$u(\pm 1) = 0\,, \quad u_x^{(n)}(-1) = 0\,, \quad u_x^{(m)}(1) = 0\,, \quad n, m \in \{1, 2, 3\}\,, \tag{5.15b}$$

where $f(x)$ is a given smooth function.

The interpolating polynomial, and the requirement that $u_{xxxx}(x) = f(x)$ on the interior $N-3$ grid points, yields the expressions, given by (5.9) and (5.10) respectively. The four boundary conditions, given by (5.15b), imply that

$$p_{N-1}^{(n)}(-1) = \sum_{j=1}^{N-1} u_j \phi_j^{(n)}(x_N) = 0\,, \quad p_{N-1}^{(m)}(1) = \sum_{j=1}^{N-1} u_j \phi_j^{(m)}(x_0) = 0\,.$$

Therefore the resulting linear system of $N-1$ equations, can be written in matrix form as

$$\begin{pmatrix} \phi_1^{(m)}(x_0) & \phi_2^{(m)}(x_0) & \ldots & \phi_{N-1}^{(m)}(x_0) \\ \phi_1''''(x_2) & \phi_2''''(x_2) & \ldots & \phi_{N-1}''''(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1''''(x_{N-2}) & \phi_2''''(x_{N-2}) & \ldots & \phi_{N-1}''''(x_{N-2}) \\ \phi_1^{(n)}(x_N) & \phi_2^{(n)}(x_N) & \ldots & \phi_{N-1}^{(n)}(x_N) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ f_2 \\ \vdots \\ f_{N-2} \\ 0 \end{pmatrix}.$$

For consistency with the notation that has been used throughout, we let

$$\left.\begin{aligned} \phi_i^{\{m,n\}}(x_j) &= D_N^{m,n}(j, i)\,, \\ \phi_i''''(x_k) &= \tilde{D}_N^4(k,\, i)\,, \end{aligned}\right\} \qquad \begin{aligned} & i = 1, 2, \ldots, N-1\,, \;\; j = 0 \text{ or } N\,, \\ & k = 2, 3, \ldots, N-2\,, \end{aligned}$$

and give the alternative form of the matrix system:

$$\begin{pmatrix} D_N^m(0,1) & D_N^m(0,2) & \dots & D_N^m(0,N-1) \\ \tilde{D}_N^4(2,1) & \tilde{D}_N^4(2,2) & \dots & \tilde{D}_N^4(2,N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{D}_N^4(N-2,1) & \tilde{D}_N^4(N-2,2) & \dots & \tilde{D}_N^4(N-2,N-1) \\ D_N^n(N,1) & D_N^n(N,2) & \dots & D_N^n(N,N-1) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} 0 \\ f_2 \\ \vdots \\ f_{N-2} \\ 0 \end{pmatrix}.$$

The approximation to the solution on the interior grid points is therefore easily obtainable from inverting the matrix.

This approach can be extended to accommodate more complicated boundary conditions. As a final example, we consider problem (5.15a) with the boundary conditions

$$\begin{aligned} u(1) + u_x(1) &= a_+\,, & u(-1) + u_x(-1) &= a_-\,, \\ u_{xx}(1) + u_{xxx}(1) &= b_+\,, & u_{xx}(-1) + u_{xxx}(-1) &= b_-\,, \end{aligned} \tag{5.16}$$

for some given constants $a_\pm$ and $b_\pm$. The boundary conditions are imposed numerically by taking the system

$$D_N^4 \mathbf{u}_j = \mathbf{f}_j\,, \quad 0 \leqslant j \leqslant N\,, \tag{5.17}$$

where $\mathbf{u}_j = (u(x_0), u(x_1), \dots, u(x_N))^T$ and $\mathbf{f}_j = (f(x_0), f(x_1), \dots, f(x_N))^T$, and replacing the first two and last two rows of D_N^4 by appropriate linear combinations of D_N^0, D_N, D_N^2 and D_N^3, as indicated by the boundary conditions. Explicitly, the system of equations that results is given as follows:

$$\begin{pmatrix} d(0,0) & d(0,1) & \dots & d(0,N) \\ d(1,0) & d(1,1) & \dots & d(1,N) \\ D_N^4(2,0) & D_N^4(2,1) & \dots & D_N^4(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ D_N^4(N-2,0) & D_N^4(N-2,1) & \dots & D_N^4(N-2,N) \\ d(N-1,0) & d(N-1,1) & \dots & d(N-1,N) \\ d(N,0) & d(N,1) & \dots & d(N,N) \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \\ u_N \end{pmatrix} = \begin{pmatrix} a_+ \\ b_+ \\ f_2 \\ \vdots \\ f_{N-2} \\ b_- \\ a_- \end{pmatrix},$$

where

$$\left.\begin{aligned} d(0,i) &= D_N^0(0,i) + D_N(0,i)\,, \\ d(1,i) &= D_N^2(0,i) + D_N^3(0,i)\,, \\ d(N-1,i) &= D_N^2(N,i) + D_N^3(N,i)\,, \\ d(N,i) &= D_N^0(N,i) + D_N(N,i)\,, \end{aligned}\right\} \quad i = 0, 1, \ldots, N\,.$$

Remark 5.2.3. This approach can be extended to any linear operator acting on $u(x)$. For example, if the problem was given by $u_{xxxx}(x) + u_{xx}(x) = f(x)$, with the boundary conditions given by (5.16), then the initial system, given by (5.17), would be replaced by

$$\left(D_N^4 + D_N^2\right)\mathbf{u}_j = \mathbf{f}_j\,, \quad 0 \leqslant j \leqslant N\,,$$

and the remainder of the method for imposing the boundary conditions is unchanged.

Remark 5.2.4. The general approach of the numerical schemes for imposing the boundary conditions, given by (5.16), can be adapted to accommodate other, more complicated, boundary conditions.

5.3 Numerical Transforms

We conclude this chapter by discussing an alternative approach for solving boundary value problems, involving the implicit imposition of the boundary conditions.

The use of numerical transforms will be discussed in further detail in Chapter 6, in the context of periodic problems. Indeed, the use of the discrete Fourier transform, which can be implemented using the fast Fourier transform algorithm, offers a fast and efficient approach for solving such problems. However, our attention now turns to the problem of solving linear boundary value problems by using the appropriate transform.

The most common boundary conditions are when either the solution vanishes, or when the derivatives vanish, at the two endpoints of the interval. i.e., when the boundary conditions are either of Dirichlet or Neumann type. In these cases, the two transforms that are employed, for the implicit imposition of the boundary conditions, are the discrete sine transform and the discrete cosine transform respectively, and these will be the focus for the remainder of the chapter.

5.3.1 The Sine Transform and the Cosine Transform

Let us begin with the continuous case. Recall the Fourier transform pair of a function $u(x)$, $x \in \mathbb{R}$, defined in Section 1.2.1 by

$$\begin{aligned} \hat{u}(k) &= Fu(x) = \int_{-\infty}^{\infty} e^{-ikx} u(x)\, \mathrm{d}x\,, \quad x\,, k \in \mathbb{R}\,, \\ u(x) &= F^{-1}\hat{u}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{u}(k)\, \mathrm{d}k\,. \end{aligned}$$

In the case where $u(x)$ is either an odd or an even function, the Fourier transform pair reduces to the sine transform pair or the cosine transform pair respectively. Furthermore, if $u(x)$ is either an odd or an even function, then $\hat{u}(k)$ will be either odd or even too:

i.) Fourier Sine Transform: If $u(x)$ is an odd function, i.e., $u(x) = -u(-x)$, $x \in \mathbb{R}$, then the Fourier sine transform $\hat{u}_s(k)$, and the inverse Fourier sine transform $u_s(x)$, are defined by

$$\hat{u}_s(k) = Fu_s(x) = \int_{0}^{\infty} u_s(x)\, \sin(kx)\, \mathrm{d}x\,, \quad x\,, k \in \mathbb{R}\,, \tag{5.18a}$$

$$u_s(x) = F^{-1}\hat{u}_s(k) = \frac{2}{\pi} \int_{0}^{\infty} \hat{u}_s(k)\, \sin(kx)\, \mathrm{d}k\,, \tag{5.18b}$$

where

$$u(x) = \frac{1}{\sqrt{2}}\, u_s(x)\,, \quad \hat{u}(k) = -\sqrt{2}\, i \hat{u}_s(k)\,.$$

ii.) Fourier Cosine Transform: If $u(x)$ is an even function, i.e., $u(x) = u(-x)$, $x \in \mathbb{R}$, then the Fourier cosine transform $\hat{u}_c(k)$, and the inverse Fourier cosine transform $u_c(x)$, are defined by

$$\hat{u}_c(k) = Fu_c(x) = \int_{0}^{\infty} u_c(x)\, \cos(kx)\, \mathrm{d}x\,, \quad x\,, k \in \mathbb{R}\,, \tag{5.19a}$$

$$u_c(x) = F^{-1}\hat{u}_c(k) = \frac{2}{\pi} \int_{0}^{\infty} \hat{u}_c(k)\, \cos(kx)\, \mathrm{d}k\,, \tag{5.19b}$$

where

$$u(x) = \frac{1}{\sqrt{2}}\, u_c(x)\,, \qquad \hat{u}(k) = \sqrt{2}\, \hat{u}_c(k)\,.$$

The formulae used to obtain the Fourier sine/cosine transform of a derivative, are achieved by integration by parts of equations (5.18a) and (5.19a) respectively. For example, the first derivatives are given by

$$\widehat{u'_s}(k) = -k\hat{u}_c(k)\,, \qquad \widehat{u'_c}(k) = \hat{u}_s(k) - u(0)\,,$$

and the second derivatives are given by

$$\widehat{u''_s}(k) = -k^2\hat{u}_s(k)\,, \qquad \widehat{u''_c}(k) = -k^2\hat{u}_c(k)\,.$$

Higher order derivatives are calculated analogously.

Let us now consider the discrete case, and assume that we are given a function $u(x)$, defined on the interval $[0, \pi]$. We consider the discrete sine transform and the discrete cosine transform separately:

i.) Discrete Sine Transform: *Matlab* has two built in functions for implementing the discrete sine transform and the inverse discrete sine transform, that we shall refer to as the *dst* and *idst* respectively. The formula for the *dst* is given by

$$F_s u_j = \hat{u}_k = \sum_{j=1}^{N} u_j \sin\left(\frac{\pi k j}{N+1}\right)\,, \quad k = 1, \ldots, N\,.$$

i.e., $F_s u_j = \hat{u}_k$ computes the discrete sine transform of the function $u(x)$.

It may be convenient to think of the discrete sine transform as a matrix $\times$ vector multiplication. The discrete sine transform of the vector $u(x) = (u_1, u_2, \ldots, u_N)^T$, is given by $\hat{u}_k = (\hat{u}_{k_1}, \hat{u}_{k_2}, \ldots, \hat{u}_{k_N})^T$, and can be performed according to the following system:

$$\begin{pmatrix} \hat{u}_{k_1} \\ \hat{u}_{k_2} \\ \vdots \\ \hat{u}_{k_N} \end{pmatrix} = \begin{pmatrix} \sin\left(\frac{\pi}{N+1}\right) & \sin\left(\frac{2\pi}{N+1}\right) & \ldots & \sin\left(\frac{N\pi}{N+1}\right) \\ \sin\left(\frac{2\pi}{N+1}\right) & \sin\left(\frac{4\pi}{N+1}\right) & \ldots & \sin\left(\frac{2N\pi}{N+1}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sin\left(\frac{N\pi}{N+1}\right) & \sin\left(\frac{2N\pi}{N+1}\right) & \ldots & \sin\left(\frac{N^2\pi}{N+1}\right) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}.$$

The *idst* function, that is used to transform back to physical space, is given by

$$F_s^{-1}\hat{u}_k = u_j = \frac{2}{N+1}\sum_{k=1}^{N} \hat{u}_k \sin\left(\frac{\pi j k}{N+1}\right)\,, \quad j = 1, \ldots, N\,,$$

and is given explicitly by the following matrix $\times$ vector multiplication:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \frac{2}{N+1} \begin{pmatrix} \sin\left(\frac{\pi}{N+1}\right) & \sin\left(\frac{2\pi}{N+1}\right) & \ldots & \sin\left(\frac{N\pi}{N+1}\right) \\ \sin\left(\frac{2\pi}{N+1}\right) & \sin\left(\frac{4\pi}{N+1}\right) & \ldots & \sin\left(\frac{2N\pi}{N+1}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sin\left(\frac{N\pi}{N+1}\right) & \sin\left(\frac{2N\pi}{N+1}\right) & \ldots & \sin\left(\frac{N^2\pi}{N+1}\right) \end{pmatrix} \begin{pmatrix} \hat{u}_{k_1} \\ \hat{u}_{k_2} \\ \vdots \\ \hat{u}_{k_N} \end{pmatrix}.$$

Hence, the second derivative of a function, satisfying Dirichlet boundary conditions, is given by

$$u_j'' \approxeq F_s^{-1}\left(-k^2 F_s u_j\right)\,, \quad j = 1, \ldots, N\,. \tag{5.20}$$

Remark 5.3.1. *Matlab* stores the wavenumbers in the order $1, \ldots, N/2+1, -N/2+2, \ldots, 0$.

ii.) Discrete Cosine Transform: *Matlab* has built in functions for implementing the discrete cosine transform and the inverse discrete cosine transform, that we shall refer to as the *dct* and *idct* respectively. The formula for the *dct* is given by

$$F_c u_j = \hat{u}_k = y_k \sum_{j=1}^{N} u_j \cos\left(\frac{\pi(2j-1)(k-1)}{2N}\right)\,, \quad k = 1, \ldots, N\,,$$

where

$$y_k = \begin{cases} \sqrt{\frac{1}{N}}\,, & k = 1\,, \\ \sqrt{\frac{2}{N}}\,, & 2 \leqslant k \leqslant N\,. \end{cases} \tag{5.21}$$

i.e., $F_c u_j = \hat{u}_k$ computes the discrete cosine transform of the function $u(x)$. This can be written in matrix $\times$ vector form as follows:

$$\begin{pmatrix} \hat{u}_{k_1} \\ \hat{u}_{k_2} \\ \hat{u}_{k_3} \\ \vdots \\ \hat{u}_{k_N} \end{pmatrix} = \sqrt{\frac{2}{N}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \cos\left(\frac{\pi}{2N}\right) & \cos\left(\frac{3\pi}{2N}\right) & \cdots & \cos\left(\frac{\pi(2N-1)}{2N}\right) \\ \cos\left(\frac{2\pi}{2N}\right) & \cos\left(\frac{6\pi}{2N}\right) & \cdots & \cos\left(\frac{2\pi(2N-1)}{2N}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \cos\left(\frac{\pi(N-1)}{2N}\right) & \cos\left(\frac{3\pi(N-1)}{2N}\right) & \cdots & \cos\left(\frac{\pi(2N-1)(N-1)}{2N}\right) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{pmatrix}.$$

The *idct* function, that is used to transform back to physical space, is given by

$$F_c^{-1}\hat{u}_k = u_j = \sum_{k=1}^{N} y_k\, \hat{u}_k \cos\left(\frac{\pi(2j-1)(k-1)}{2N}\right)\,, \quad j = 1, \ldots, N\,,$$

where y_k is given by (5.21), and is given explicitly by the following matrix $\times$ vector multiplication:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{pmatrix} = \frac{2}{\sqrt{N/2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \cos\left(\frac{\pi}{2N}\right) & \cdots & \cos\left(\frac{\pi(N-1)}{2N}\right) \\ \frac{1}{\sqrt{2}} & \cos\left(\frac{3\pi}{2N}\right) & \cdots & \cos\left(\frac{3\pi(N-1)}{2N}\right) \\ \frac{1}{\sqrt{2}} & \cos\left(\frac{5\pi}{2N}\right) & \cdots & \cos\left(\frac{5\pi(N-1)}{2N}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2}} & \cos\left(\frac{\pi(2N-1)}{2N}\right) & \cdots & \cos\left(\frac{\pi(2N-1)(N-1)}{2N}\right) \end{pmatrix} \begin{pmatrix} \hat{u}_{k_1} \\ \hat{u}_{k_2} \\ \hat{u}_{k_3} \\ \vdots \\ \hat{u}_{k_N} \end{pmatrix}.$$

Hence the second derivative of a function, satisfying Neumann boundary conditions, is given by

$$u_j'' \approxeq F_c^{-1}\left(-k^2 F_c u_j\right)\,, \quad j = 1, \dots, N\,.$$

Remark 5.3.2. *Matlab* stores the wavenumbers in the order $0, \dots, N/2, -N/2+1, \dots, -1$.

Example: We conclude this chapter with a simple example to illustrate the use of the discrete sine transform algorithm and consider the following fourth order beam equation with Dirichlet boundary conditions:

$$u_{tt}(x,t) + u_{xxxx}(x,t) = 0\,, \quad u(x,0) = u_0(x)\,, \quad t > 0\,, \quad x \in [0,\pi]\,, \tag{5.22a}$$

$$u(0,t) = 0\,, \quad u(L,t) = 0\,, \quad u_{xx}(0,t) = 0\,, \quad u_{xx}(L,t) = 0\,, \tag{5.22b}$$

for some given smooth function $u_0(x)$.

Remark 5.3.3. Whilst (5.22a) is not of the form of the fourth order PDEs studied in Chapter 4, it has a physical application to the real life situation of a vibrating beam with stationary end points. Furthermore, the focus of the example is on the approximation of the fourth order spatial derivative and the imposition of the boundary conditions, and the form of both in (5.22) is the same for the analogous PDE with a first order time derivative.

For the time derivative we use a simple leap-frog formula, and we approximate the fourth order spatial derivative using the discrete sine transform formula, analogous to expression (5.20). Hence

$$u_{xxxx}(x,t) \approxeq F_s^{-1}\left(k^4 F_s u(x,t)\right)\,, \quad j = 1, \dots, N\,,$$

and therefore

$$u(x, t+\Delta t) = 2u(x,t) - u(x, t-\Delta t) - (\Delta t)^2 F_s^{-1}\left(k^4 F_s u(x,t)\right)\,.$$

The domain was discretised by 256 grid points and the program was run with $\Delta t = 0.00001$ and the initial solution

$$u(x,0) = e^{-100\left(x-\frac{\pi}{2}\right)^2}\,, \tag{5.23}$$

and the numerical output is given in Figure 5.1.

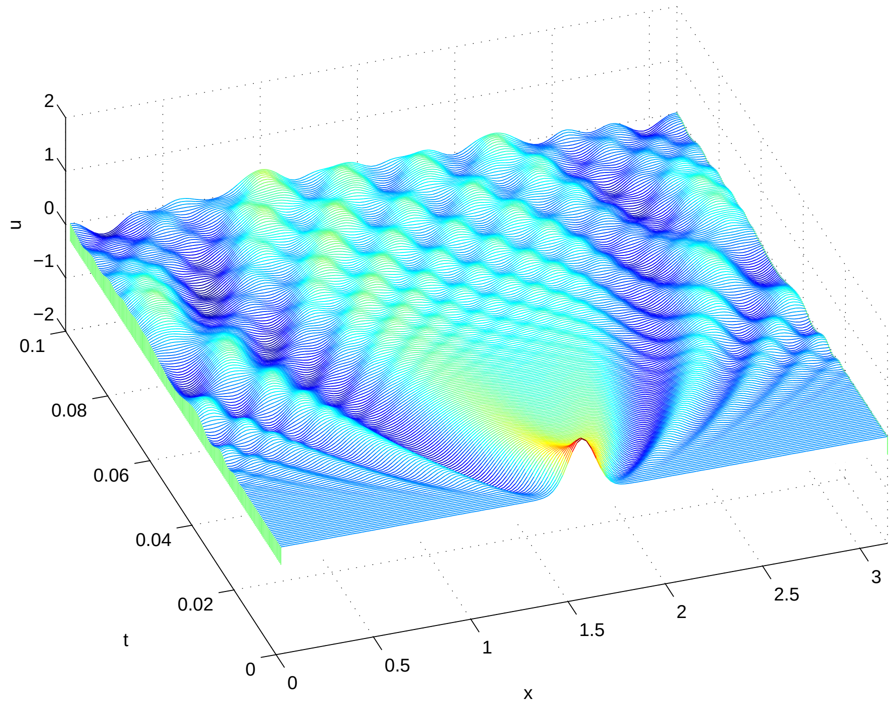


Figure 5.1: The PDE $u_{tt}(x,t) + u_{xxxx}(x,t) = 0$ with initial condition given by (5.23), solved using 256 grid points, $\Delta t = 0.00001$ and the discrete sine transform for the implicit imposition of the Dirichlet boundary conditions.

Remark 5.3.4. Numerical transforms offer an alternative approach for the implicit imposition of boundary conditions and the discrete Fourier transform will be used in the next chapter to model the periodic KdV equation. The concept of developing this approach for the imposition of more complicated boundary conditions, for example Robin or coupled boundary conditions, is left as an open problem and a topic for future work.

Chapter 6

Nonlinear Numerical Results

In this chapter we consider the KdV equation posed on a finite interval. To be precise, we consider the following problem

$$u_t + uu_x + u_{xxx} = 0\,, \quad t > 0\,, \quad x \in [0, L]\,, \tag{6.1a}$$

$$u(x,0) = u_0(x)\,, \quad x \in [0, L]\,, \tag{6.1b}$$

where $u_0(x)$ is a given smooth function, L is a positive constant, and it is assumed that appropriate boundary conditions are imposed, see Theorem 2.1.1. Equation (6.1) is not quite the KdV equation, as the linear term $u_x(x,t)$ is missing. However, (6.1) is equivalent to the KdV equation on $\mathbb{R}$, hence for our periodic examples we can easily add the consideration of the extra term in all computations.

