Analysis and Computation of a Simple Glacier Model using Moving Grids

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Dale Partridge

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Abstract

In this dissertation we are concerned with the study of glaciers, and analysing a simple model that determines the basis for the glacier to flow. We set up a PDE model with diffusion and source terms in one-dimension. From this model we define a nodal velocity associated with mass conservation, leading to the movement of a grid.

We assess the velocity using two methods, the first by assuming that a subdomain will hold the same properties as the whole domain, and the second by assuming the normalised ice volume remains constant in time. The analytical and computational benefits of each is considered.

Next we allow for surface elevation, using the subdomain assumption. In addition the velocity will be changed to allow for the effect of basal sliding. The velocity satisfies a Burgers-like equation, and the theory of characteristics and shocks is used to try and determine our aim of finding out the movement time.

Finally, consideration will be given to where this work can be taken next.

Acknowledgements

This year has reignited my passion for maths, and I have all the lecturers and my peers to thank for that! Also without funding from the NERC none of this would be possible so thank you to them. Finally special thanks to my supervisor Mike Baines, who is a complete legend and the most helpful guy imaginable!

Declaration

I confirm that this is my own work and the use of all materials from other sources have been properly and fully acknowledged.

Signed.................................. Date............................

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Chapter 1

Introduction

Glaciers and Ice Sheets are a hot topic at the moment with global warming causing the considerable retreat of ice. Sometimes in order to understand them better we need to take a step back and look at what drives the movement and sliding of the glacier in the first place. This is the basis for this dissertation.

To do this we start by looking at glaciers from a physical point of view with no mathematics involved. Then in Chapter 3 we consider a standard simple PDE model of glacier movement. We then define a velocity, which can be achieved in a number of ways. Chapter 3 will therefore act as a hub to the next few chapters.

The ultimate goal of this dissertation will be to uncover the conditions the glacier needs to meet before it begins to move, where we note that glaciers experience a waiting time, (see Stojsavljevic[10]), as with other non-linear diffusion equation solutions. Over the subsequent chapters two different methods for estimating the velocity on a moving grid will be analysed and computed, with results and comments as to the benefits of each.

Next we shall look at the impact of making the model more complex, with the addition of surface elevation and basal sliding, using the first velocity method.

In Chapter 7 we derive a Burgers-like equation for the velocity. We then examine the application of characteristics theory and the potential benefits such results could supply, with the aim of estimating a shock, leading to movement at the boundary of the glacier. Finally, consideration is given to ways the model may be extended, and the impacts these extensions may have on the results we have concluded, leading the way to potential further work to be undertaken in the topic.

For most of the project we are not concerned with physical values for the variables, but more with the methodology and the theory behind why we might see a certain behaviour. Hence computation will be performed with initial conditions chosen purely because they meet the required boundary conditions, and ran for a sufficient length of time.

Chapter 2

The Physics of Glaciers

The term glacier is well known, though most only know that it has something to do with ice, so what exactly is one? The official definition from the AMS glossary [14] says:

> *"'A glacier is a mass of land ice, formed by the further recrystallisation of firn, flowing continuously from higher to lower elevations."'*

So according to the definition, a glacier is essentially a river of ice flowing down a mountain, where the ice is made up from firn (German for old snow) recrystallising. Knowing how they form is important if we are interested in how they move. In order for glaciers to form they first need enough snow over the winter period to be able to survive through the summer, i.e. more accumulation of snow than is lost through melting and evaporation. This needs to be repeated over a number of succesive years, and as more snow builds up, the weight increases and pressure compresses the firn into ice. Once this ice is thick enough, gravity, amongst other forces, causes the ice to flow down the mountain. This is a long, complex process which takes less time in regions where temperature changes quicker, such as the Alps and North America.

On a global scale, ice quantities vary considerably. At present glaciers make up around 2% of the Earths water, but during an ice age this vastly increases. Either way they have a large impact on the climate system, and are becoming increasingly affected by climate change. If all this ice melted into the oceans, there would be a sea level rise of around 70m. We are interested in glaciers for more than just the climate change reasons, as they can have a large effect on the local terrain, causing events such as landslides and flash floods.

One of the main things we need to take into consideration when modelling glaciers is the idea of mass balance, and where on the glacier mass is gained or lost. In the top zone of Figure 2.1 the accumulation of snow is greater than the ablation (melting/evaporation), so the mass increases. Further down the mountain the ablation becomes greater than the accumulation, and the mass decreases. However ice can build up in the lower zone due to ice flow coming from the glacier's upper zone.