The first difficulty is caused by the lack of symmetry, characteristic to any third order boundary value problem, and the second by the nonlinearity. The numerical schemes we use are all spectral. In particular, we shall use the FFT described in Section 1.2.1, as well as the Toeplitz differentiation matrix approach, see Section 1.2.1. To set the stage, we begin by addressing the periodic problem, and consider the evolution of an initial solitary wave of the form

$$u(x,0) = 3A^2 \mathrm{sech}^2\left(\frac{1}{2}Ax\right)\,, \tag{6.2}$$

where A is a constant that determines both the amplitude and speed. All schemes are based on either the one-step method of Fornberg and Whitham [25] or the split-step method of Tappert [45]. We use them to study the interaction of solitary solutions. We also consider the issue of numerical conservation of energy. This is satisfied to a very good accuracy, when the problem is periodic.

We then consider the non-periodic boundary value problem. We first discuss the well known issue of the necessity for unevenly spaced grid points in the construction of our spectral methods. All the non-periodic problems considered are solved using Chebyshev differentiation matrices and the split-step method of Tappert [45]. The focus of the non-periodic results will be on the implementation of the boundary conditions, in view of studying their effect on the evolution of the solitary solutions of the form (6.2).

6.1 The Periodic Problem

In this section, we consider two numerical methods for solving the third order periodic problem for the KdV equation (6.1). The first scheme is a one step method while the second uses a split step approach.

6.1.1 Method One: The One-Step Fourier Method by Fornberg and Whitham

Fornberg and Whitham [25] consider the nonlinear wave equations of the form

$$u_t + f(u)u_x + Lu = 0\,,$$

where $f(u)$ is a given function, and L is a linear operator with constant coefficients. They present an efficient numerical method for solving such nonlinear wave equations using a simple leap-frog scheme in time, along with a Fourier transform treatment of the space dependence. The equation that will be the focus of our attentions is equation (6.1), hence $f(u) = u(x,t)$ and $L = \frac{\partial^3}{\partial x^3}$.

The spatial period is $[0, 2\pi]$ discretised by N points. Using a leap-frog scheme in time, equation (6.1) is approximated by

$$\frac{u(x,t+\Delta t) - u(x,t-\Delta t)}{2\Delta t} + u(x,t)F^{-1}\left(ikFu(x,t)\right) - F^{-1}\left(ik^3Fu(x,t)\right) = 0\,,$$

where $F^{-1}\left(ikFu(x,t)\right)$ and $-F^{-1}\left(ik^3Fu(x,t)\right)$, according to Section 1.2.1, are the Fourier approximations to $u_x(x,t)$ and $u_{xxx}(x,t)$ respectively. Rearranging yields

$$u(x,t+\Delta t) - u(x,t-\Delta t) + 2i\,u(x,t)\,\Delta t\,F^{-1}\left(kFu(x,t)\right) - 2i\,\Delta t\,F^{-1}\left(k^3Fu(x,t)\right) = 0\,. \tag{6.3}$$

The accuracy of all consistent difference approximations to a differential equation decreases rapidly as the wavenumbers increase. This is particularly true for the leap-frog scheme in time. However, the accuracy of the scheme for high wavenumbers can be improved by modification of the last term to give

$$u(x,t+\Delta t) - u(x,t-\Delta t) + 2i\,u(x,t)\,\Delta t\,F^{-1}\left(kFu(x,t)\right) \\ - 2i\,F^{-1}\big(\sin(k^3\Delta t)Fu(x,t)\big) = 0\,, \tag{6.4}$$

noting that the two methods are identical in the limit Δt decreasing to zero. The computational cost for both equations, (6.3) and (6.4), is three fast Fourier transforms per time step, and since the scheme is second order accurate in time, when Δt is halved the overall error due to the time discretisation can be expected to decrease by a factor of four.

For low wavenumbers k, the difference between (6.3) and (6.4) is only $O(k^3)$. If equation (6.3) is considered for high wavenumbers k, then the term approximating u_{xxx} dominates uu_x, and equation (6.3) is essentially

$$u(x,t+\Delta t) - u(x,t-\Delta t) - 2i\,\Delta t\,F^{-1}\big(k^3Fu(x,t)\big) = 0\,,$$

which approximates the linear equation

$$u_t(x,t) + u_{xxx}(x,t) = 0\,. \tag{6.5}$$

In comparison equation (6.4) becomes

$$u(x,t+\Delta t) - u(x,t-\Delta t) - 2i\,F^{-1}\big(\sin(k^3\Delta t)Fu(x,t)\big) = 0\,,$$

for large wavenumbers, and is exactly satisfied by any solution of the linear equation (6.5). To see this, consider the solution $u(x,t) = e^{ikx+ik^3t}$ of equation (6.5). It follows that

$$u(x,t+\Delta t) = e^{ik^3\Delta t}u(x,t)\,, \quad u(x,t-\Delta t) = e^{-ik^3\Delta t}u(x,t)\,,$$

and hence

$$u(x,t+\Delta t) - u(x,t-\Delta t) - 2i\,\sin(k^3\Delta t)\,u(x,t) = 0\,.$$

Furthermore, the linearised stability condition for (6.4), which is discussed in detail in [25], is given by

$$\frac{\Delta t}{\Delta x^3} < \frac{3}{2\pi^2} \approxeq 0.1520\,.$$

Hence, for the spatial domain $[0, 2\pi]$ it is required that

$$\Delta t < \left(\frac{2\pi}{N}\right)^3 0.1520\,.$$

This method is used to study the interaction of solitary waves, and we begin by modelling the evolution of a single soliton. The initial solution takes the form

$$u(x,0) = 3A^2\text{sech}^2\left(\frac{1}{2}A(x-\pi+2)\right)\,, \tag{6.6}$$

where the parameter A determines both the amplitude and speed. The program was run using 128 grid points with $\Delta t = 0.00001$ and $A = 15$.

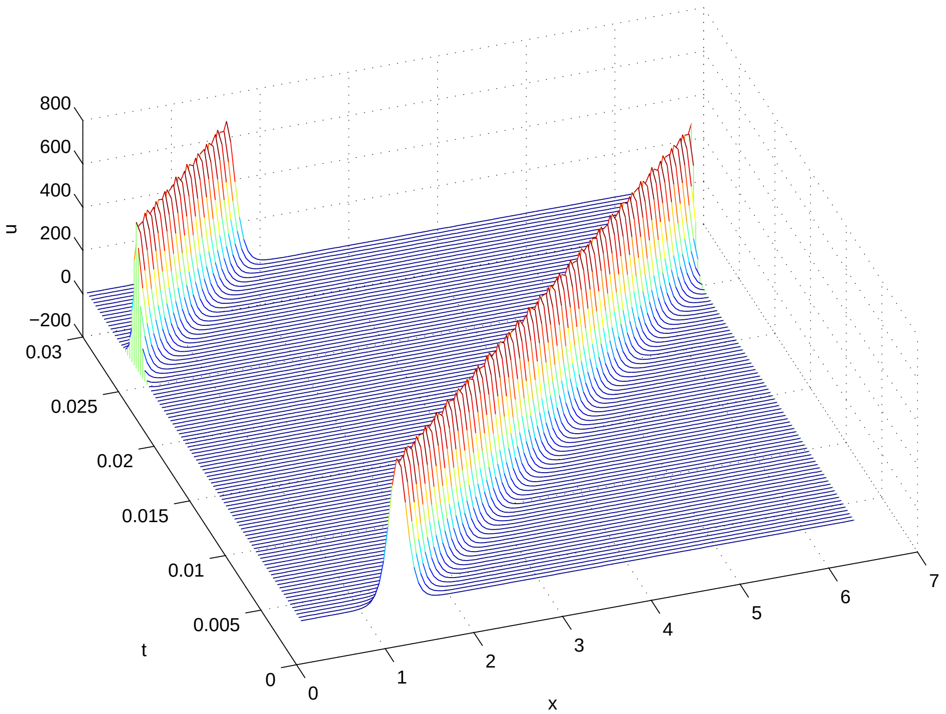


Figure 6.1: Fornberg and Whitham's one-step method for the periodic KdV equation with the single soliton initial solution (6.6), solved using 128 grid points, $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

The numerical output, given by Figure 6.1, confirms the known behaviour of the soliton. On a periodic grid, the soliton travels without losing energy, collides with the boundary, emerges from the opposing boundary and continues to travel in the same direction, with the same speed. The stability properties of the solution $u(x,t)$ can be analysed by examining the Euclidean norm

$$||u(x,t)||_2^2 = \int_0^L |u(x,t)|^2 \, \mathrm{d}x \, ,$$

where $x \in [0,L]$. We begin with equation (6.1), multiply throughout by $u(x,t)$ and integrate with respect to x over the domain to give

$$\int_0^L \left(uu_t + u^2 u_x + uu_{xxx}\right) \, \mathrm{d}x = 0 \, .$$

Integration by parts yields the following

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u(x,t)||_2^2 = \left[\frac{1}{2}u_x^2 - uu_{xx} - \frac{1}{3}u^3\right]_0^L \, . \tag{6.7}$$

It follows trivially that if the problem is periodic, then the following conservation of energy law must be satisfied:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |u(x,t)|^2 \, \mathrm{d}x = 0 \, . \tag{6.8}$$

We now consider the numerical form

$$\sum_{i=1}^{N} |u_i(x,t)|^2 = \sum_{i=1}^{N} |u_i(x,t+\Delta t)|^2 \, , \tag{6.9}$$

which is the discrete analogue of (6.8). Hence numerically the Euclidean norm, for each solution $u(x,t)$, is computed using the formula

$$||u(x,t)||_2^2 = \sum_{i=1}^{N} |u_i(x,t)|^2 \, ,$$

and the normalised Euclidean norms are calculated using the formula

$$||u(x,t)||_2^2 = \frac{1}{||u(x,0)||_2^2}\sum_{i=1}^{N} |u_i(x,t)|^2 \, . \tag{6.10}$$

The same program is used to demonstrate the clean interaction of two waves (Figure 6.2) and repeated for three waves (Figure 6.3). Since these problems are also periodic, the energy is conserved and therefore in both cases the solitons travel without losing energy. The initial solutions take the form

$$u(x,0) = 3A^2 \mathrm{sech}^2\left(\frac{1}{2}A(x-\pi+2)\right) + 3B^2 \mathrm{sech}^2\left(\frac{1}{2}B(x-\pi)\right) \, , \tag{6.11}$$

where $B = 10$ for the double soliton initial solution, and

$$u(x,0) \;=\; 3A^2\text{sech}^2\left(\frac{1}{2}A(x-\pi+2)\right)+3B^2\text{sech}^2\left(\frac{1}{2}B(x-\pi)\right)$$
$$+3C^2\text{sech}^2\left(\frac{1}{2}C(x-\pi-1)\right)\,, \tag{6.12}$$

where $C = 8$ for the three soliton initial solution. All other parameters remained as for the single soliton example.

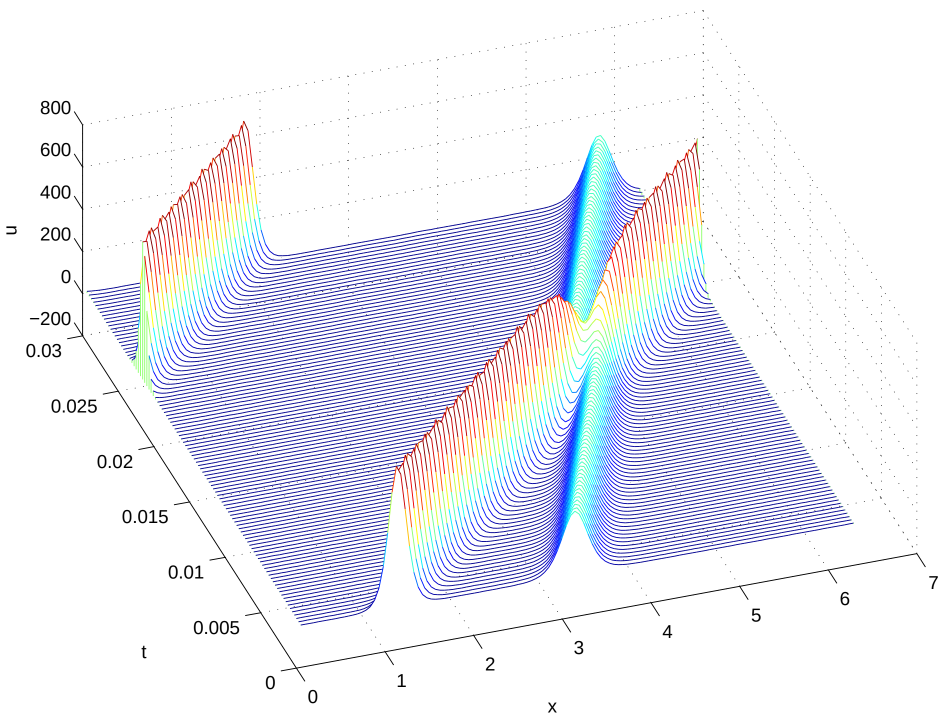


Figure 6.2: Fornberg and Whitham's one-step method for the periodic KdV equation with the double soliton initial solution (6.11), solved using 128 grid points, $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

The programs return favourable results, showing the clean interaction of solitary waves. It can be seen in Figure 6.2 that for the double soliton, since the wave speed is proportional to the amplitude, the wave starting furthest left has greater amplitude than the smaller soliton to its right and therefore travels faster, catches up with the smaller soliton, passes thorough its path, and continues to travel in the same direction and at the same speed as it was travelling before the collision. The interaction is clean and the waves propagate without any distortion. The interaction of the three solitons, demonstrated in Figure 6.3, follows analogously to the interaction of the two soliton example.

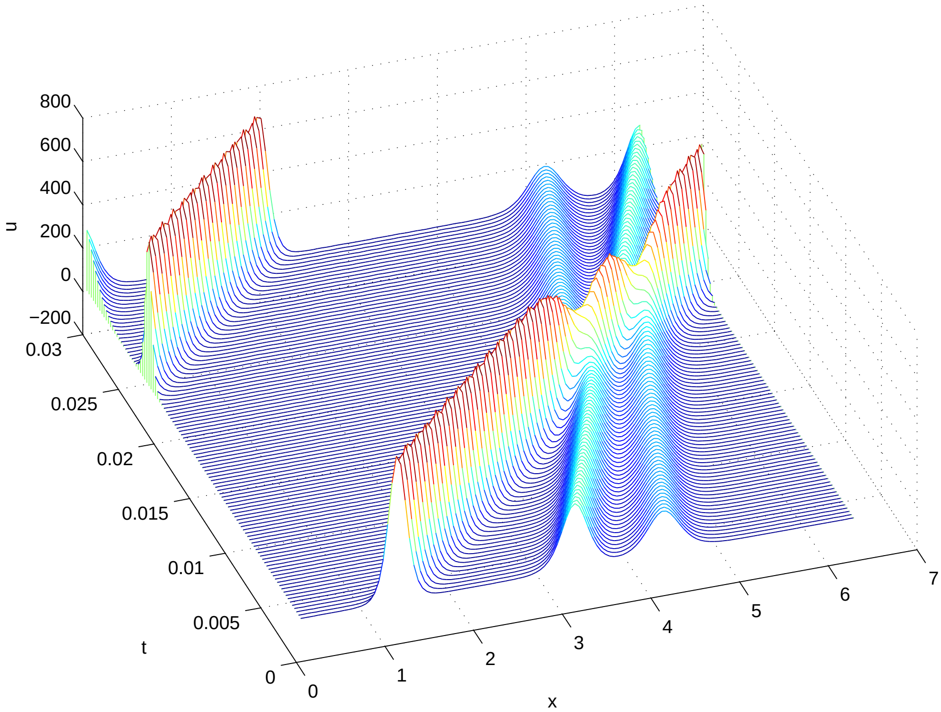


Figure 6.3: Fornberg and Whitham's one-step method for the periodic KdV equation with the triple soliton initial solution (6.12), solved using 128 grid points, $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

Table 6.1, gives the normalised numerical Euclidean norms, calculated using equation (6.10), at the initial solution and thereafter in time intervals of 0.003 up to the final solution at time $t = 0.030$. The values obtained agree favourably with the conservation of energy law, given by (6.9).

For completeness, the previous three examples, for the single, double and triple soliton initial solutions, were repeated using the same one-step approach to numerically model equation (6.1) but rather than using the Fourier approach to approximate the derivatives, the spectral derivatives were calculated using the Toeplitz differentiation matrices, (see (1.11)). Using a leap-frog scheme in time, the approximation to equation (6.1) is given by

$$u(x, t + \Delta t) = u(x, t - \Delta t) - 2\Delta t\, u(x, t) D_N u(x, t) - 2\Delta t\, D_N^3 u(x, t)\,,$$

where $D_N u(x, t)$ and $D_N^3 u(x, t)$ are the spectral approximations to $u_x(x, t)$ and $u_{xxx}(x, t)$ respectively.

This approach was successfully applied with the single soliton solution, given by (6.6), used in method one and also to demonstrate the interaction of two solitons, given by

t	Single Soliton	Double Soliton	Triple Soliton
0	1.0000	1.0000	1.0000
0.003	1.0037	1.0030	1.0027
0.006	1.0043	1.0035	1.0032
0.009	1.0011	1.0008	1.0006
0.012	0.9972	0.9981	0.9984
0.015	0.9962	1.0009	0.9998
0.018	0.9985	0.9986	1.0017
0.021	1.0019	0.9982	0.9998
0.024	1.0035	0.9994	0.9975
0.027	1.0024	1.0007	0.9980
0.030	0.9996	1.0015	0.9998

Table 6.1: The normalised Euclidean norms of the numerical results, given by Figures 6.1, 6.2 and Figure 6.3, for the periodic KdV equation, solved using Fornberg and Whitham's one-step method with $N = 128$, $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

(6.11), and three solitons, given by (6.12). In all cases the spatial domain was discretised by 128 grid points and $\Delta t = 0.000001$ and the results are shown in Figures 6.4, 6.5 and Figure 6.6.

The normalised numerical Euclidean norms for the results given by Figures 6.4, 6.5 and Figure 6.6 are presented in Table 6.2, for the initial solution and every solution thereafter in time intervals of 0.003, up to the final solution at time $t = 0.030$.

If we compare the results from Table 6.2 to Table 6.1, we see that the normalised numerical Euclidean norms calculated at comparative time intervals, suggest that the Toeplitz differentiation matrix approach yields more accurate results.

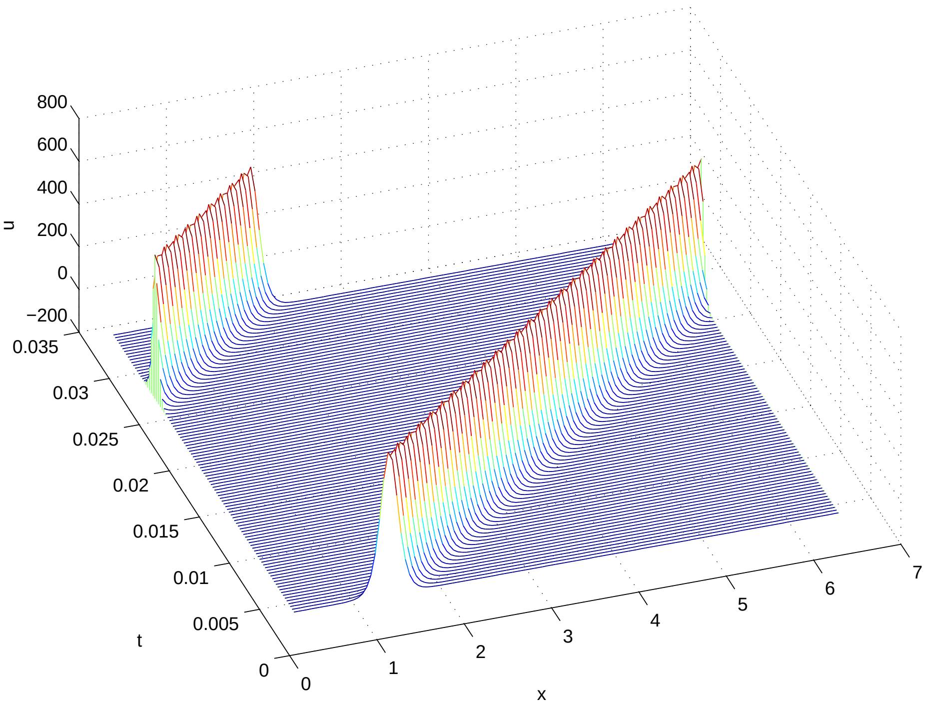


Figure 6.4: Fornberg and Whitham's one-step method for the periodic KdV equation with the single soliton initial solution (6.6), solved using 128 grid points, $\Delta t = 0.000001$ and Toeplitz differentiation matrices to approximate the spatial derivatives.

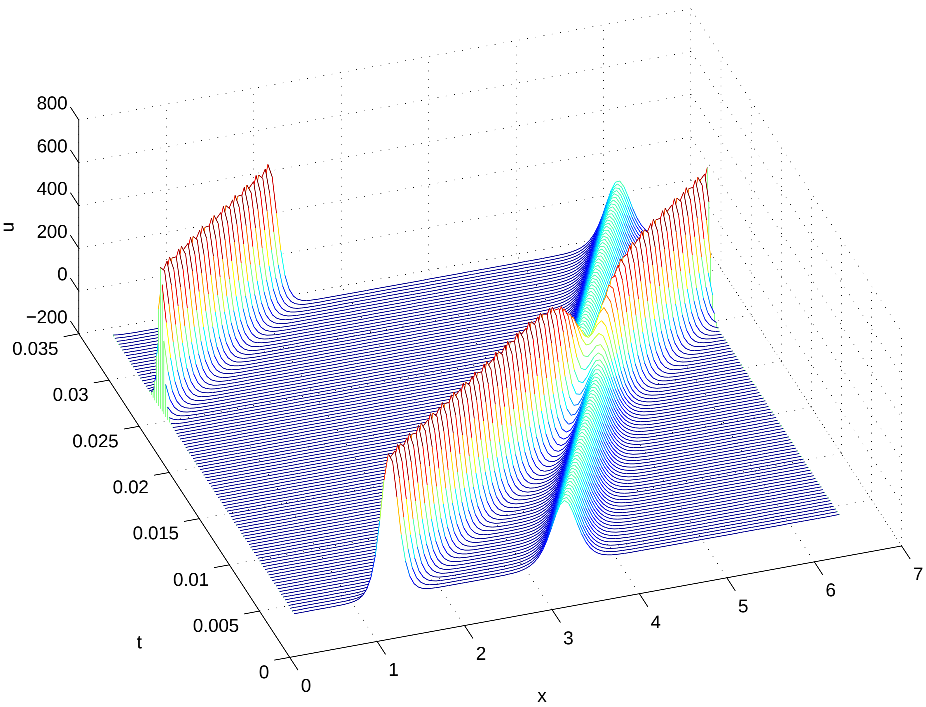


Figure 6.5: Fornberg and Whitham's one-step method for the periodic KdV equation with the double soliton initial solution (6.11), solved using 128 grid points, $\Delta t = 0.000001$ and Toeplitz differentiation matrices to approximate the spatial derivatives.

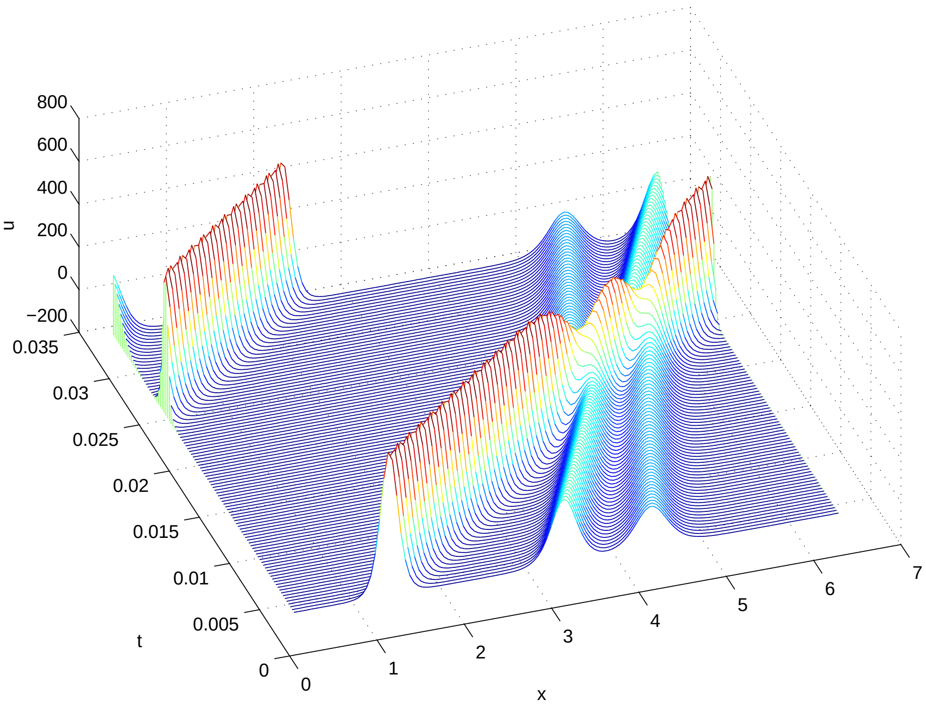


Figure 6.6: Fornberg and Whitham's one-step method for the periodic KdV equation with the triple soliton initial solution (6.12), solved using 128 grid points, $\Delta t = 0.000001$ and Toeplitz differentiation matrices to approximate the spatial derivatives.

t	Single Soliton	Double Soliton	Triple Soliton
0	1.0000	1.0000	1.0000
0.003	1.0004	1.0003	1.0003
0.006	1.0004	1.0003	1.0003
0.009	1.0000	1.0000	1.0000
0.012	0.9996	0.9998	0.9998
0.015	0.9996	1.0000	0.9999
0.018	1.0000	0.9998	1.0001
0.021	1.0003	0.9998	0.9999
0.024	1.0003	1.0001	0.9997
0.027	1.0000	1.0002	0.9999
0.030	0.9997	1.0001	1.0001

Table 6.2: The normalised Euclidean norms of the numerical results, given by Figures 6.4, 6.5 and Figure 6.6, for the periodic KdV equation, solved using Fornberg and Whitham's one-step method with $N = 128$, $\Delta t = 0.000001$ and Toeplitz differentiation matrices to approximate the spatial derivatives.

6.1.2 Method Two: The Split-Step Fourier Method by Tappert

We now present a scheme that discretises the time variable using a split-step Fourier method. This method is due to Tappert. We follow the presentation given by Taha and Ablowitz [45]. The idea is to split the problem into the linear and nonlinear parts and alternate between the steps. The first step advances the solution half of the time step using only the nonlinear term, and the second half advances the new solution the final half of the time-step using only the linear term. The scheme can be considered as successfully solving the equations

$$u_t + uu_x = 0\,, \quad u_t + u_{xxx} = 0\,,$$

where the solution of the former equation is used as the initial condition for the latter. The advantage of this method is the fact that it avoids solving a nonlinear system of equations at each time level, and the fact that the linear step can be solved exactly.

We begin by considering the problem with spatial domain $[0, 2\pi]$ and periodic boundary conditions. The spatial variable as usual, is discretised spectrally.

For the first step, we transform the problem into Fourier space, advance the solution using the classical fourth-order Runge-Kutta formula, and transform back to physical space using the standard inversion formula. We begin by rewriting the equation as follows

$$u_t + uu_x = 0 \quad \Rightarrow \quad u_t + \frac{1}{2}\left(u^2\right)_x = 0\,.$$

Transforming into Fourier space yields

$$\hat{u}_t + \frac{1}{2} ik\widehat{u^2} = 0\,,$$

where the hat denotes the Fourier transform. Therefore

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = -\frac{1}{2} ikF(u^2) = f(\hat{u})\,.$$

The fourth-order Runge-Kutta formula is given by

$$\hat{u}_{n+1} = \hat{u}_n + \frac{1}{6}\left(d^{(1)} + 2(d^{(2)} + d^{(3)}) + d^{(4)}\right)\,, \tag{6.13}$$

where

$$\begin{aligned}
d^{(1)} &= \frac{\Delta t}{2} f\left(\hat{u}_n\right) , \\
d^{(2)} &= \frac{\Delta t}{2} f\left(\hat{u}_n + \frac{d^{(1)}}{2}\right) , \\
d^{(3)} &= \frac{\Delta t}{2} f\left(\hat{u}_n + \frac{d^{(2)}}{2}\right) , \\
d^{(4)} &= \frac{\Delta t}{2} f\left(\hat{u}_n + d^{(3)}\right) .
\end{aligned}$$

To implement the second step, the solution is advanced using only the linear term by means of the FFT. Taking the Fourier transform of the equation $u_t + u_{xxx} = 0$, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_k = ik^3\hat{u}_k .$$

Integrating with respect to t implies

$$\hat{u}_k = \hat{u}_0 e^{ik^3 t} ,$$

and thus

$$\hat{u}_k\left(t + \frac{\Delta t}{2}\right) = \hat{u}_k e^{ik^3 \Delta t/2} .$$

Therefore, for the second step, we advance the solution using the formula

$$u(x_j, t+\Delta t) = F^{-1}\left(e^{ik^3\Delta t/2} F u\left(x_j, t + \frac{\Delta t}{2}\right)\right) .$$

The split-step method is successful and the results achieved are more accurate than those from the one-step approach of Fornberg and Whitham. The first program was run using 128 grid points, $\Delta t = 0.00001$ and an initial solution given by

$$u(x,0) = 3A^2 \mathrm{sech}^2\left(\frac{1}{2}A(x-\pi+2)\right) , \tag{6.14}$$

where $A = 15$. The program was then repeated for the double soliton solution given by

$$u(x,0) = 3A^2 \mathrm{sech}^2\left(\frac{1}{2}A(x-\pi+4)\right) + 3B^2 \mathrm{sech}^2\left(\frac{1}{2}B(x-\pi+3)\right) , \tag{6.15}$$

where $B = 10$, and the triple soliton solution given by

$$\begin{aligned}
u(x,0) &= 3A^2 \mathrm{sech}^2\left(\frac{1}{2}A(x-\pi+4)\right) + 3B^2 \mathrm{sech}^2\left(\frac{1}{2}B(x-\pi+3)\right) \\
&\quad + 3C^2 \mathrm{sech}^2\left(\frac{1}{2}C(x-\pi+2)\right) , \qquad (6.16)
\end{aligned}$$

where $C = 8$. The results for the three cases are given in Figures 6.7, 6.8 and Figure 6.9 respectively.

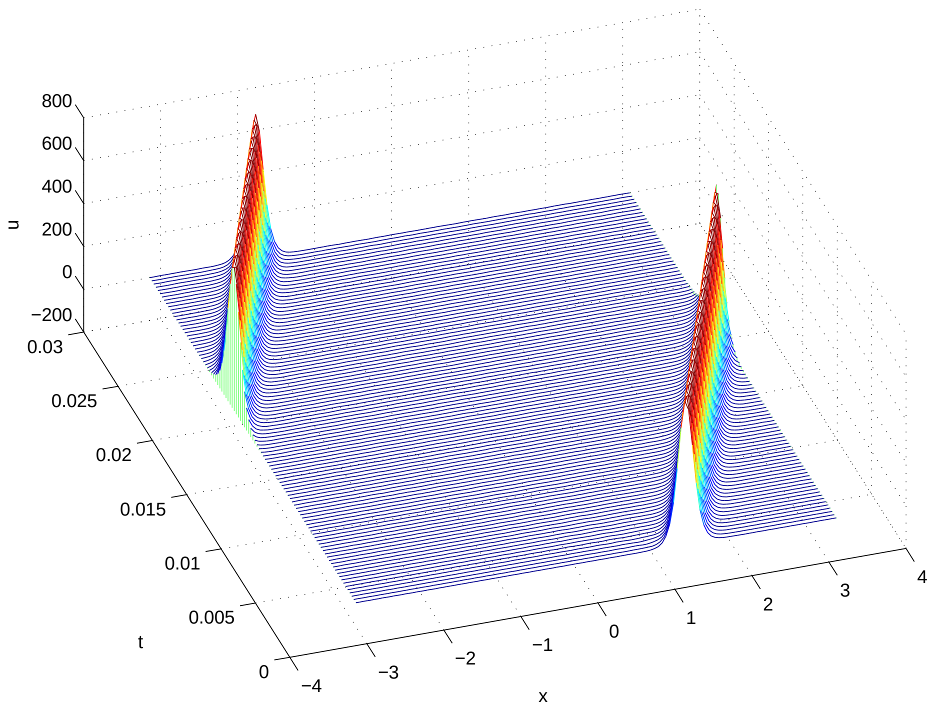


Figure 6.7: Tappert's split-step method for the periodic KdV equation with the single soliton initial solution (6.14), solved using 128 grid points, $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

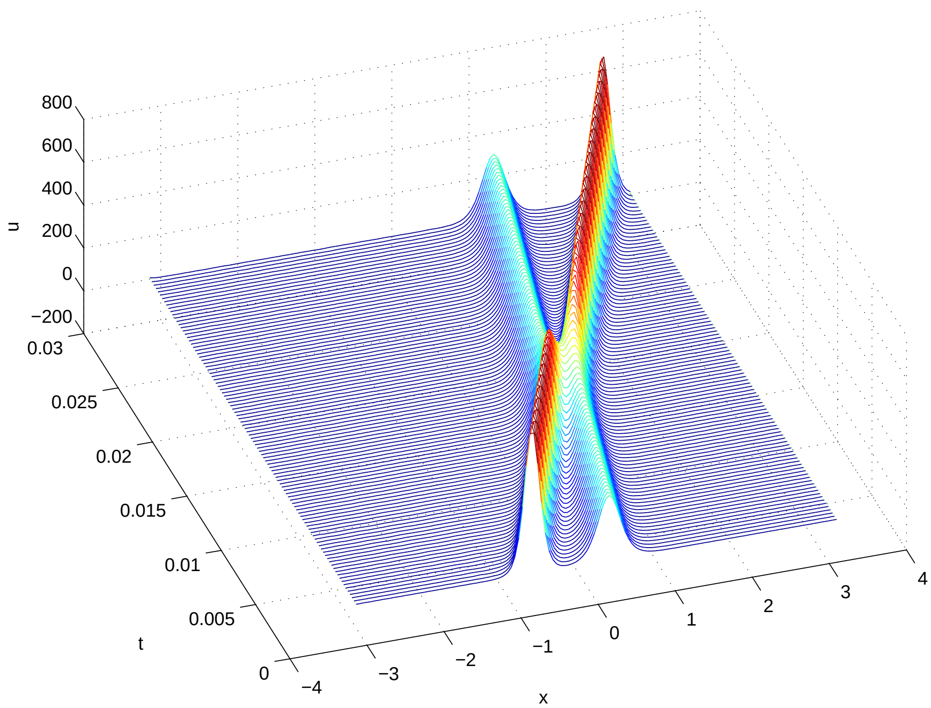


Figure 6.8: Tappert's split-step method for the periodic KdV equation with the double soliton initial solution (6.15), solved using 128 grid points, $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

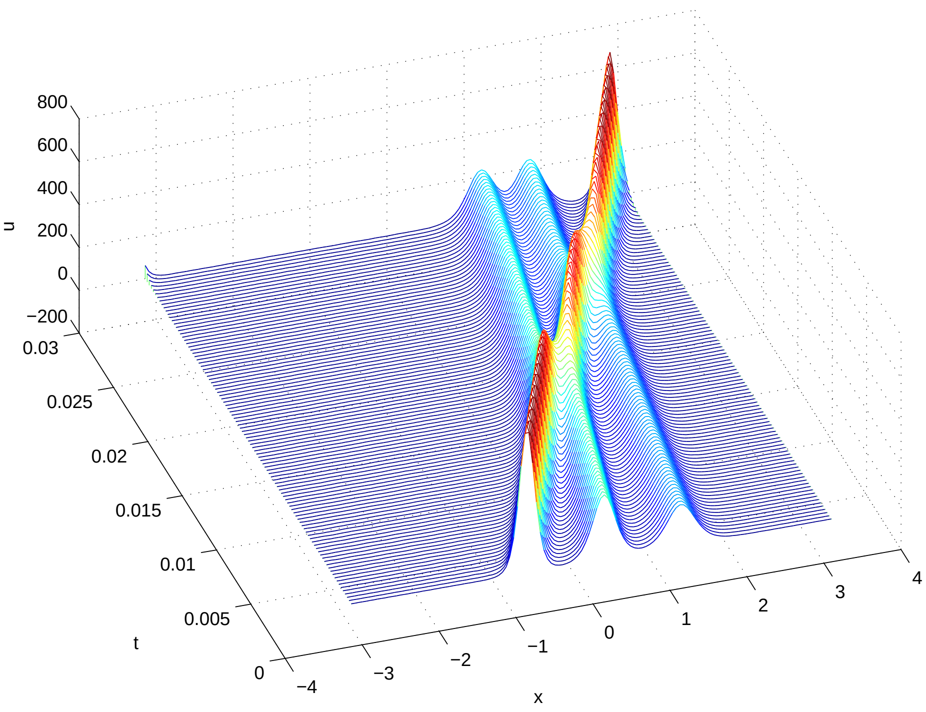


Figure 6.9: Tappert's split-step method for the periodic KdV equation with the triple soliton initial solution (6.16), solved using 128 grid points, $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

The normalised numerical Euclidean norms are given in Table 6.3 for the initial solution and every solution thereafter in time intervals of 0.003 up to the final solution at time $t = 0.030$. The results show that the conservation of energy law, given by (6.9), is satisfied exactly. In comparison to the one-step method of Fornberg and Whitham, we conclude that the split-step approach yields more accurate results.

As with the first method, we now solve equation (6.1) using the split step approach of Tappert, but using Toeplitz differentiation matrices, as opposed to the FFT, to approximate the spatial derivatives. The problem we consider is equation (6.1), with spatial domain $[-\pi, \pi]$ and periodic boundary conditions. For step one we begin by rewriting the equation $u_t + uu_x = 0$ in the form

$$u_t = -\frac{1}{2}\left(u^2\right)_x = f(u)\,,$$

and use the classical fourth-order Runge-Kutta formula and approximate the spatial derivative using the Toeplitz differentiation matrix D_N given by (1.11).

For step two the solution is advanced using an implicit Crank-Nicolson formula given

t	Single Soliton	Double Soliton	Triple Soliton
0	1	1	1
0.003	1	1	1
0.006	1	1	1
0.009	1	1	1
0.012	1	1	1
0.015	1	1	1
0.018	1	1	1
0.021	1	1	1
0.024	1	1	1
0.027	1	1	1
0.030	1	1	1

Table 6.3: The normalised Euclidean norms of the numerical results, given by Figures 6.7, 6.8 and Figure 6.9, for the periodic KdV equation, solved using Tappert's split-step method with $N = 128$, $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

by

$$u(x_j, t + \Delta t) = u\left(x_j, t + \frac{\Delta t}{2}\right) - \frac{\Delta t}{4}\left(u_{xxx}(x_j, t + \Delta t) + u_{xxx}\left(x_j, t + \frac{\Delta t}{2}\right)\right).$$

The program was run using 128 grid points, $\Delta t = 0.00001$ and an initial solution given by

$$u(x, 0) = 3A^2 \text{sech}^2\left(\frac{1}{2}A\left(x - 2\pi + 2\right)\right), \tag{6.17}$$

where $A = 15$, and the numerical output is given by Figure 6.10.

The normalised numerical Euclidean norms are given in Table 6.4, for the initial solution and every solution thereafter in time intervals of 0.003 up to the final solution at time $t = 0.030$. The results show that the scheme satisfies the conservation of energy law, given by (6.9), exactly.

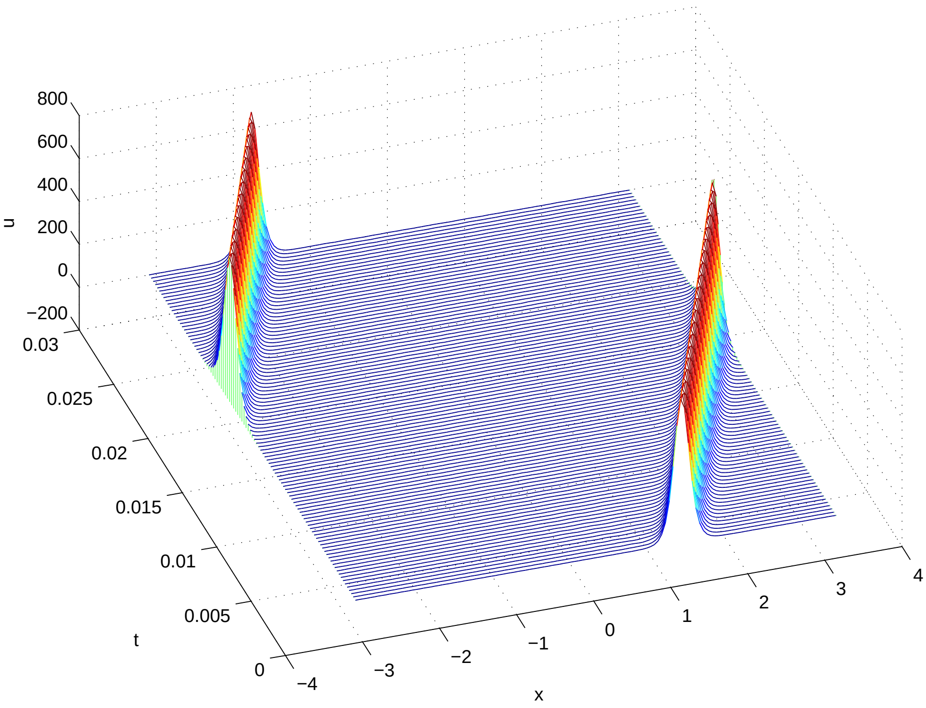


Figure 6.10: Tappert's split-step method for the periodic KdV equation with the single soliton initial solution (6.17), solved using 128 grid points, $\Delta t = 0.00001$ and Toeplitz differentiation matrices to approximate the spatial derivatives.

t	Single Soliton
0	1
0.003	1
0.006	1
0.009	1
0.012	1
0.015	1
0.018	1
0.021	1
0.024	1
0.027	1
0.030	1

Table 6.4: The normalised Euclidean norms of the numerical results, given by Figure 6.10, for the periodic KdV equation, solved using Tappert's split-step method with $N = 128$, $\Delta t = 0.00001$ and Toeplitz differentiation matrices to approximate the spatial derivatives.

An Alternative Approach to the Split-Step Method

In the split-step scheme, as proposed by Tappert [45], the first half-step was taken by solving the nonlinear problem. Here we show that the order of the steps can be inverted at no cost to the computation. Hence we advance the solution according to the linear terms first. We do this because for some boundary value problems the imposition of the boundary conditions is more difficult at one step than at the other.