The front-most end of the glacier is known as the snout, which rarely moves straight away; it waits until the velocity behind it is great enough to push it down the mountain. It is this feature which is of special interest in this dissertation.

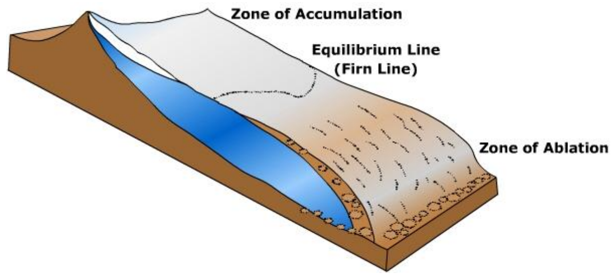


Figure 2.1: *Glacier Zones*

Each individual glacier tends to have reasonably steady flow, but between glaciers this flow can vary considerably. Some glaciers are even prone to surges, where they barely move for years before advancing very quickly; generally a few kilometres over a few months.

One of the nice things about glaciers is that they move similarly to a viscous fluid, though with a very high viscosity of around $10^{12} Pa.s$, and for comparison this is roughly 10^{15} times that of water [8]. However we cannot use viscous theory to measure flow, since glaciers are unique in experiencing something called *basal sliding*. As the ice is flowing down the mountain, friction is generated, melting the ice which makes contact with the surface, causing the base to slide. This can also be caused by geothermal heat below the surface. These factors can be used to set up a mathematical model.

Chapter 3

A One-Dimensional Model

Consider a glacier on a flat bed occupying the region $[0, b(t)]$ as shown in Fig.3.1. Let H be the thickness of the ice. At the ends of this domain we have two boundary conditions, $H = 0$ at $x = b(t)$ and $\frac{\partial H}{\partial x} = 0$ at $x = 0$.

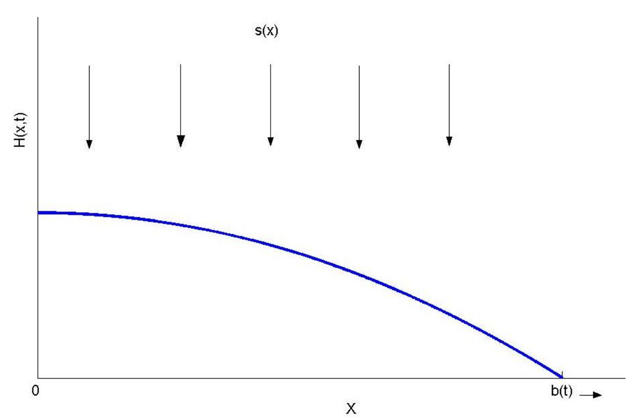


Figure 3.1: *One-Dimensional Domain*

As a starting point we are going to derive a simple model for glaciers. This takes the form

$$\begin{aligned} \frac{\partial H(x,t)}{\partial t} &= \frac{\partial}{\partial x}\left[cH(x,t)^5 H_x(x,t)^3\right] + s(x) \\ &= (D(H,H_x))_x + s(x) \end{aligned} \tag{3.1}$$

where we define $H(x,t)$ as the thickness of the ice and $s(x)$ as a source term (in our case we shall call this the snow term). The constant c is a parameter taken to be 0.000022765 (from derivation). This model was proposed by Oerlemans [2] in 1984.

3.1 Model Derivation

In one dimension the continuity equation for ice can be written as

$$\frac{\partial H}{\partial t} = -\frac{\partial (Hu)}{\partial x} + s - s_b, \tag{3.2}$$

where H is the ice thickness, s is the accumulation rate of snow, s_b is the basal melting rate and u is the vertical mean velocity. Initially we shall assume that the basal melting rate is zero.
The vertical mean velocity u is given by [1]

$$u = \frac{2AH}{n+2}\tau_{dx}^n,$$

with τ_{dx} the stress term, and parameters A and n taken from Glen's flow law, which is an established general law for steady state ice deformation [3].

From Van Der Veen[4] the driving stress is given by

$$\tau_{dx} = -\rho g H \frac{\partial h}{\partial x},$$

with ρ the ice density, g representing gravity, and h representing the ice thickness plus the surface elevation. On a flat bed there is no surface elevation so we may put $h = H$. Putting all of these terms together we get an equation for the vertical mean velocity

$$u = -\frac{2AH}{n+2}\rho^n g^n H^n \frac{\partial H}{\partial x}^n.$$

Most of the parameters in the model may be set as constant to simplify the model as much as possible, giving us

$$u = cH^{n+1}\frac{\partial H}{\partial x}^n.$$

From Roberts [1] we set $c = 0.000022765$. Putting the velocity back in to equation 3.2 we get the model equation

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[cH^5 H_x^3\right] + s(x), \tag{3.3}$$

which incorporates non-linear diffusion and a source term.