To analyse this effect, we consider the third order nonlinear KdV equation, given by (6.1), and consider the scheme as successfully solving the equations

$$u_t + u_{xxx} = 0\,, \quad u_t + uu_x = 0\,,$$

where the solution of the former equation is used as the initial condition for the latter. For completeness we repeat two of the examples. The first is the example, whose results are given by Figure 6.7 and whose Euclidean norms can be found in Table 6.3. The program uses 128 grid points, $\Delta t = 0.00001$ and initial solution

$$u(x,0) = 3A^2 \text{sech}^2 \left(\frac{1}{2} A(x - \pi + 2) \right)\,, \tag{6.18}$$

with $A = 15$. The first half advances the solution using only the linear term by means of the FFT, and the second half advances the solution according to the nonlinear terms using the classical fourth-order Runga-Kutta formula. The new results are given by Figure 6.11 and the normalised Euclidean norms are given in Table 6.5. The results are identical to those given in Table 6.3, achieved from the standard split-step approach.

We also repeat the example using the split-step method and the Toeplitz differentiation matrices, whose results are given by Figure 6.10 and Table 6.4, but with the order of the steps reversed. Hence the first step advances the solution according to the linear terms using an implicit Crank-Nicolson formula, and the second step advances the updated solution according to the nonlinear terms using the classical fourth-order Runga-Kutta formula. The program was run using 128 grid points, $\Delta t = 0.00001$ and initial solution given by

$$u(x,0) = 3A^2 \text{sech}^2 \left(\frac{1}{2} A(x - 2\pi + 2) \right)\,, \tag{6.19}$$

where $A = 15$, and the numerical output is given by Figure 6.12.

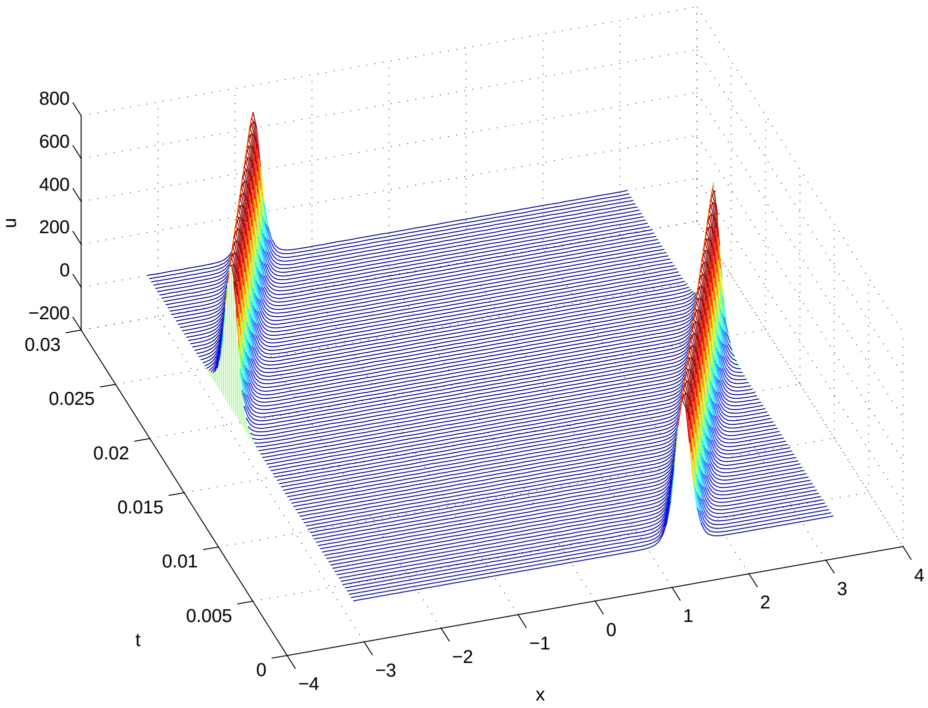


Figure 6.11: Tappert's split-step method, with the order of the steps reversed, for the periodic KdV equation with the single soliton initial solution (6.18), solved using 128 grid points and $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

t	Single Soliton
0	1
0.003	1
0.006	1
0.009	1
0.012	1
0.015	1
0.018	1
0.021	1
0.024	1
0.027	1
0.030	1

Table 6.5: The normalised Euclidean norms of the numerical results, given by Figure 6.11, for the periodic KdV equation, solved using Tappert's split-step method with the order of the steps reversed, $N = 128$, $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

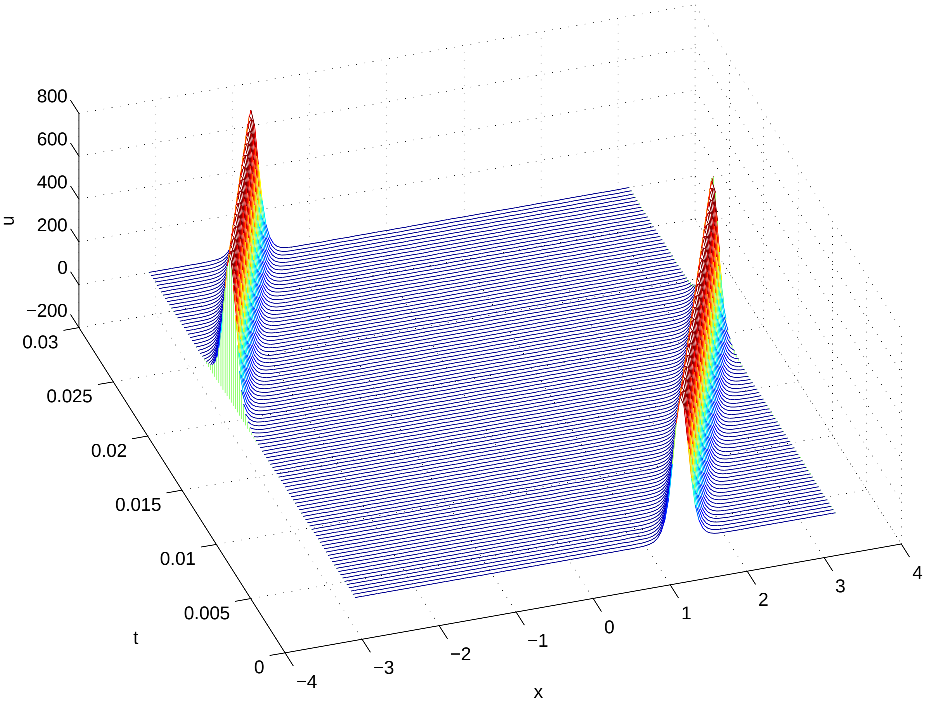


Figure 6.12: Tappert's Split-Step method with the order of the steps swapped, for the periodic KdV equation with the single soliton initial solution (6.19), solved using 128 grid points, $\Delta t = 0.00001$ and Toeplitz differentiation matrices to approximate the spatial derivatives.

The normalised numerical Euclidean norms are given in Table 6.6, for the initial solution and every solution thereafter in time intervals of 0.003, up to the final solution at time $t = 0.030$. This example also demonstrates that swapping the order of the steps has no effect on the scheme, and using either approach the conservation of energy law is exactly satisfied.

Before concluding this section, we remark that further approaches to the split-step method can be employed. One such scheme, splits each time step Δt into three parts, and can be considered as successfully solving the equations

$$u_t + uu_x = 0\,, \quad u_t + u_{xxx} = 0\,, \quad u_t + uu_x = 0\,.$$

The first step, advances the solution a quarter of the time step according to the nonlinear terms. The second step then uses the solution achieved from step one as the initial condition, and advances it half of the time step using only the linear term. The final step then advances the solution from the second stage, the remaining quarter time step according to the nonlinear terms.

We will use this approach of swapping the order of the steps later, for solving some two-point boundary value problems.

t	Single Soliton
0	1
0.003	1
0.006	1
0.009	1
0.012	1
0.015	1
0.018	1
0.021	1
0.024	1
0.027	1
0.030	1

Table 6.6: The normalised Euclidean norms of the numerical results, given by Figure 6.12 for the periodic KdV equation, solved using Tappert's split-step method with the order of the steps reversed, $N = 128$, $\Delta t = 0.00001$ and Toeplitz differentiation matrices to approximate the spatial derivatives.

6.1.3 Alternative Initial Conditions

In this section we consider once again the one-step method of Fornberg and Whitham, and the split-step numerical scheme of Tappert for solving the periodic KdV equation, given by (6.1), but with the imposition of a Gaussian type initial condition of the form

$$u(x,0) = 3A^2 e^{-10\left(x-\frac{L}{2}\right)^2}, \quad x \in [0, L]\,. \tag{6.20}$$

for some constant A. To begin with, we repeat the one-step method of Section 6.1.1 solved using a leap-frog scheme in time, with the FFT approach to approximate the spatial derivatives and then using the Toeplitz differentiation matrix approach for the numerical approximation of the spatial derivatives (see Figures 6.1, 6.2, 6.3 and Figures 6.4, 6.5, 6.6 respectively).

All of the programs were run using $N = 128$ grid points and $A = 15$. The first program using the FFT approach was run with $\Delta t = 0.00001$ and the second program using the Toeplitz differentiation matrices was run using $\Delta t = 0.000001$. The results are given in Figure 6.13 and Figure 6.14 respectively.

These results were then compared to the split-step Fourier method of Tappert, described in Section 6.1.2, and the programs using both the FFT (Figures 6.7, 6.8 and Figure 6.9), and the Toeplitz differentiation matrix approach (Figure 6.10) were run with the initial condition of the form (6.20). The results are given in Figure 6.15 and Figure 6.16 respectively.

In all four examples, we see that as the initial solution begins to propagate, it immediately splits and forms two soliton type solutions. The energy that remains forms a trail of smaller dispersive waves. The soliton that is to the left is smaller in amplitude to the wave to its right, and therefore travels slower. Both waves travel without losing energy or changing in form. The wave whose amplitude is greatest collides with the boundary first and then reappears from the opposing boundary. This behaviour is then repeated by the second wave. When the two waves collide, the interaction is clean and the waves continue to travel in the direction they were travelling before the collision.

The normalised Euclidean norms, calculated at time intervals of 0.005 up to the final time of $t = 0.050$ for all four programs, are given in Table 6.7.

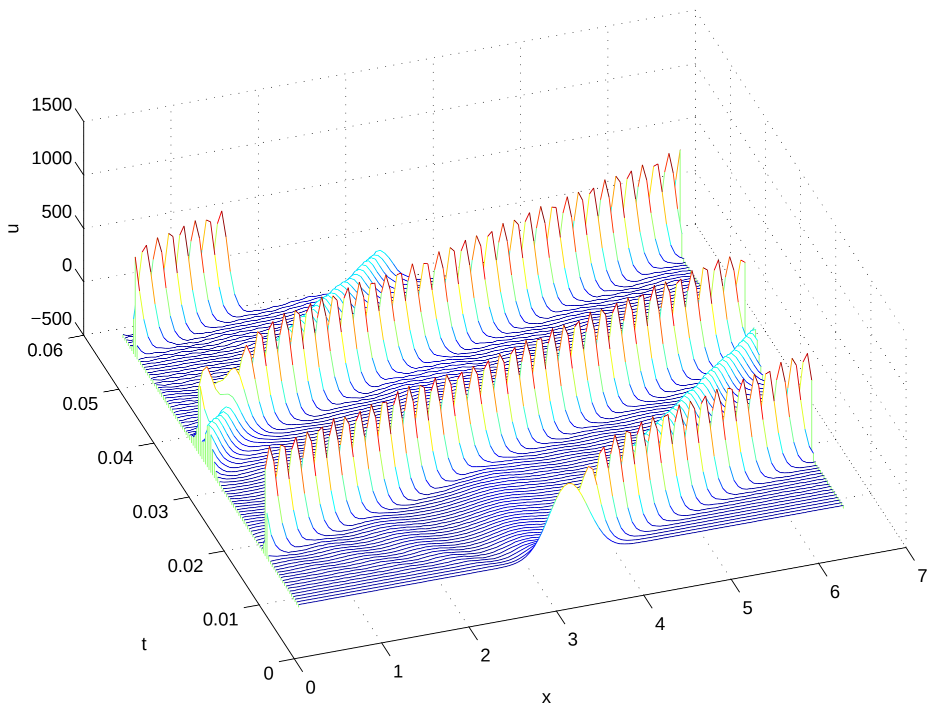


Figure 6.13: Fornberg and Whitham's one-step method for the periodic KdV equation with the initial condition $u(x, 0) = 3A^2 e^{-10(x-\pi)^2}$, solved using 128 grid points, $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

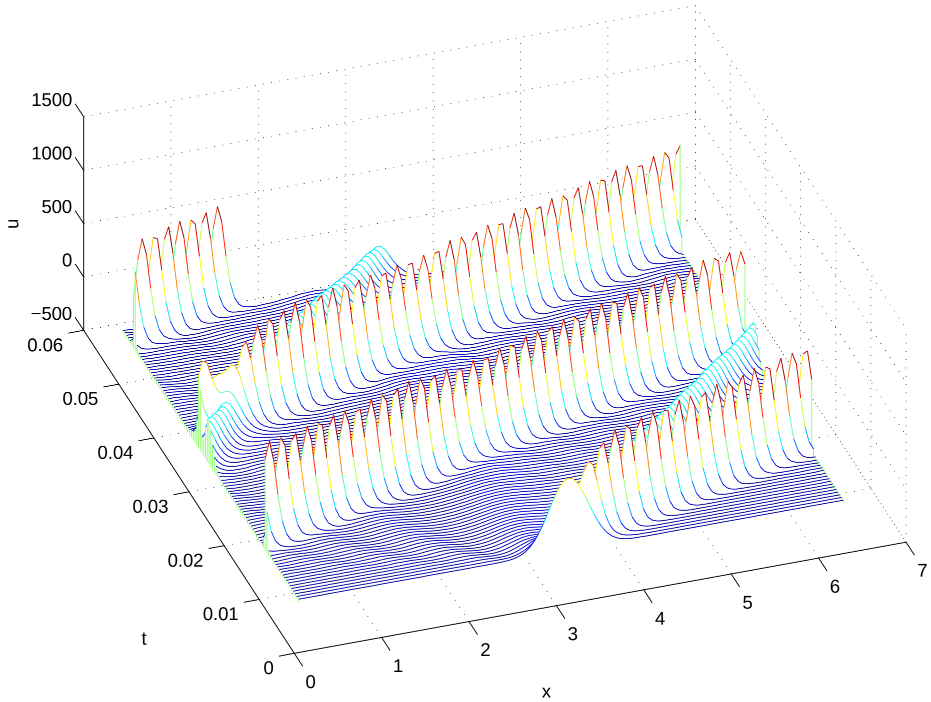


Figure 6.14: Fornberg and Whitham's one-step method for the periodic KdV equation with the initial condition $u(x, 0) = 3A^2e^{-10(x-\pi)^2}$, solved using 128 grid points, $\Delta t = 0.000001$ and Toeplitz differentiation matrices to approximate the spatial derivatives.

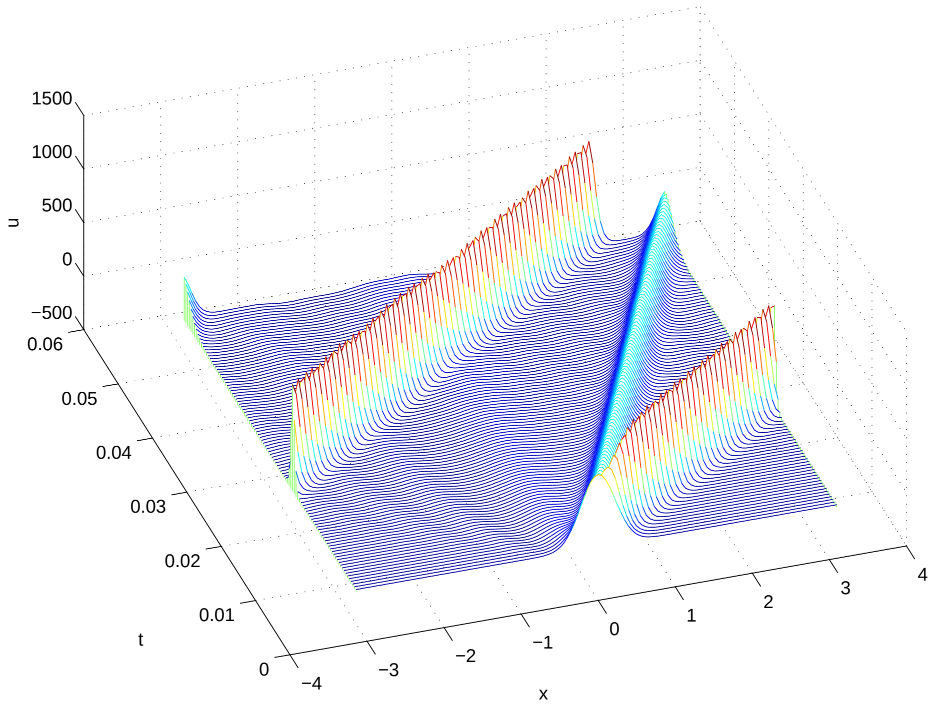


Figure 6.15: Tappert's split-step method for the periodic KdV equation with the initial condition $u(x, 0) = 3A^2e^{-10x^2}$, solved using 128 grid points, $\Delta t = 0.00001$ and the FFT to approximate the spatial derivatives.

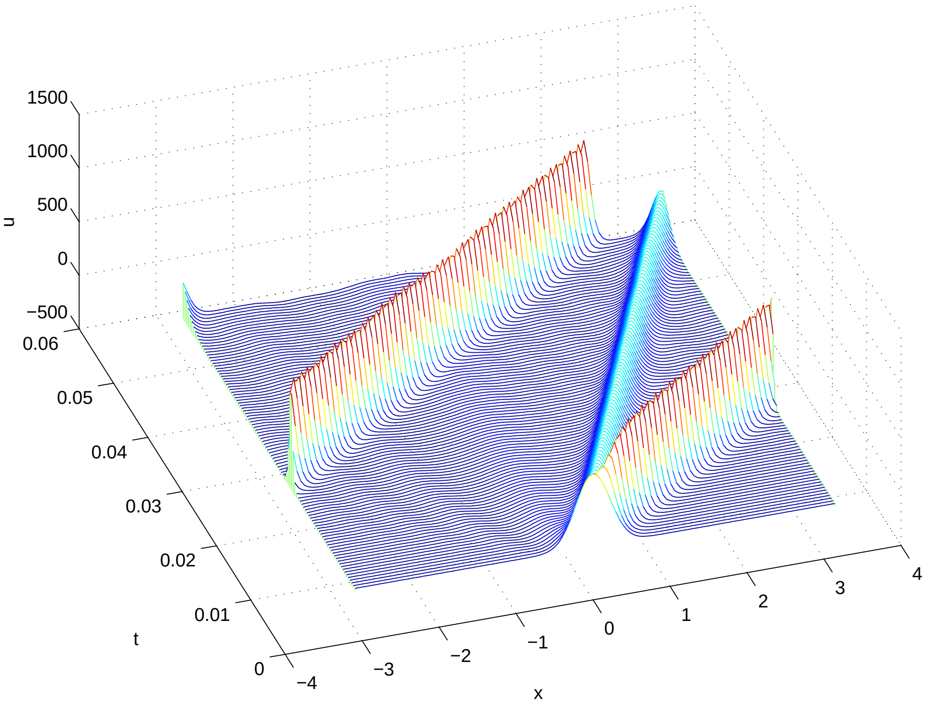


Figure 6.16: Tappert's split-step method for the periodic KdV equation with the initial condition $u(x,0) = 3A^2e^{-10x^2}$, solved using 128 grid points, $\Delta t = 0.00001$ and Toeplitz differentiation matrices to approximate the spatial derivatives.

The results show that the one-step method, using both the FFT approach and the Toeplitz differentiation matrix approach for the numerical approximation of the spatial derivatives, achieves adequate results. In comparison, the results given in Table 6.7 for the two numerical schemes using the split-step method of Tappert, show that the solution satisfies the conservation of energy law exactly.

We conclude this section by remarking that all of the periodic problems studied in Section 6.1 using the split-step method, satisfy the conservation of energy law exactly.

	One-Step		Split-Step	
t	FFT	Toeplitz	FFT	Toeplitz
0	1.0000	1.0000	1.0000	1.0000
0.005	1.0008	1.0000	1.0000	1.0000
0.010	0.9988	1.0000	1.0000	1.0000
0.015	0.9994	0.9999	1.0000	1.0000
0.020	0.9995	1.0000	1.0000	1.0000
0.025	0.9980	0.9997	1.0000	1.0000
0.030	1.0009	1.0000	1.0000	1.0000
0.035	1.0004	1.0000	1.0000	1.0000
0.040	0.9991	0.9999	1.0000	1.0000
0.045	1.0031	1.0000	1.0000	1.0000
0.050	1.0000	1.0001	1.0000	1.0000

Table 6.7: The normalised Euclidean norms of the numerical results, given by Figures 6.13, 6.14, 6.15 and Figure 6.16 for the periodic KdV equation with a Gaussian initial condition of the form (6.20), solved using $N = 128$.

6.2 The Non-Periodic Problem

The focus of the remainder of the third order numerical results is on the imposition of a variety of boundary conditions. All of the schemes presented will combine the split-step approach outlined in the previous section for discretising the time variable, with the methods developed in Chapter 5 for the linear boundary value problems for the imposition of the boundary conditions.

6.2.1 Uncoupled Boundary Conditions

We begin by considering the third order nonlinear problem

$$u_t + uu_y + u_{yyy} = 0\,, \quad u(y,0) = u_0(y)\,, \quad t > 0\,, \quad y \in [-L, L]\,,$$

$$u(-L,t) = 0\,, \quad u(L,t) = 0\,, \quad u_y(L,t) = 0\,.$$

The spatial domain is transformed to $[-1,1]$ be letting $x = y/L$ to give

$$u_t + \tfrac{1}{L} u u_x + \left(\tfrac{1}{L}\right)^3 u_{xxx} = 0\,, \quad u(x,0) = u_0(x)\,, \quad t > 0\,, \quad x \in [-1,1]\,,$$

$$u(-1,t) = 0\,, \quad u(1,t) = 0\,, \quad u_x(1,t) = 0\,.$$

If we compare the results of this section with the results of the periodic problem, we see that for the non-periodic problem, the solitary wave solutions travel without losing energy until colliding with the boundary. After colliding with the boundary the waves then start to lose energy and disperse. This can be easily proved by considering the Euclidean norm for this problem. Equation (6.7), evaluated for $x \in [-1,1]$, reduces to the following

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u(x,t)||_2^2 = -\frac{1}{2}u_x^2(-1,t) \leqslant 0\,,$$

proving that the problem is dispersive, with energy seeping away through the boundary.

Split-Step method

The split-step approach of Tappert is used to numerically model this problem, and all three boundary conditions are imposed numerically at both stages of the step. For the first step, the solution is advanced by using the nonlinear part of the equation

$$u_t + \frac{1}{L} u u_x = 0\,,$$

and the boundary conditions $u(\pm 1, t) = 0$ and $u_x(1,t) = 0$ are imposed using the polynomial trick

$$u(x,t) = (1-x)q(x,t)\,, \quad q(\pm 1, t) = 0\,. \tag{6.21}$$

(This method of imposing the boundary conditions was described in detail in Section 5.1.1, for the third order linear ODE $u_{xxx}(x) = f(x)$ given by (5.2)).

So, given an initial condition $u(x,0)$, $q(x,0)$ is computed according to equation (6.21), and then the equivalent problem in $q(x,0)$, with Dirichlet boundary conditions, is advanced in time using the classical fourth-order Runge-Kutta formula, given by (6.13). The spatial derivatives are imposed using the Chebyshev differentiation matrix D_N, defined by Theorem 1.2.1. The problem is formulated as follows

$$q_t(x,t) = \frac{1}{L}q^2(x,t) - \frac{1}{2L}(1-x)(q^2)_x(x,t)\,, \qquad q(\pm 1, t) = 0\,.$$

The Runge-Kutta formula, given by (6.13), then implies

$$
\begin{aligned}
q_{n+1}(x,t) &= q_n(x,t) + \frac{1}{6}\left(d^{(1)} + 2\left(d^{(2)} + d^{(3)}\right) + d^{(4)}\right) , \\
d^{(1)} &= \frac{\Delta t}{2L} q_n^2(x,t) - \frac{\Delta t}{4L}(1-x)(q_n^2)_x(x,t) , \\
d^{(2)} &= \frac{\Delta t}{2L}\left(q_n(x,t) + \frac{1}{2}d^{(1)}\right)^2 - \frac{\Delta t}{4L}(1-x)\left(q_n(x,t) + \frac{1}{2}d^{(1)}\right)^2_x , \\
d^{(3)} &= \frac{\Delta t}{2L}\left(q_n(x,t) + \frac{1}{2}d^{(2)}\right)^2 - \frac{\Delta t}{4L}(1-x)\left(q_n(x,t) + \frac{1}{2}d^{(2)}\right)^2_x , \\
d^{(4)} &= \frac{\Delta t}{2L}\left(q_n(x,t) + d^{(3)}\right)^2 - \frac{\Delta t}{4L}(1-x)\left(q_n(x,t) + d^{(3)}\right)^2_x .
\end{aligned}
$$

Once $q(x,t)$ is known at the appropriate time level, $u(x,t)$ is trivially computed using equation (6.21). This solution then becomes the initial solution that is advanced the remaining half of the time-step according to the linear part of the equation

$$u_t + \left(\frac{1}{L}\right)^3 u_{xxx} = 0 ,$$

and the boundary conditions $u(\pm 1,t) = 0$ and $u_x(1,t) = 0$. For the time derivative we use an implicit Crank-Nicolson formula, which yields the following approximation:

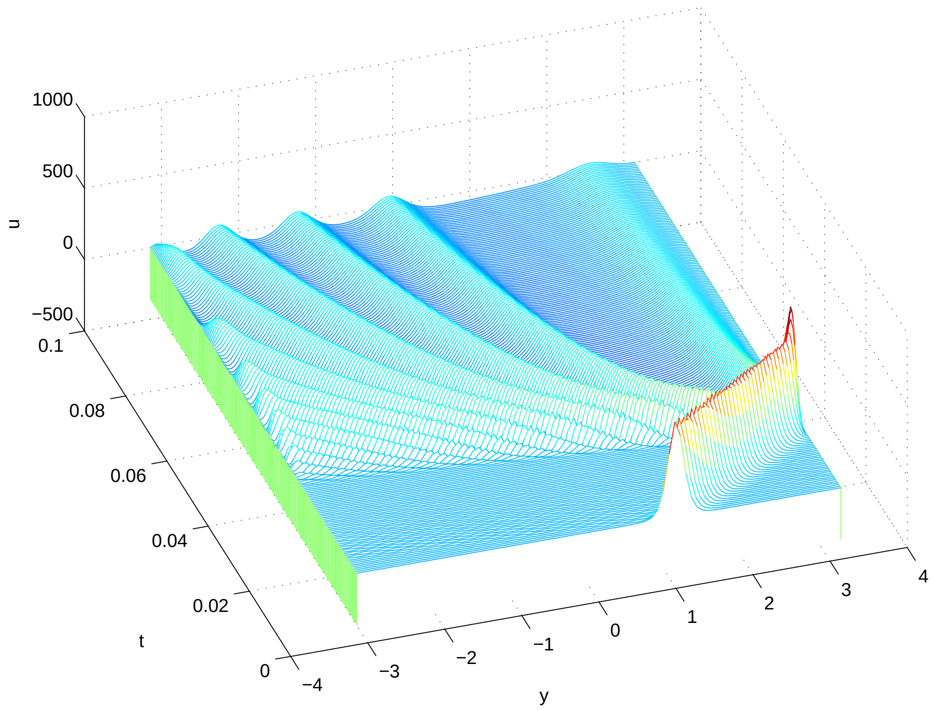
$$
\begin{aligned}
-\frac{\Delta t}{4L^3} u_{xxx}(x,t+\Delta t) + u(x,t+\Delta t) &= -\frac{\Delta t}{4L^3} u_{xxx}\left(x, t+\frac{\Delta t}{2}\right) + u\left(x, t+\frac{\Delta t}{2}\right) , \\
u(\pm 1, t+\Delta t) = 0 \quad &, \quad u_x(1, t+\Delta t) = 0 .
\end{aligned}
$$

This is a time independent problem, which can be solved by rewriting the problem in terms of $q(x,t)$ using (6.21) evaluated at $t+\Delta t$, and adapting the method presented in Section 5.1.1 for the imposition of the same boundary conditions on equation (5.2).

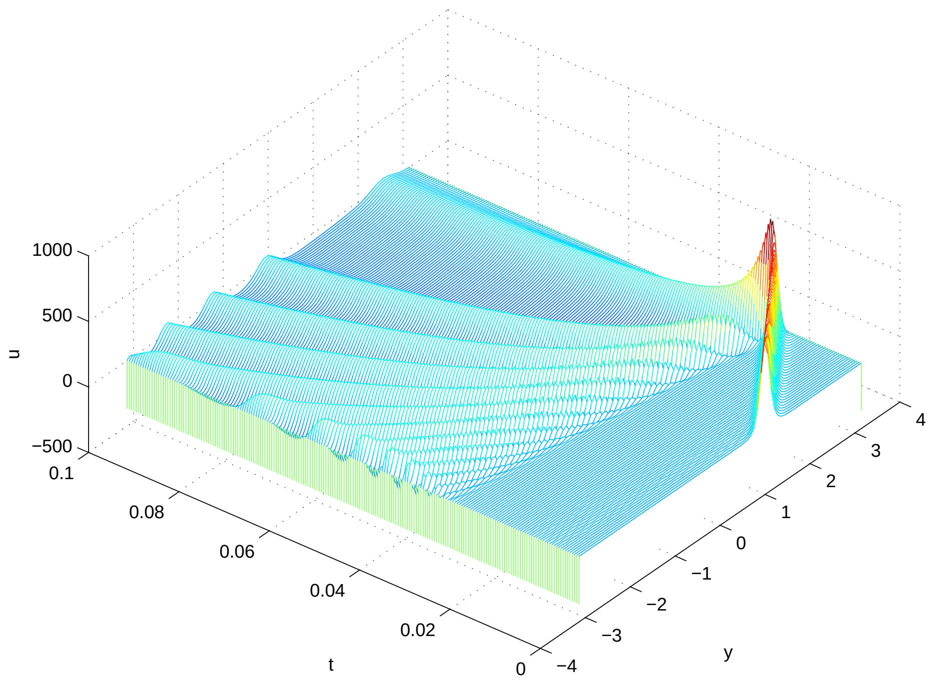
The program was run using 128 grid points, $\Delta t = 0.00001$, $L = \pi$ and initial solution given by

$$u(x,0) = 3A^2 \mathrm{sech}^2\left(\frac{1}{2}A(x\pi - 1)\right) , \tag{6.22}$$

where $A = 15$. The numerical output is given by Figure 6.17(a), which is then rotated to produce Figure 6.17(b).



(a)



(b)

Figure 6.17: The non-periodic KdV equation with the boundary conditions $u(\pm L, t) = 0$ and $u_y(L, t) = 0$ and the single soliton initial solution (6.22), solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$.

If we compare these results with the results of the periodic problem (Figures 6.1, 6.4, 6.7 and Figure 6.10) we see that the behaviour of the solitary wave is similar until reaching the boundary. In the periodic case the soliton reaches the boundary and emerges from the opposing boundary with the same amplitude and speed and continues to travel in the same direction. In the non-periodic case (Figure 6.17) the soliton travels with constant speed and amplitude until it reaches the boundary, at time $t \approxeq 0.0185$, and then it is reflected and it starts to lose energy and disperse.

The program was then repeated for the double soliton initial solution of the form

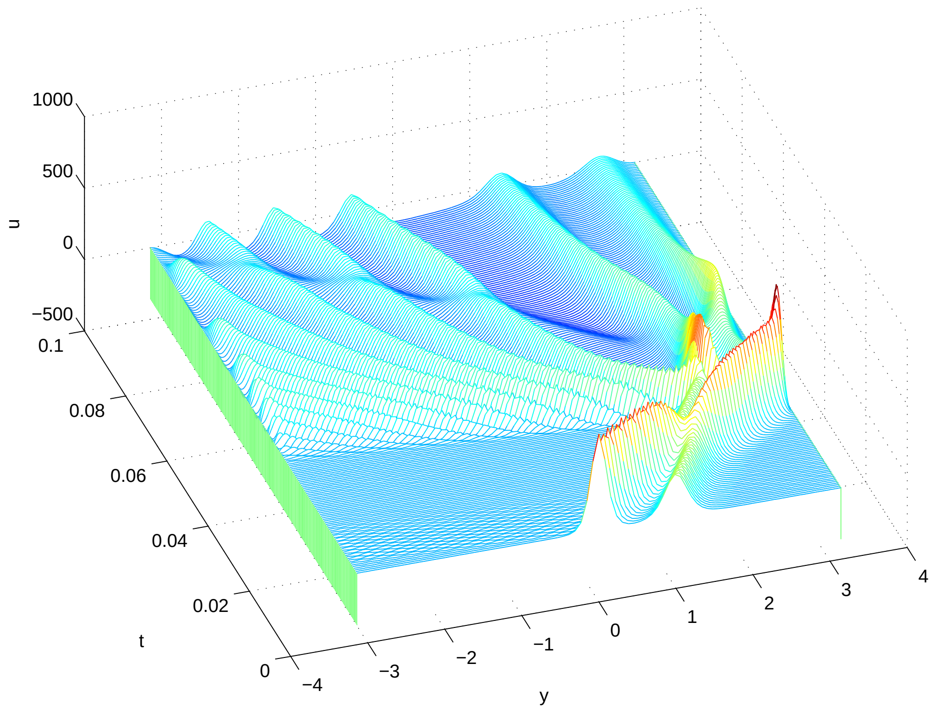
$$u(x,0) = 3A^2\mathrm{sech}^2\left(\frac{1}{2}Ax\pi\right) + 3B^2\mathrm{sech}^2\left(\frac{1}{2}B(x\pi - 1)\right), \tag{6.23}$$

where $A = 15$ and $B = 10$, and the output is given by Figure 6.18(a), which is then rotated to produce Figure 6.18(b).

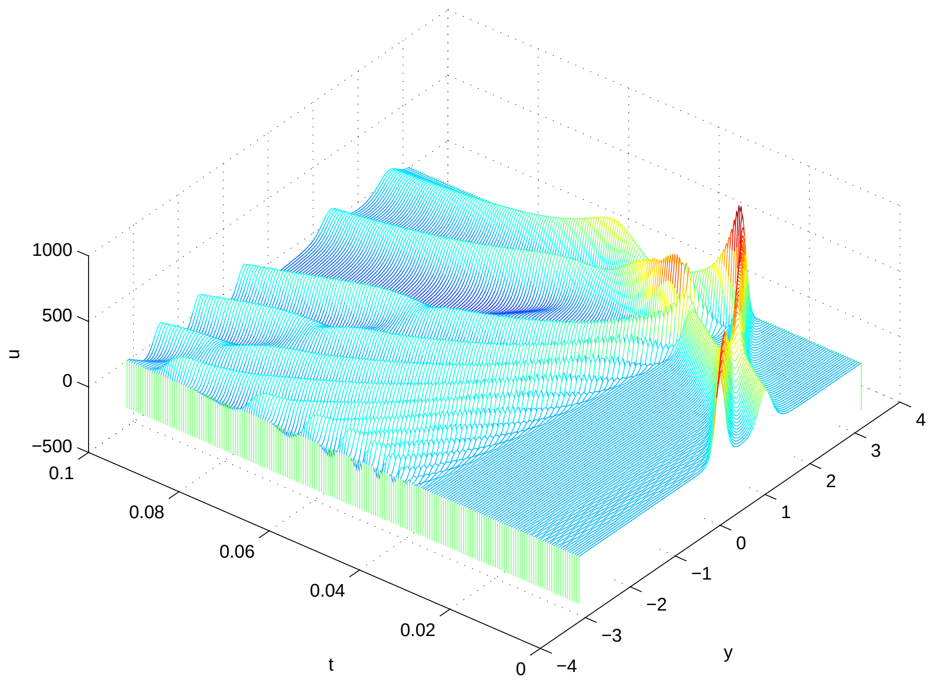
If we compare these results to the periodic problem (Figures 6.2, 6.5 and Figure 6.8) we see that for the non-periodic problem (Figure 6.18), both solitons travel without losing energy until they collide with the boundary. After colliding with the boundary, at time $t \approxeq 0.0255$, the waves start to lose energy and disperse.

The single soliton solution collides with the boundary when $t \approxeq 0.0185$, and the numerical Euclidean norm calculations, given in Table 6.8, show that after the solution is reflected back from the boundary, it continues to lose energy and disperse. This is reflected both in the graphical output, given by Figure 6.17, and in the numerical norm calculations, given in Table 6.8 at time intervals of $t = 0.005$ up to $t = 0.100$. Table 6.9 gives the normalised Euclidean norm calculations for the single soliton solution for the period $0.0150 \leqslant t \leqslant 0.0200$ within which the collision with the boundary occurs, in time intervals of 0.0005.

Similarly, the double soliton initial solution travels uniformly until the collision with the boundary at $t \approxeq 0.0255$, and thereafter the Euclidean norm values decrease as the solution disperses. Table 6.10 gives the normalised Euclidean norms for the double soliton solution for the period $0.0230 \leqslant t \leqslant 0.0280$ within which the collision with the boundary occurs.



(a)



(b)

Figure 6.18: The non-periodic KdV equation with the boundary conditions $u(\pm L, t) = 0$ and $u_y(L, t) = 0$ and the double soliton initial solution (6.23), solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$.

t	Single Soliton	Double Soliton
0	1.0000	1.0000
0.005	1.0457	1.0111
0.010	1.1356	1.0391
0.015	1.3603	1.1005
0.020	1.5898	1.2269
0.025	1.2219	1.6562
0.030	1.1074	1.2972
0.035	1.0806	1.1532
0.040	1.0417	1.1309
0.045	0.9610	1.1327
0.050	0.8503	1.1227
0.055	0.7717	1.0892
0.060	0.7072	1.0025
0.065	0.6760	0.9013
0.070	0.6721	0.8605
0.075	0.5751	0.7765
0.080	0.5802	0.7501
0.085	0.5217	0.7237
0.090	0.5322	0.7321
0.095	0.5243	0.6515
0.100	0.4454	0.6686

Table 6.8: The normalised Euclidean norms for the non-periodic KdV equation, with the boundary conditions $u(\pm L, t) = 0$ and $u_y(L, t) = 0$, given by Figure 6.17 and Figure 6.18, solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$.

t	Single Soliton
0.0150	1.3603
0.0155	1.4052
0.0160	1.4601
0.0165	1.5277
0.0170	1.6078
0.0175	1.6889
0.0180	1.7437
0.0185	1.7501
0.0190	1.7131
0.0195	1.6539
0.0200	1.5898

Table 6.9: The normalised Euclidean norms for the non-periodic KdV equation with the boundary conditions $u(\pm L, t) = 0$ and $u_y(L, t) = 0$, given by Figure 6.17, for the single soliton initial solution (6.22) and the period $0.0150 \leqslant t \leqslant 0.0200$ including the collision of the soliton with the boundary at $t = 0.0185$.

t	Double Soliton
0.0230	1.4168
0.0235	1.4752
0.0240	1.5440
0.0245	1.6124
0.0250	1.6562
0.0255	1.6582
0.0260	1.6248
0.0265	1.5745
0.0270	1.5213
0.0275	1.4718
0.0280	1.4271

Table 6.10: The normalised Euclidean norms for the non-periodic KdV equation with the boundary conditions $u(\pm L, t) = 0$ and $u_y(L, t) = 0$, given by Figure 6.18, for the double soliton initial solution (6.23) and the period $0.0230 \leqslant t \leqslant 0.0280$ including the collision with the boundary at $t = 0.0255$.

An Alternative Split-Step Method

We now consider the effect of swapping the order of the steps and advancing the solution according to the linear terms first. Since the boundary conditions are imposed at each half-step, we expect the swapping to have no effect, and this is indeed what we find. The problem we consider is given by

$$u_t + uu_y + u_{yyy} = 0\,, \quad u(y,0) = u_0(y)\,, \quad t > 0\,, \quad y \in [-L, L]\,,$$

$$u(-L,t) = 0\,, \quad u(L,t) = 0\,, \quad u_y(L,t) = 0\,.$$

We follow the example for the single soliton exactly, whose results are given by Figure 6.17, Table 6.8 and Table 6.9, but reverse the order of the steps. Hence the first step advances the solution according to the linear terms using an implicit Crank-Nicolson scheme, and the second half advances the updated solution according to the nonlinear terms, using the classical fourth-order Runge-Kutta scheme. The program was run using 128 grid points, $\Delta t = 0.00001$ and $L = \pi$. The problem was transformed to the domain $[-1, 1]$, and the initial solution was given by

$$u(x, 0) = 3A^2 \text{sech}^2 \left(\frac{1}{2} A(x\pi - 1) \right) , \tag{6.24}$$

where $A = 15$. The numerical output is given by Figure 6.19.

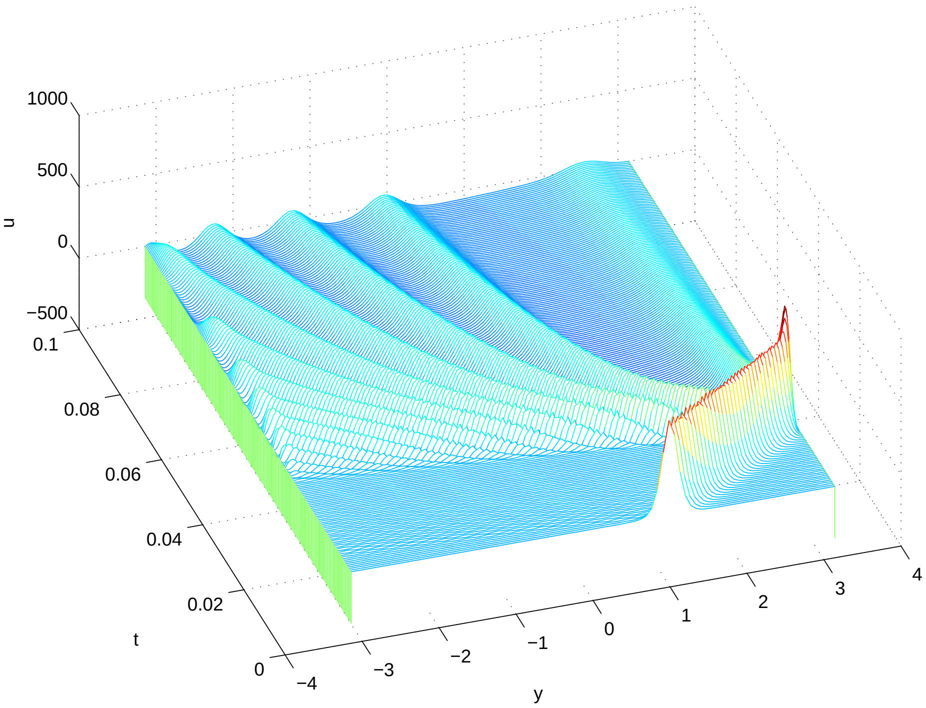


Figure 6.19: The non-periodic KdV equation with the boundary conditions $u(\pm L, t) = 0$ and $u_y(L, t) = 0$ and the single soliton initial solution (6.24), solved using Tappert's split-step method with the order of the steps reversed, $N = 128$ and $\Delta t = 0.00001$.

The numerical calculations for the normalised Euclidean norms are given in Table 6.11 at time intervals of $t = 0.005$ up to $t = 0.100$, and can be compared to the results for the single soliton given in Table 6.8. Table 6.12 gives the values for the time period $0.0150 \leqslant t \leqslant 0.0200$, including the collision of the soliton with the boundary at time $t = 0.0185$, and can be compared to the results in Table 6.9.

If we compare the two schemes we conclude that preliminary results indicate that swapping the steps appears to have very little effect on the dissipation of energy of the solution.

t	Single Soliton
0	1.0000
0.005	1.0457
0.010	1.1356
0.015	1.3605
0.020	1.5947
0.025	1.2227
0.030	1.1076
0.035	1.0811
0.040	1.0423
0.045	0.9611
0.050	0.8513
0.055	0.7718
0.060	0.7076
0.065	0.6763
0.070	0.6716
0.075	0.5752
0.080	0.5803
0.085	0.5220
0.090	0.5324
0.095	0.5246
0.100	0.4454

Table 6.11: The normalised Euclidean norms of the numerical results, for the non-periodic KdV equation with the boundary conditions $u(\pm L, t) = 0$ and $u_y(L, t) = 0$, given by Figure 6.19, solved using Tappert's split-step method with the order of the steps reversed, $N = 128$ and $\Delta t = 0.00001$.

t	Single Soliton
0.0150	1.3605
0.0155	1.4054
0.0160	1.4603
0.0165	1.5282
0.0170	1.6091
0.0175	1.6920
0.0180	1.7494
0.0185	1.7574
0.0190	1.7203
0.0195	1.6600
0.0200	1.5947

Table 6.12: The normalised Euclidean norms for the non-periodic KdV equation with the boundary conditions $u(\pm L, t) = 0$ and $u_y(L, t) = 0$, given by Figure 6.19, for the single soliton initial solution (6.24) and the period $0.0150 \leqslant t \leqslant 0.0200$ including the collision of the soliton with the boundary at $t = 0.0185$.

6.2.2 Non-Homogeneous Uncoupled Boundary Conditions

We now consider the third order nonlinear problem

$$u_t + uu_y + u_{yyy} = 0\,, \quad u(y,0) = u_0(y)\,, \quad t > 0\,, \quad y \in [-L, L]\,,$$

$$u(-L,t) = a\,, \quad u(L,t) = b\,, \quad u_y(L,t) = c\,,$$

for some given smooth function $u_0(y)$ and constants a, b and c. Transforming the problem to $[-1, 1]$, by substituting $x = y/L$, gives

$$u_t + \tfrac{1}{L}uu_x + \left(\tfrac{1}{L}\right)^3 u_{xxx} = 0\,, \quad u(x,0) = u_0(x)\,, \quad t > 0\,, \quad x \in [-1,1]\,, \quad (6.25a)$$

$$u(-1,t) = a\,, \quad u(1,t) = b\,, \quad u_x(1,t) = Lc = d\,, \qquad (6.25b)$$

for some given smooth function $u_0(x)$ and constant d. This problem is solved numerically, using exactly the same approach that was used in Section 6.2, but the imposition of the boundary conditions follows the method that was presented in Section 5.1.1 for equation (5.4).

The first step is the advancement of the solution according to the nonlinear term. The boundary conditions are imposed according to equation (5.5). The equivalent problem in terms of $q(x,t)$ is given by

$$q_t = -\frac{1}{L}\left[\frac{1}{2}(x-1)\left(q^2\right)_x + q^2 + \frac{1}{(x-1)}qh + q_x h + qh_x + \frac{1}{(x-1)}hh_x\right] = f(q)\,,$$

where

$$h(x) = \left(\frac{a+2d-b}{4}\right)x^2 + \left(\frac{b-a}{2}\right)x + \left(\frac{3b+a-2d}{4}\right)\,,$$

and $q(\pm 1, t) = 0$. Given an initial condition $u(x,0)$ we define $q(x,0)$ (on the interior points) as $q(x_j) = \frac{u(x_j)-h(x_j)}{(x_j-1)}$, and following the standard procedure, advance the solution in time using the classical fourth-order Runge-Kutta formula (6.13). The resulting solution $u(x,t)$ is then used as the initial solution for the second half of the step and advanced according to the linear term in equation (6.25a). A backward Euler scheme is used and the boundary conditions imposed according to equation (5.5). The numerical solution to this follows immediately from the general form of the problem solved in Section 5.1.2.

The first result, given by Figure 6.20(a) and rotated to produce Figure 6.20(b), was obtained by imposing the initial condition

$$u(x,0) = 3A^2\text{sech}^2\left(\frac{1}{2}A\left(x\pi - \frac{5}{2}\right)\right), \tag{6.26}$$

where $A = 15$, $\Delta t = 0.00001$, $L = \pi$ and $N = 128$. The boundary conditions were chosen with $a = 0$ and $b = 0.1785$, according to the exact values of (6.26) at the two end points $x = -1$ and $x = 1$ respectively, and $d = -8.4118$ which is the exact value of $u_x(1,0)$. It follows that for the corresponding problem on $[-L, L]$, $c = -2.6776$.

The program was then repeated for the initial condition

$$u(x,0) = 3A^2\text{sech}^2\left(\frac{1}{2}A\left(x\pi + \frac{5}{2}\right)\right), \tag{6.27}$$

with the boundary conditions given by $a = 0.1785$, $b = 0$ and $d = 0$. The numerical output is given by Figure 6.21(a), which was then rotated to produce Figure 6.21(b).

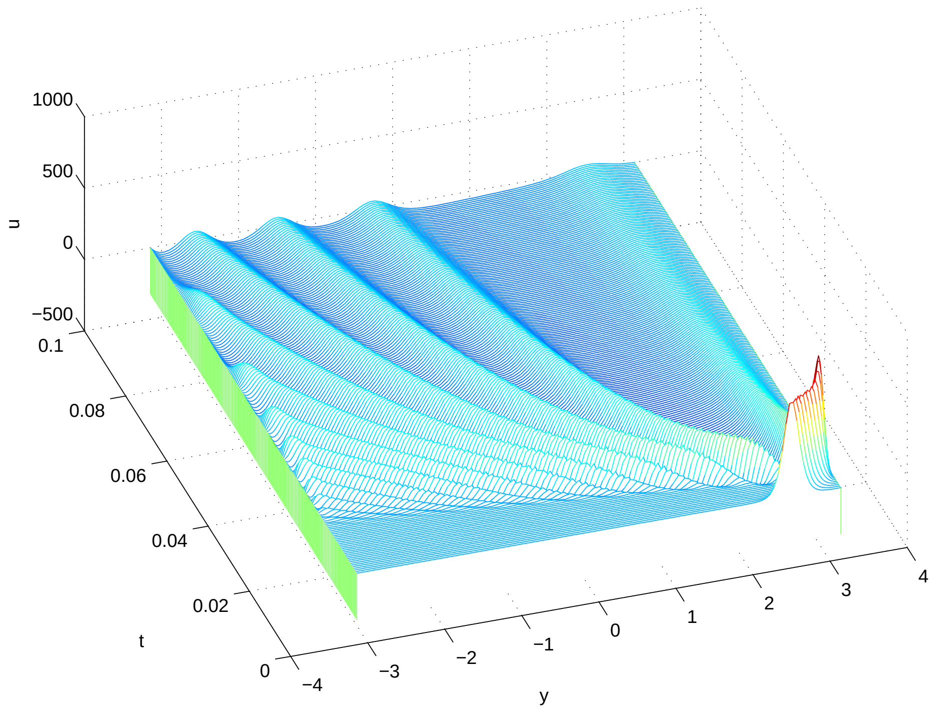
The normalised Euclidean norms for both examples, are given in Table 6.13. The results demonstrate the dispersive nature of the solution, with similar behaviour to the initial solution of the equivalent homogeneous problem (Figure 6.17, Table 6.8).

The particular choice of a, b and c was motivated by the initial condition. To analyse the example in greater detail, we examine the Euclidean norm given by

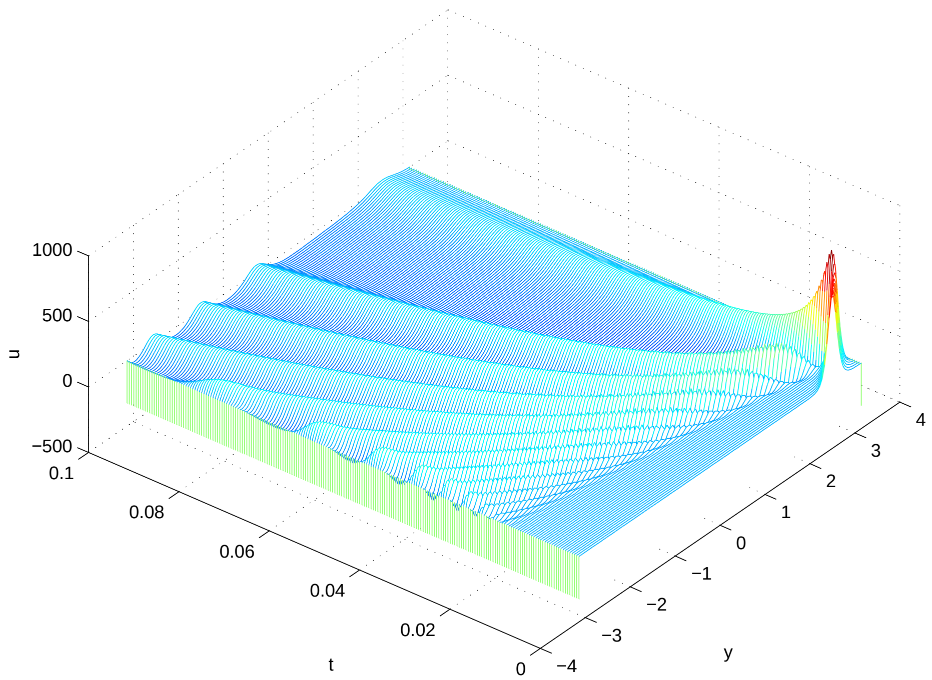
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u(y,t)||_2^2 = \left[\frac{1}{2}u_y^2 - uu_{yy} - \frac{1}{3}u^3\right]_{-L}^{L}. \tag{6.28}$$

Imposing the boundary conditions $u(-L,t) = a$, $u(L,t) = b$ and $u_y(L,t) = c$ reduces (6.28) to the following

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u(y,t)||_2^2 = \frac{1}{2}c^2 - bu_{yy}(L,t) - \frac{1}{3}b^3 - \frac{1}{2}u_y^2(-L,t) + au_{yy}(-L,t) + \frac{1}{3}a^3.$$

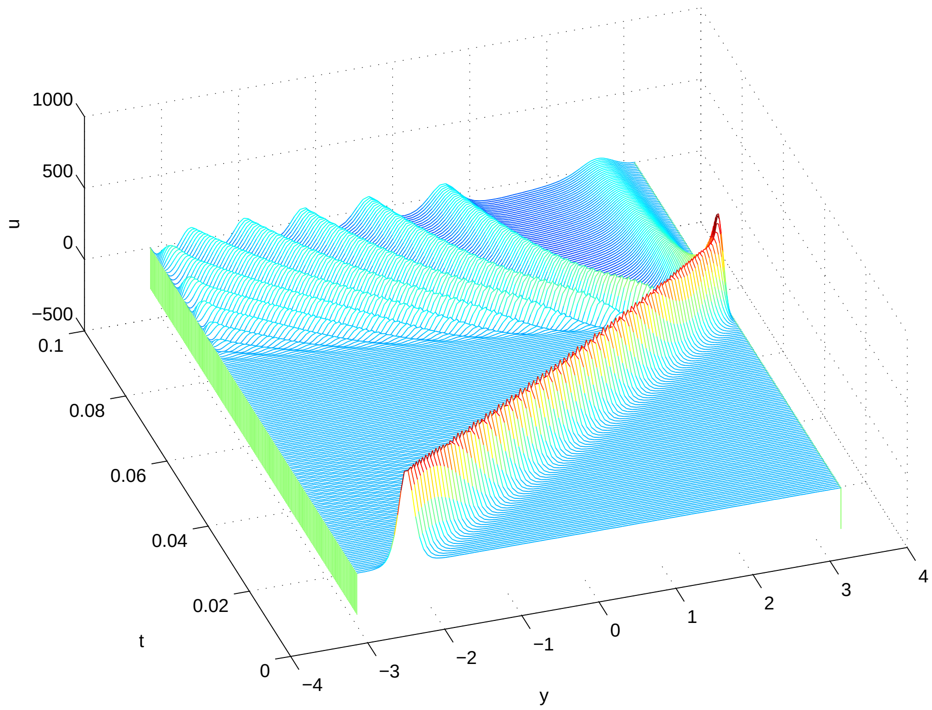


(a)

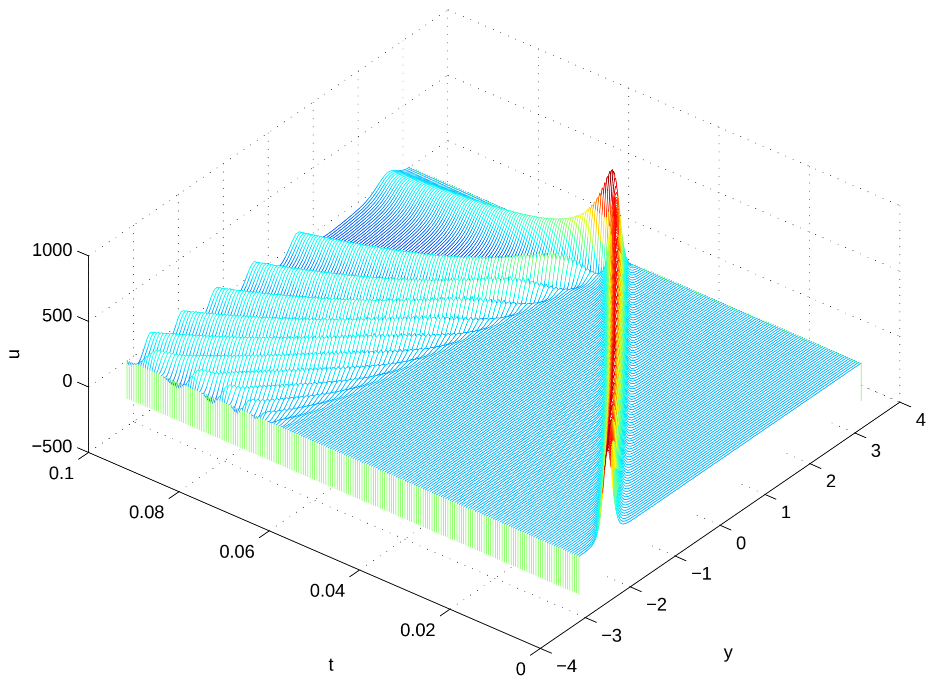


(b)

Figure 6.20: The non-periodic KdV equation with the boundary conditions $u(-L,t) = 0$, $u(L,t) = 0.1785$ and $u_y(L,t) = -2.6776$ with the single soliton initial solution (6.26), solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$.



(a)



(b)

Figure 6.21: The non-periodic KdV equation with the boundary conditions $u(-L, t) = 0.1785$, $u(L, t) = 0$ and $u_y(L, t) = 0$ with the single soliton initial solution (6.27), solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$.

	Single Soliton	
t	$a = 0\,, b = 0.1785\,, c = -2.6776$	$a = 0.1785\,, b = 0\,, c = 0$
0	1.0000	1.0000
0.005	1.3469	0.8643
0.010	0.9314	0.7988
0.015	0.7950	0.7616
0.020	0.7423	0.7406
0.025	0.7215	0.7316
0.030	0.6803	0.7330
0.035	0.6289	0.7462
0.040	0.5886	0.7760
0.045	0.5639	0.8366
0.050	0.5161	0.9875
0.055	0.4644	1.0943
0.060	0.4582	0.8308
0.065	0.4297	0.7326
0.070	0.3976	0.6910
0.075	0.3796	0.6664
0.080	0.3942	0.6280
0.085	0.3417	0.6177
0.090	0.3400	0.5600
0.095	0.3346	0.5445
0.100	0.3095	0.5017

Table 6.13: The normalised Euclidean norms of the numerical results for the non-periodic KdV equation with the boundary conditions $u(-L,t) = a$, $u(L,t) = b$ and $u_y(L,t) = c$, given by Figure 6.20 and Figure 6.21, solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$.

The particular boundary conditions chosen according to the initial solution (6.26), generalise to the case where $a = 0$, $b > 0$ and $c < 0$. Hence for this example

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u(y,t)||_2^2 = \frac{1}{2}c^2 - bu_{yy}(L,t) - \frac{1}{3}b^3 - \frac{1}{2}u_y^2(-L,t)\,.$$

Similarly, the particular boundary conditions chosen according to the initial solution (6.27), generalise to the case where $a > 0$, $b = 0$ and $c = 0$. Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u(y,t)||_2^2 = -\frac{1}{2}u_y^2(-L,t) + au_{yy}(-L,t) + \frac{1}{3}a^2\,.$$

Given we do not know the signs of $u_{yy}(\pm L,t)$ or $u_y(-L,t)$ we cannot predict for either example how the wave should evolve, but the numerical investigation for both cases indicates strongly that the problems are dispersive in nature.

If we consider again equation (6.28), we see that the problematic terms in analysing the sign of $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u(y,t)||_2^2$ are primarily those involving $u_{yy}(\pm L,t)$. However, even if $a = b = 0$ then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u(y,t)||_2^2 = \frac{1}{2}c^2 - \frac{1}{2}u_y^2(-L,t)\,,$$

and we could still not conclude anything about $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u(y,t)||_2^2$ unless we knew how the value of $\frac{1}{2}u_y^2(-L,t)$ compared to the value of c.

6.2.3 Non-Homogeneous Time-Dependent Uncoupled Boundary Conditions

We now consider the following third order nonlinear problem

$$u_t + uu_y + u_{yyy} = 0\,, \quad u(y,0) = u_0(y)\,, \quad t > 0\,, \quad y \in [-1,1]\,,$$

$$u(-1,t) = f(t)\,, \quad u(1,t) = 0\,, \quad u_y(1,t) = 0\,,$$

where $f(t)$ is a prescribed time-dependent function. For the first half of the split-step procedure, the boundary conditions are imposed using the following polynomial trick

$$u(y,t) = \frac{1}{2}(1-y)q(y,t)\,, \qquad q(1,t) = 0\,, \quad q(-1,t) = f(t)\,.$$

The equivalent problem in terms of $q(y,t)$ is therefore given by

$$q_t(y,t) = -\frac{1}{4}(1-y)(q^2)_y(y,t) + \frac{1}{2}q^2(y,t) = f(q,t)\,.$$

Given an initial condition $u(y_j,0)$, $0 \leqslant j \leqslant N$, define $q(y_j,0) = \frac{2u(y_j,0)}{1-y_j}$, $1 \leqslant j \leqslant N-1$. The standard fourth order Runge-Kutta formulae, given by (6.13), is used to advance the solution half of the time-step:

$$
\begin{aligned}
d^{(1)} &= \frac{\Delta t}{2} f(\hat{q}_n) = -\frac{\Delta t}{8}(1-y)(q_n^2)_y + \frac{\Delta t}{4} q_n^2 \,, \\
d^{(2)} &= \frac{\Delta t}{2} f\left(\hat{q}_n + \frac{d^{(1)}}{2}\right) = -\frac{\Delta t}{8}(1-y)\left(q_n + \frac{1}{2}d^{(1)}\right)_y^2 + \frac{\Delta t}{4}\left(q_n + \frac{1}{2}d^{(1)}\right)^2 \,, \\
d^{(3)} &= \frac{\Delta t}{2} f\left(\hat{q}_n + \frac{d^{(2)}}{2}\right) = -\frac{\Delta t}{8}(1-y)\left(q_n + \frac{1}{2}d^{(2)}\right)_y^2 + \frac{\Delta t}{4}\left(q_n + \frac{1}{2}d^{(2)}\right)^2 \,, \\
d^{(4)} &= \frac{\Delta t}{2} f(\hat{q}_n + d^{(3)}) = -\frac{\Delta t}{8}(1-y)\left(q_n + d^{(3)}\right)_y^2 + \frac{\Delta t}{4}\left(q_n + d^{(3)}\right)^2 \,.
\end{aligned}
$$

Therefore

$$
q\left(y_j, t + \frac{\Delta t}{2}\right) = \left[0;\ q\left(y_j, t + \frac{\Delta t}{2}\right);\ f(t)\right], \quad 1 \leqslant j \leqslant N-1 \,,
$$

and thus

$$
u\left(y_j, t + \frac{\Delta t}{2}\right) = \frac{1}{2}(1-y_j) q\left(y_j, t + \frac{\Delta t}{2}\right), \quad 0 \leqslant j \leqslant N \,.
$$

To advance the solution the remaining half of the time step a Backward Euler formula is used:

$$
u_{yyy}(y_j, t + \Delta t) + \frac{2}{\Delta t} u(y_j, t + \Delta t) = \frac{2}{\Delta t} u\left(y_j, t + \frac{\Delta t}{2}\right),
$$

where

$$
u(-1, t + \Delta t) = f(t) \,, \quad u(1, t + \Delta t) = 0 \,, \quad u_y(1, t + \Delta t) = 0 \,.
$$

The boundary conditions are imposed by letting

$$
u(y_j, t + \Delta t) = (y_j - 1) q(y_j, t + \Delta t) + f(t)\left(\frac{y_j^2}{4} - \frac{y_j}{2} + \frac{1}{4}\right),
$$

where $q(\pm 1, t) = 0$. Hence

$$
\begin{aligned}
&(y_j - 1) q_{yyy}(y_j, t + \Delta t) + 3 q_{yy}(y_j, t + \Delta t) + \frac{2}{\Delta t}(y_j - 1) q(y_j, t + \Delta t) \\
&+ \frac{2}{\Delta t} f(t)\left(\frac{y_j^2}{4} - \frac{y_j}{2} + \frac{1}{4}\right) = \frac{2}{\Delta t} u\left(y_j, t + \frac{\Delta t}{2}\right).
\end{aligned}
$$

This equation is solved for $q\left(y_j, t+\Delta t\right)$ using the formula

$$q(y_j, t+\Delta t) = L^{-1} f\,,$$

where

$$\begin{aligned} L &= \operatorname{diag}(y_j-1)\tilde{D}_N^3 + 3\tilde{D}_N^2 + \frac{2}{\Delta t}\operatorname{diag}(y_j-1)\tilde{D}_N^0\,, \\ f &= \frac{2}{\Delta t}u\left(y_j, t+\frac{\Delta t}{2}\right) - \frac{2}{\Delta t}f(t)\left(\frac{y_j^2}{4} - \frac{y_j}{2} + \frac{1}{4}\right). \end{aligned}$$

The solution $u\left(y_j, t+\Delta t\right)$ is therefore given by

$$u(y_j, t+\Delta t) = (y_j-1)q(y_j, t+\Delta t)\left(\frac{y_j^2}{4} - \frac{y_j}{2} + \frac{1}{4}\right).$$

The spatial domain was discretised by 128 grid points, a time step $\Delta t = 0.00001$ was used along with $f(t) = \sin(t)$ and the initial solution was taken to be

$$u(y,0) = 3A^2 \operatorname{sech}^2\left(\frac{1}{2}Ay\right), \tag{6.29}$$

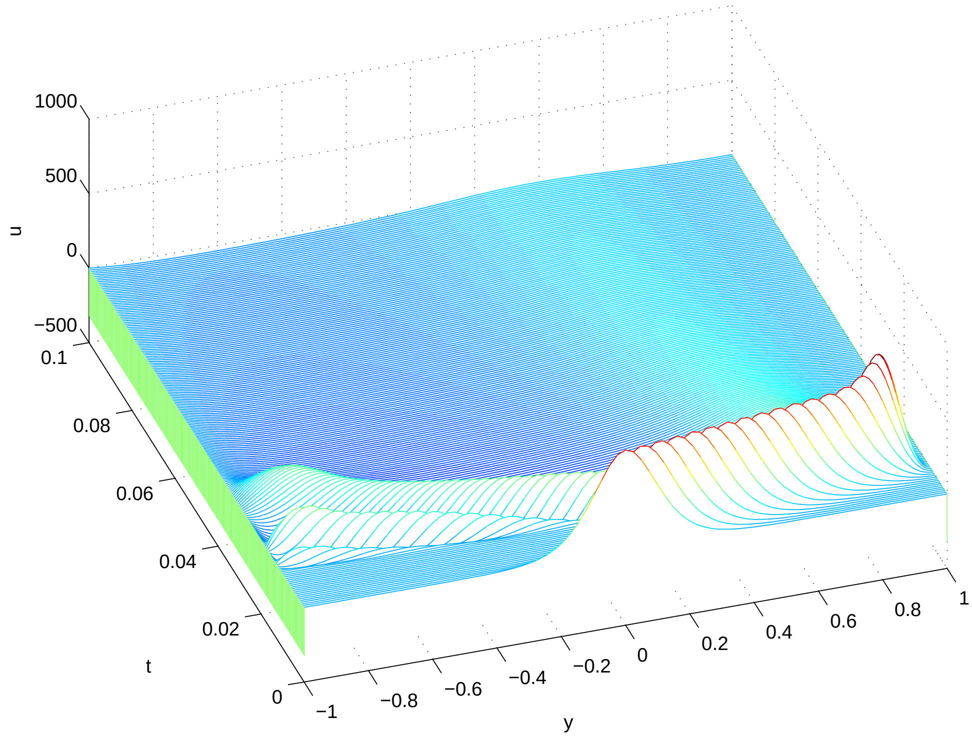
where $A = 15$. The output obtained is given by Figure 6.22(a), which is then rotated and presented for time up to $t = 0.025$ only, to produce Figure 6.22(b).

The normalised numerical Euclidean norms were calculated at every time interval, and Table 6.14 shows the values obtained starting with the initial solution, and thereafter in time intervals of $t = 0.005$.

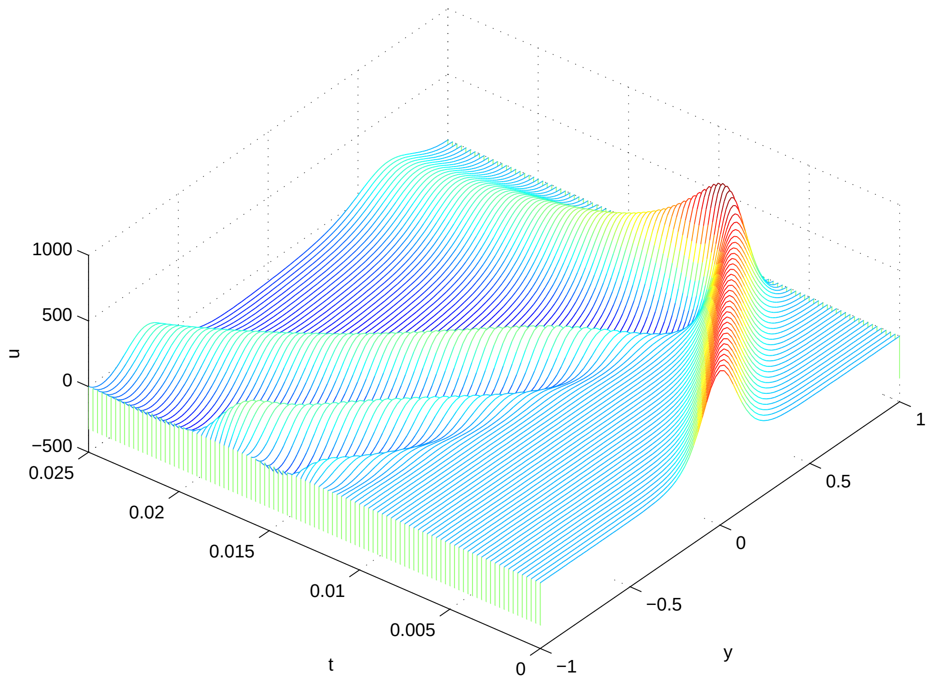
If we consider the formula for the Euclidean norm, given by (6.7) for the domain $[-1, 1]$, then the boundary conditions $u(-1,t) = \sin(t)$, $u(1,t) = 0$ and $u_y(1,t) = 0$ reduce equation (6.7) to the following

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u(y,t)||_2^2 = -\frac{1}{2}u_y^2(-1,t) + \sin(t)u_{yy}(-1,t) + \frac{1}{3}(\sin(t))^3\,,$$

and therefore, with just the knowledge of the boundary conditions, we cannot conclude anything about the sign of $\frac{\mathrm{d}}{\mathrm{d}t}||u(y,t)||_2^2$. However, the numerical analysis shows the problem to be highly dispersive in nature.



(a)



(b)

Figure 6.22: The non-periodic KdV equation with the time-dependent boundary conditions $u(-1,t) = \sin(t)$, $u(1,t) = 0$ and $u_y(1,t) = 0$ and the initial solution (6.29), solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$.

t	Single Soliton
0	1.0000
0.005	1.0875
0.010	1.1426
0.015	0.9268
0.020	0.7717
0.025	0.5755
0.030	0.5881
0.035	0.4148
0.040	0.4114
0.045	0.3748
0.050	0.3328
0.055	0.2973
0.060	0.2681
0.065	0.2438
0.070	0.2235
0.075	0.2062
0.080	0.1914
0.085	0.1785
0.090	0.1673
0.095	0.1573
0.100	0.1485

Table 6.14: The normalised Euclidean norms of the numerical results for the non-periodic KdV equation with the time-dependent boundary conditions $u(-1,t) = \sin(t)$, $u(1,t) = 0$ and $u_y(1,t) = 0$, given by Figure 6.22, solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$.

6.2.4 Coupled Boundary Conditions

We now construct a numerical method for approximating the solution of the KdV equation with coupled boundary conditions. The problem that we consider is given by

$$u_t + uu_y + u_{yyy} = 0\,, \quad u(y,0) = u_0(y)\,, \quad t > 0\,, \quad [-L, L]\,,$$

$$u(-L,t) = 0\,, \quad u(L,t) = 0\,, \quad u_y(L,t) = \alpha u_y(-L,t)\,.$$

This problem was considered in Section 3.2.4 where it was explained that α can take any value. It is the most difficult of the examples examined in terms of numerical modelling.

The split-step method of Tappert is used to solve this problem, and the imposition of the coupled boundary condition is restricted to the linear step. Hence at each time level, the solution is advanced according to the nonlinear terms of the equation, but only the Dirichlet boundary conditions are imposed. The solution that then results is treated as the initial solution for the advancement of the solution the final half of the step according to the linear terms, and all three boundary conditions are imposed. We therefore anticipate a lack of accuracy from this scheme.

The problem is transformed to $[-1, 1]$ by letting $x = y/L$ to give

$$u_t + \tfrac{1}{L}uu_x + \left(\tfrac{1}{L}\right)^3 u_{xxx} = 0\,, \quad u(x,0) = u_0(x)\,, \quad t > 0\,, \quad [-1, 1]\,, \tag{6.30a}$$

$$u(-1,t) = 0\,, \quad u(1,t) = 0\,, \quad u_x(1,t) = \alpha u_x(-1,t)\,. \tag{6.30b}$$

For the first half of the step, the fourth-order Runge-Kutta formula is applied to $u(x, 0)$, defined on the interior grid points of the interval $[-1, 1]$, according to the formula

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{6}\left(d^{(1)} + 2\left(d^{(2)} + d^{(3)}\right) + d^{(4)}\right)\,,$$

where

$$\begin{aligned}
d^{(1)} &= -\frac{\Delta t}{4L}\left(u_n^2(x,t)\right)_x\,,\\
d^{(2)} &= -\frac{\Delta t}{4L}\left(u_n(x,t) + \frac{1}{2}d^{(1)}\right)_x^2\,,\\
d^{(3)} &= -\frac{\Delta t}{4L}\left(u_n(x,t) + \frac{1}{2}d^{(2)}\right)_x^2\,,\\
d^{(4)} &= -\frac{\Delta t}{4L}\left(u_n(x,t) + d^{(3)}\right)_x^2\,.
\end{aligned}$$

This achieves the solution $u\left(x, t+\frac{\Delta t}{2}\right)$. The solution is advanced the remaining half of the time step using the linear part of the equation and a Backward Euler scheme, according to the equation

$$u_t + \left(\frac{1}{L}\right)^3 u_{xxx} = 0\,.$$

Advancing the solution half a time-step, implies that

$$\left(D_N^3 + L^3\left(\frac{2}{\Delta t}\right)D_N^0\right)u\left(x, t+\Delta t\right) = L^3\left(\frac{2}{\Delta t}\right)u\left(x, t+\frac{\Delta t}{2}\right)\,.$$

Hence, the problem to be solved is of the form

$$\left(D_N^3 + L^3\left(\frac{2}{\Delta t}\right)D_N^0\right)\mathbf{u}_j = \mathbf{f}_j\,, \quad 0 \leqslant j \leqslant N\,,$$

where $\mathbf{u}_j = u\left(x_j, t+\Delta t\right)$ and $\mathbf{f}_j = L^3\left(\frac{2}{\Delta t}\right)u\left(x_j, t+\frac{\Delta t}{2}\right)$. The procedure for imposing all three boundary conditions, follows identically the method applied to equation (5.6), by replacing D_N^3 with $\left(D_N^3 + L^3\left(\frac{2}{\Delta t}\right)D_N^0\right)$. This achieves the solution $u(x, t+\Delta t)$.

The analysis of the Euclidean norm yields expression (3.44), evaluated for the domain $[-L, L]$, and shows that the stability of the solution $u(x,t)$ depends on the value of the constant α:

- $|\alpha| > 1 \;\Rightarrow$ the energy of the solution $u(x,t)$ grows as $t \to \infty$,
- $\alpha = 1 \;\Rightarrow$ the energy of the solution $u(x,t)$ is conserved as $t \to \infty$,
- $|\alpha| < 1 \;\Rightarrow$ the energy of the solution $u(x,t)$ is dispersed as $t \to \infty$.

The program was run using 128 grid points and a time-step of $\Delta t = 0.00001$. The initial soliton solution, given by

$$u(x,0) = 3A^2\text{sech}^2\left(\frac{1}{2}A(x\pi - 1)\right)\,, \tag{6.31}$$

where $A = 15$, was used in all cases and the results plotted on the domain $[-\pi, \pi]$. The outputs, for different values of α, are given by Figures 6.23, 6.24, 6.25 and Figure 6.26. The normalised Euclidean norms, for all the cases, are given in Table 6.15, for time increments of 0.005 up to the time $t = 0.070$.

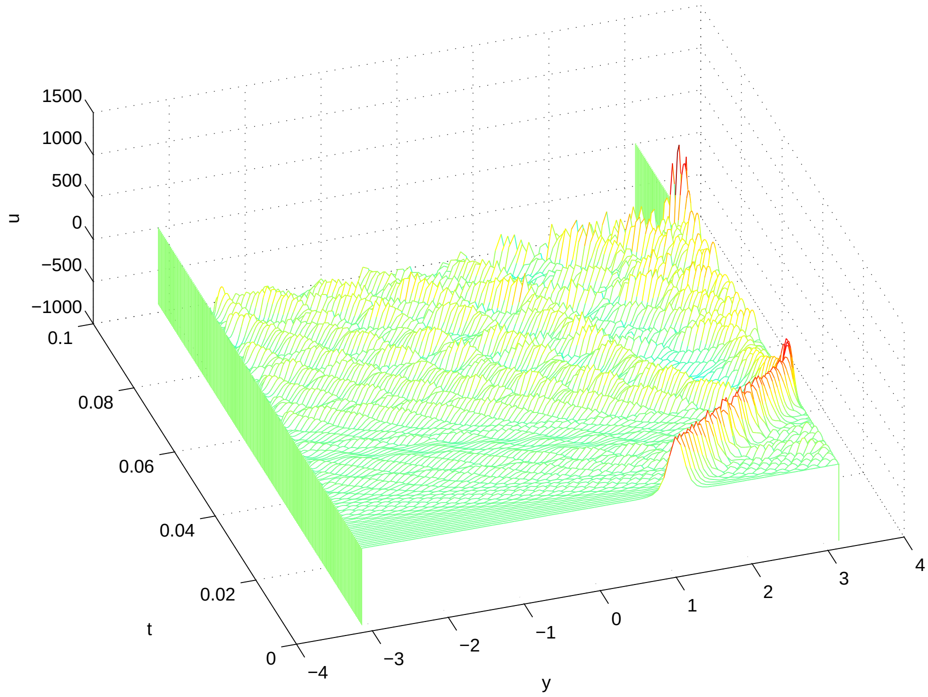


Figure 6.23: The non-periodic KdV equation with initial solution (6.31) and the coupled boundary conditions (6.30b) with $\alpha = -2$, solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$. The soliton reaches the boundary when $t \approxeq 0.0195$.

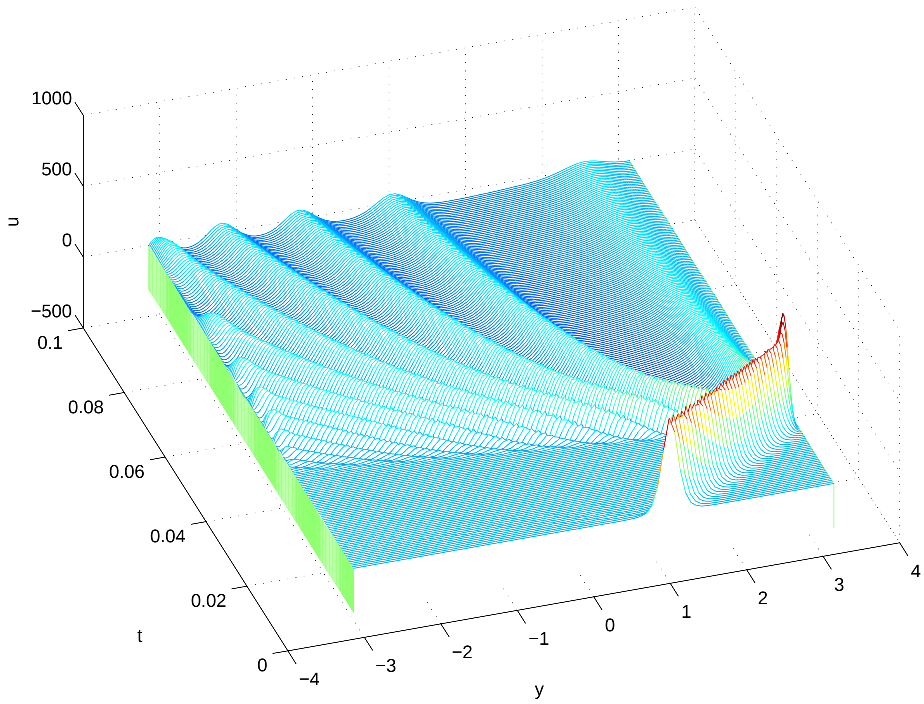


Figure 6.24: The non-periodic KdV equation with initial solution (6.31) and the coupled boundary conditions (6.30b) with $\alpha = 0$, solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$. The soliton reaches the boundary when $t \approxeq 0.0185$.

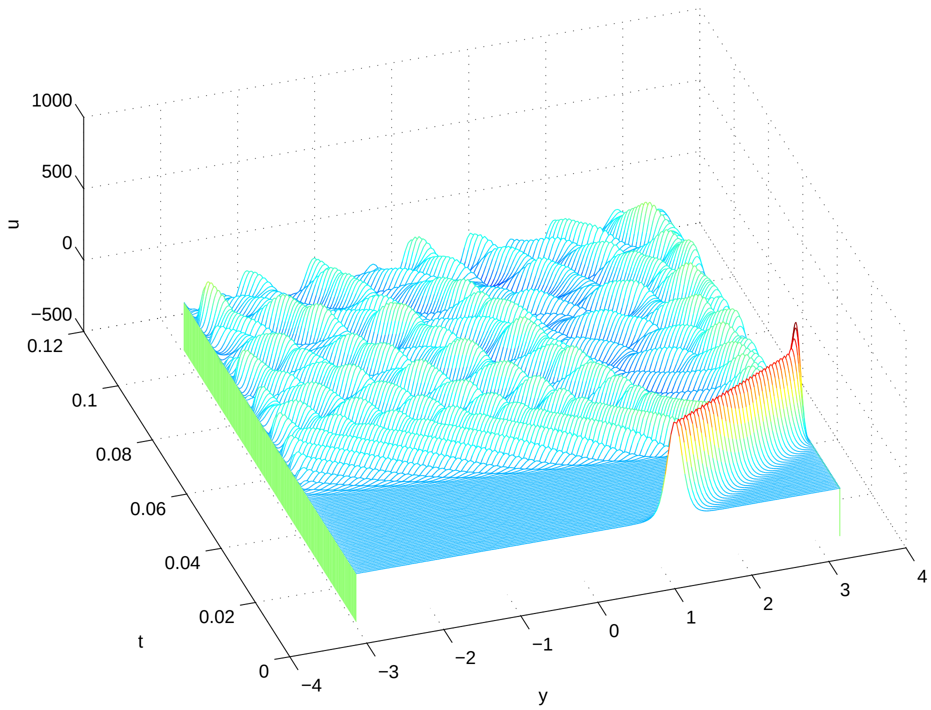


Figure 6.25: The non-periodic KdV equation with initial solution (6.31) and the coupled boundary conditions (6.30b) with $\alpha = 1$, solved using Tappert's split-step method with $N = 256$ and $\Delta t = 0.000005$. The soliton reaches the boundary when $t \approxeq 0.0185$.

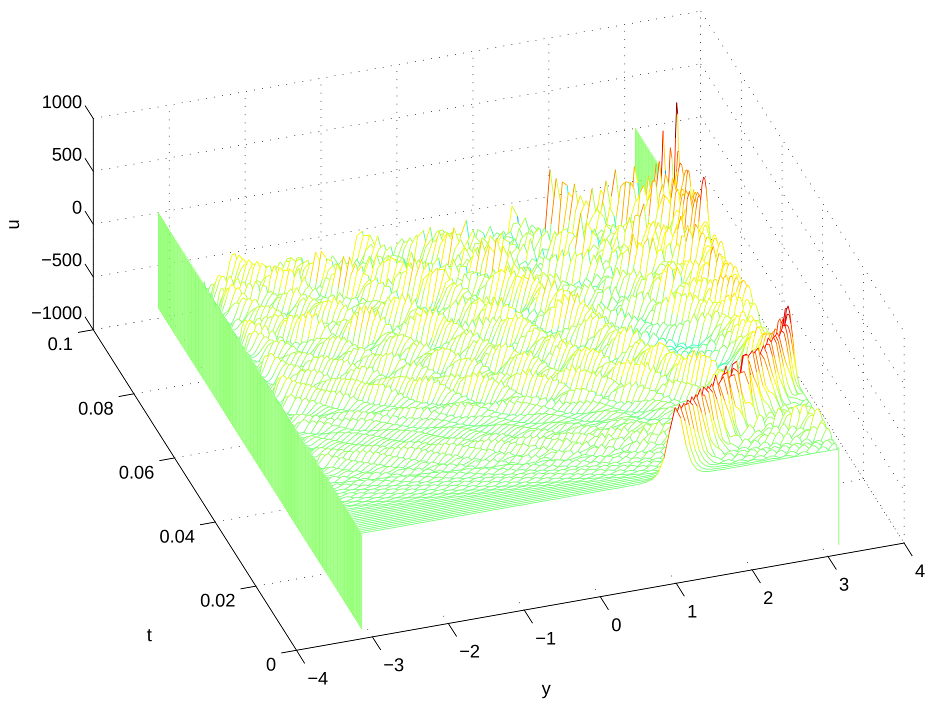


Figure 6.26: The non-periodic KdV equation with initial solution (6.31) and the coupled boundary conditions (6.30b) with $\alpha = 2$, solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$. The soliton reaches the boundary when $t \approxeq 0.0190$.

t	$\alpha = -2$	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$
0	1.0000	1.0000	1.0000	1.0000
0.005	1.1189	1.0312	1.0380	1.0628
0.010	1.1429	1.1022	1.1179	1.1786
0.015	1.2877	1.2825	1.3182	1.2954
0.020	1.5816	1.4981	1.5361	1.5112
0.025	1.1281	1.0940	1.1490	1.1827
0.030	1.0206	0.9583	1.0212	1.0228
0.035	0.9760	0.9048	0.9943	1.0623
0.040	1.2215	0.8639	0.9975	1.1312
0.045	1.2089	0.8071	0.9934	1.1855
0.050	1.3162	0.7908	1.0293	1.3926
0.055	1.3996	0.7074	1.0226	1.3528
0.060	1.4683	0.6809	1.0214	1.6641
0.065	1.5837	0.6344	0.9425	1.6473
0.070	1.7094	0.5807	1.0109	1.7304

Table 6.15: The normalised Euclidean norms of the numerical results for the non-periodic KdV equation with the single soliton initial solution (6.31) and the coupled boundary conditions $u(\pm 1, t) = 0$ and $u_x(1, t) = \alpha u_x(-1, t)$, given by Figures 6.23, 6.24, 6.25 and Figure 6.26, solved using Tappert's split-step method.

The program was then repeated under the same conditions, but with the double soliton initial solution of the form

$$u(x, 0) = 3A^2\text{sech}^2\left(\frac{1}{2}Ax\pi\right) + 3B^2\text{sech}^2\left(\frac{1}{2}B(x\pi - 1)\right), \tag{6.32}$$

where $A = 15$ and $B = 10$. The outputs are given by Figures 6.27, 6.28, 6.29 and Figure 6.30 and the numerical normalised Euclidean norms are tabulated in Table 6.16.

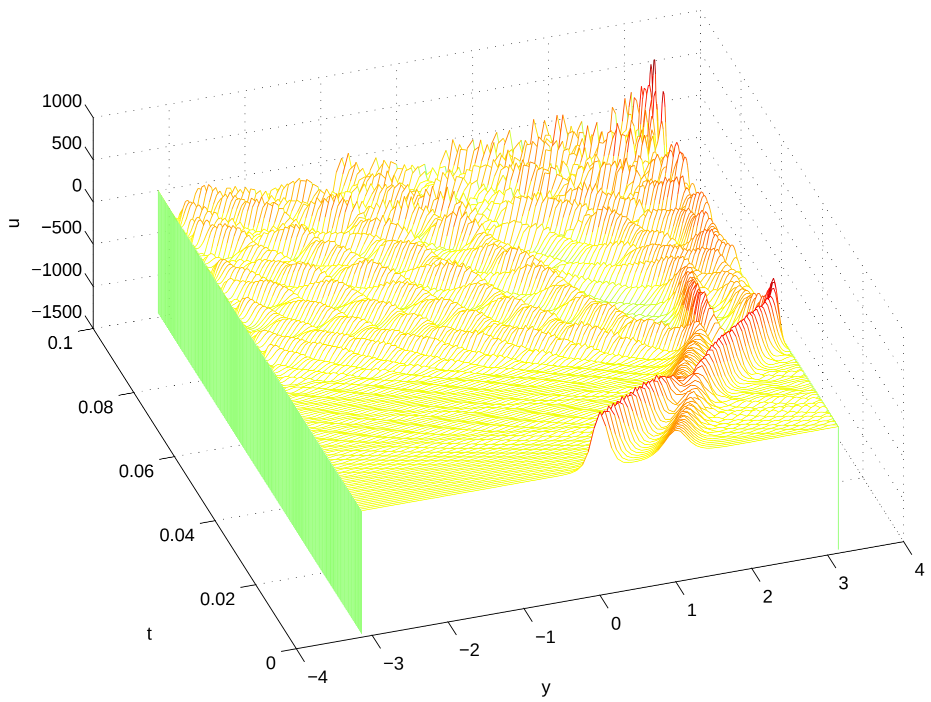


Figure 6.27: The non-periodic KdV equation with initial solution (6.32) and the coupled boundary conditions (6.30b) with $\alpha = -2$, solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$. The solitons first collide when $t \approxeq 0.0120$ and then collide with the boundary when $t \approxeq 0.0255$.

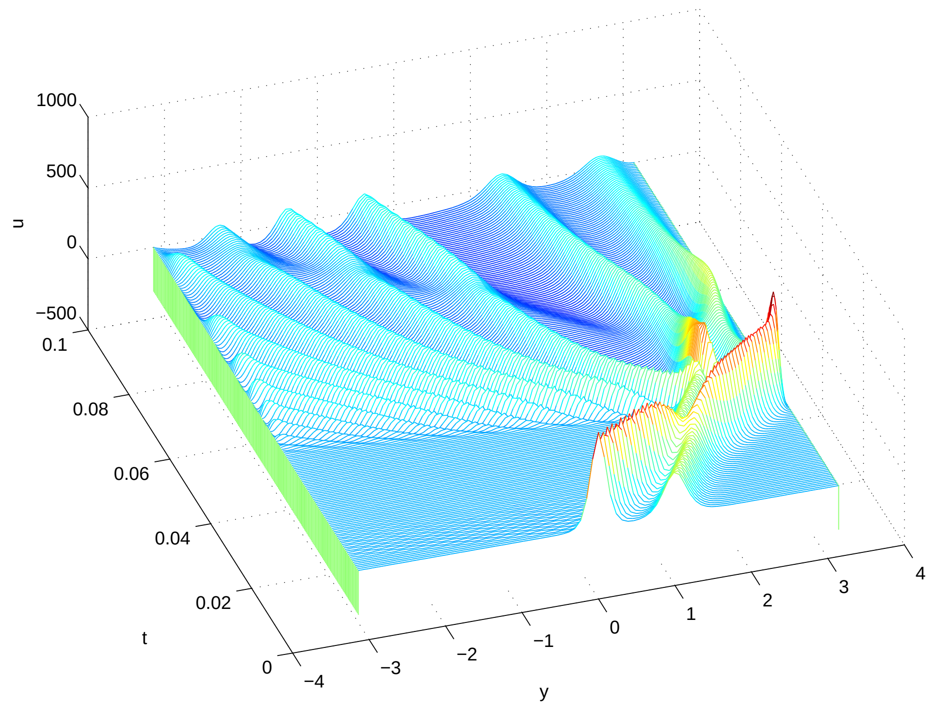


Figure 6.28: The non-periodic KdV equation with initial solution (6.32) and the coupled boundary conditions (6.30b) with $\alpha = 0$, solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$. The solitons first collide when $t \approxeq 0.0120$ and then collide with the boundary when $t \approxeq 0.0255$.

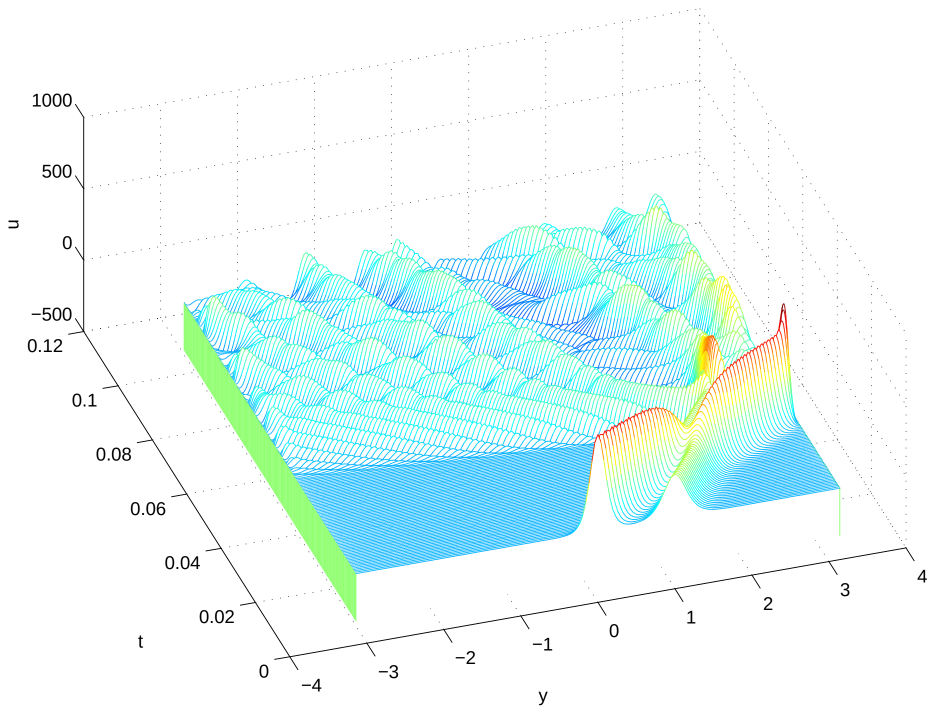


Figure 6.29: The non-periodic KdV equation with initial solution (6.32) and the coupled boundary conditions (6.30b) with $\alpha = 1$, solved using Tappert's split-step method with $N = 256$ and $\Delta t = 0.000005$. The solitons first collide when $t \approxeq 0.0120$ and then collide with the boundary when $t \approxeq 0.0255$.

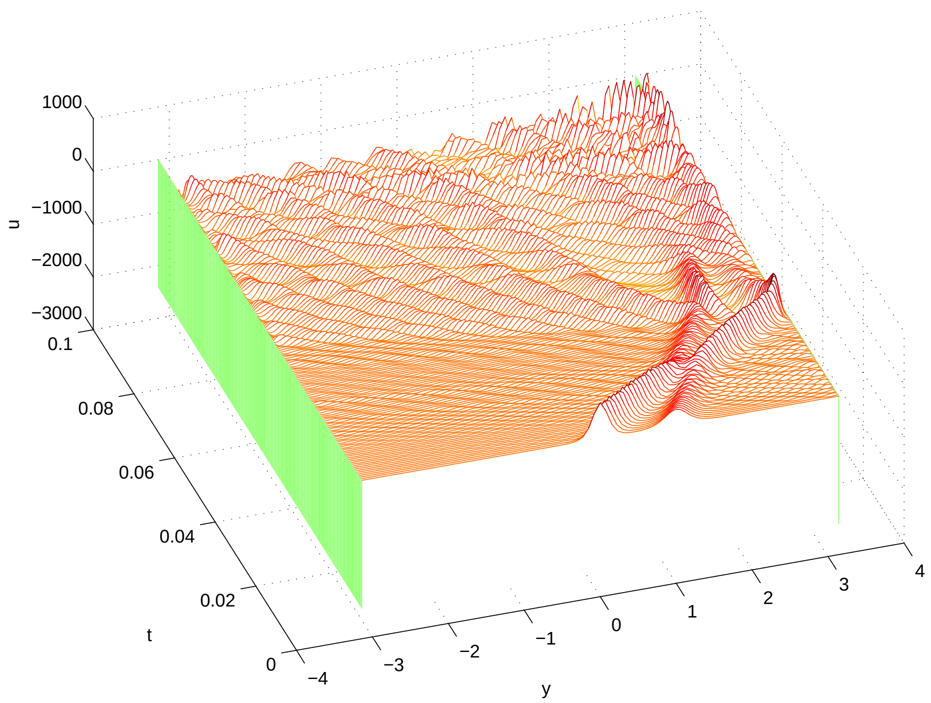


Figure 6.30: The non-periodic KdV equation with initial solution (6.32) and the coupled boundary conditions (6.30b) with $\alpha = 2$, solved using Tappert's split-step method with $N = 128$ and $\Delta t = 0.00001$. The solitons first collide when $t \approxeq 0.0120$ and then collide with the boundary when $t \approxeq 0.0260$.

t	$\alpha = -2$	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$
0	1.0000	1.0000	1.0000	1.0000
0.005	1.0033	1.0004	1.0049	1.0058
0.010	1.0267	1.0198	1.0282	1.0256
0.015	1.0752	1.0736	1.0854	1.0795
0.020	1.1763	1.1756	1.1979	1.1767
0.025	1.5182	1.5184	1.5820	1.5148
0.030	1.2036	1.2007	1.2413	1.2057
0.035	1.0410	1.0097	1.0685	1.0033
0.040	0.9894	0.9512	1.0274	0.9565
0.045	1.0095	0.9550	1.0600	1.0040
0.050	1.1485	0.9737	1.1115	1.0690
0.055	1.1776	0.9340	1.0668	1.1939
0.060	1.1604	0.8836	1.0657	1.2783
0.065	1.2678	0.8173	1.0201	1.3533
0.070	1.3998	0.7477	1.0403	1.4246
0.075	1.5278	0.7085	0.9640	1.4112
0.080	1.4182	0.6862	0.9540	1.8814
0.085	2.0672	0.6398	0.9961	1.8692
0.090	2.1647	0.6447	0.9348	4.6602

Table 6.16: The normalised Euclidean norms of the numerical results for the non-periodic KdV equation with the double soliton initial solution (6.32) and the coupled boundary conditions $u(\pm 1, t) = 0$ and $u_x(1, t) = \alpha u_x(-1, t)$, given by Figures 6.27, 6.28, 6.29 and Figure 6.30, solved using Tappert's split-step method.

The cases where $\alpha = 0$, correspond directly to the problem with boundary conditions $u(\pm L, t) = 0$ and $u_y(L, t) = 0$ studied earlier, and the outputs given by Figure 6.24 and Figure 6.28 can be compared to Figure 6.17 and Figure 6.18 respectively. The normalised Euclidean norms for $\alpha = 0$, in Table 6.15 and Table 6.16, correspond to the general case where $|\alpha| < 1$, and demonstrate the dissipative nature of the solution under this condition. The single soliton travels uniformly until reaching the boundary at time $t \approxeq 0.0185$, and thereafter continues to lose energy and disperse. The results show that the Euclidean norm increases in value up until the collision with the boundary. The double solitons travel with constant speed and direction, pass through each others path at time $t \approxeq 0.0120$ and continue to travel uniformly, with increasing Euclidean norms, until the collision with the boundary at time $t \approxeq 0.0260$. The collisions with the boundaries occur at exactly the same times for the equivalent programs with outputs given by Figure 6.17 and Figure 6.18 and coincide, for all cases, with the value of the Euclidean norm being at its maximum. For both the single and double soliton solutions, after the collision with the boundary the energy is gradually dispersed.

If we compare the results for $\alpha = 0$ in Table 6.15 and Table 6.16, with the numerical results given in Table 6.8, we see that the rate of dispersion is greater from this scheme that uses the Backward Euler approach for the linear half of the time-step. The results shown in Table 6.8, were obtained from applying the split-step method with a Crank-Nicolson scheme for advancing the solution according to the linear terms.

When $\alpha = 1$, the boundary conditions are $u(-L,t) = u(L,t) = 0$ and $u_y(-L,t) = u_y(L,t)$, and are very nearly periodic. However, they are not periodic and it is this crucial difference that makes this a much harder problem to treat numerically. Figure 6.25 and Figure 6.29, along with the norm calculations, confirm the conservative nature of the solution. In both cases, for the single and double soliton initial solutions, the program was run using 256 grid points and $\Delta t = 0.000005$. The norm values increase up until the collision of the solutions with the boundary, indicating an increase in energy. For the single soliton, given by (6.31), this collision occurs when $t \approxeq 0.0185$, and for the double soliton solution, given by (6.32), the collision with the boundary occurs when $t \approxeq 0.0255$. After the collision, the values decrease slightly but stay consistently close to 1, sufficiently satisfying the conservation of energy law, given by (6.9).

When $|\alpha| > 1$, the numerical calculations for $\alpha = -2$ and $\alpha = 2$, demonstrate the increasing energy of the wave and the instability of the schemes. For the single soliton solution, the numerical results for $\alpha = -2$ and $\alpha = 2$ become unstable for $t > 0.070$. Similarly, for the double soliton, the solution blows-up by the time $t = 0.095$ for both $\alpha = \pm 2$. We stress once again that the inaccuracy of the scheme is due to the imposition of only the Dirichlet boundary conditions for the first nonlinear step.

Chapter 7

Conclusions and Further Work

This work started from the initial idea of a study of the behaviour of the solutions of the KdV equation on a bounded domain. However, it quickly became clear that first of all it was necessary to acquire a detailed knowledge of the solution of the linearised problem, subject to a variety of boundary conditions.

Although these are classical problems and many second order examples are extensively treated in the literature, the classical methods are not naturally or easily generalisable to higher order, and in particular to odd order problems. We were led to their investigation (or re-investigation) because of the appearance of a new method for their study, due to Fokas, which offered an alternative way to consider them, as well as an alternative, integral representation of their solution, valid in general.

The research therefore naturally split into two parts: the analysis of the solution of the two-point boundary value problems for linear evolution PDEs, using the new method of Fokas and comparing it with the available classical results, and the problem of the numerical imposition of boundary conditions for linear and integrable nonlinear evolution PDEs. In particular, interesting results were obtained for third order problems, which paved the way for the numerical treatment of the KdV equation on a finite domain.

For second order two-point boundary value problems, it has been shown that, in agreement with classical theory, the method can be used to derive the infinite series representation of the solution, which always exists. This series solution can be achieved in two ways. Firstly, the algorithmic construction of the solution, and the analysis of the system of global relations, yields the appropriate basis of eigenfunctions. Using additional information on this basis, coming from classical functional analysis, one then obtains the solution as an infinite series. Alternatively, the contours of integration, in the integral representation of the solution, can be deformed to the real line. During this process one acquires a series term due to the explicit computation of the residues at the zeros of a certain function $\Delta(k)$, which is always a linear combination of exponential functions.

For third order problems the integral representation of the solution is only equivalent to an infinite discrete series, in the cases for which the boundary conditions couple the two end points of the interval. This has been illustrated for the simple case in which the boundary conditions are periodic, quasi-periodic and also for a more general case of coupled boundary conditions. It has been shown how in general, the derivation of the solution representation depends on the location in the complex k-plane of the zeros of the determinant function $\Delta(k)$. For uncoupled boundary conditions, we showed that it is not possible to deform the integral representation of the solution to an infinite series. However, this integral representation of the solution can alternatively be written as the sum of an integral term along the real line and a complex contour in the upper half complex k-plane, and a series term due to the explicit computation of the residue contributions at the zeros that lie in the lower half complex k-plane.

Finally, we studied, in some detail, the spectral representation of two-point boundary value problems for fourth order linear evolution PDEs. The examples presented illustrate the two classes of problems that arise in the case of a fourth order differential operator, characterised by whether or not the operator is self-adjoint. It has been shown that regardless of whether or not the boundary conditions give rise to a self-adjoint operator, the Fokas transform method can be used to achieve the solution as an infinite discrete series.

In summary, the only cases presented, for which the Fokas transform method fails to achieve a series representation of the solution, are the third order problems, for which the boundary conditions are uncoupled. Of all the cases considered, these are the only ones for which the classical theory does not have a positive answer: there are to our knowledge no results implying that such a series representation must exist. In all other cases the method presents a fast and efficient algorithmic approach, for the derivation of the integral representation of the solution, involving complex contours of integration, along with the infinite discrete series representation of the solution.

As a final result, it has been proven that the effective discrete spectrum of a PDE boundary value problem, defined as the set of zeros of the determinant function $\Delta(k)$, coincides with the classical discrete spectrum of the operator associated with the PDE, equipped with the same boundary conditions. A general result regarding the location in the complex k-plane of the zeros of the determinant function $\Delta(k)$, has also been addressed, and generalised for the n^{th} order case.

The secondary interest of the research presented has been the construction of numerical schemes, using spectral methods, for modelling two-point linear and nonlinear boundary value problems. In particular, we focused on the KdV equation since in this case, due to a lack of symmetry, it is more difficult to impose boundary conditions and we had to adopt a series of tricks to do so.

For third and fourth order linear ODE boundary value problems, the primary interest of the numerical schemes has been on the explicit imposition of a wide variety of boundary conditions, using either a simple polynomial trick, or via the manipulation of the specific rows and columns of the Chebyshev differentiation matrices. This approach is very versatile and all of the schemes that have been presented can be adapted to accommodate a wide variety of boundary conditions, that have not been discussed in this work. For the fourth order linear ODE boundary value problems, the work follows closely the approach outlined by Weideman [51].

This approach for the imposition of the boundary conditions is not the only method that can be used, and we have also discussed the implicit imposition of boundary conditions by means of discrete numerical transforms. The examples of the discrete sine transform and the discrete cosine transform have been presented, and in the case of Dirichlet or Neumann boundary conditions, this alternative approach offers a fast and efficient method for solving boundary value problems. Indeed, it is clear that if the boundary conditions are periodic, Dirichlet or Neumann in form, then they can be implicitly imposed by using the discrete Fourier, sine or cosine transforms respectively. It is natural then to seek transforms that can be used to impose implicitly more general boundary conditions. Since the method of Fokas, used in this work, yields constructively the appropriate eigenfunction basis, it is possible that this kind of approach will yield an alternative numerical way to solve boundary value problems, by using first appropriate approximation techniques for the eigenvalues (that cannot be computed exactly).

Another direction of investigation, departing from the usual series representation starting point, is the idea of the direct numerical evaluation of the integral representation which, we stress once more, can *always* be derived. Preliminary results in this direction are very promising, and we intend to consider this possibility further for the case of the boundary value problem analysed in this work.

List of Figures

List of Tables

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