3.2 Initial Conditions

Initially, $x \in [0, 1]$, (i.e. $b(0) = 1$), and the ice thickness is defined as

$$H = (1 - x^2)^\alpha, \tag{3.4}$$

where α here is set to 1 in the first instance, but will come into play later on. The form (3.4) is chosen since it satisfies the boundary conditions, but it is clearly not unique. The snow term will be approximated for all time by a linear function

$$s = e(1 - dx), \tag{3.5}$$

where d & e are the snow parameters, typically set to be 0.5 [1]. This results in a positive snow term up until $x = 2$, after which ablation takes over and the snow term becomes negative, making it harder for the glacier to survive further down the mountain.

An interesting physical concept to begin with is to look at what happens to the integral of the ice thickness over the whole domain (the volume) i.e.

$$\int_0^{b(t)} H(x,t)dx = \theta(t), \text{ say.} \tag{3.6}$$

and see how this changes over time.
Using Leibniz's integral rule, and applying our boundary conditions we see that

$$\begin{aligned}
\frac{d}{dt}\int_0^{b(t)} H(x,t)dx &= \int_0^{b(t)} \frac{\partial H}{\partial t}dx + H(b(t),t)\frac{db(t)}{dt} - 0 \\
&= \int_0^{b(t)} \frac{\partial}{\partial x}\left[cH^5 H_x^3\right]dx + \int_0^{b(t)} s(x)dx \\
&= \left[cH^5 H_x^3\right]_0^{b(t)} + \int_0^{b(t)} s(x)dx \\
&= \int_0^{b(t)} s(x)dx. \qquad (3.7)
\end{aligned}$$

The physical equivalent says that any change in the integral of ice thickness over the whole glacier, or equivalently any change in the ice volume, is due only to the snow term, which represents the net accumulation/ablation of snow over the whole glacier.

3.3 Velocity

In order to use a moving grid we need be able to define a velocity, v, at any arbitrary point. As with most of the variables this velocity is vertically averaged through the ice thickness. To define this velocity we need to make some form of assumption, of which there are two that will be considered here, then computationally analysed over the next few chapters.

The first method (in section 3.3.1) is to assume equation 3.7 in any moving subdomain $[0, \widehat{x}(t)]$ of $[0, b(t)]$, holds with $\widehat{x}(t)$ instead of $b(t)$. In physical terms this velocity is such that the ice volume changes only due to accumulation/ablation of snow locally. The second method (in section 3.3.2) assumes that the normalised volume $\frac{1}{\theta}\int_0^{\widehat{x}(t)} H(x,t)dx$ remains constant in time, i.e. the ice volume fraction remains constant as the glacier moves.

There are other assumptions that have been made in generating a velocity, and could be made here, such as assuming that each point in the domain is connected to its neighbours via 'springs', but they are not considered here.

3.3.1 Method 1 - Subdomain Assumption

In the first method assume that

$$\frac{d}{dt}\int_0^{\widehat{x}(t)} H(x,t)dx = \int_0^{\widehat{x}(t)} s(x)dx, \tag{3.8}$$

for each subdomain $(0, \widehat{x}(t))$. Then using Leibniz's integral rule again on the LHS of (3.8) we get

$$\begin{aligned}
\frac{d}{dt}\int_0^{\widehat{x}(t)} H(x,t)dx &= \int_0^{\widehat{x}(t)} \frac{\partial H}{\partial t}dx + H(\widehat{x}(t),t)\frac{d\widehat{x}(t)}{dt}|_0^{\widehat{x}} \\
&= \int_0^{\widehat{x}(t)} \frac{\partial}{\partial x}[D(H,H_x)]dx + \int_0^{\widehat{x}(t)} s(x)dx + H(\widehat{x}(t),t)\frac{d\widehat{x}(t)}{dt} \\
&= [D(H,H_x)]_0^{\widehat{x}(t)} + \int_0^{\widehat{x}(t)} s(x)dx + H(\widehat{x}(t),t)\frac{d\widehat{x}(t)}{dt} \\
&= D(H,H_x)|_{\widehat{x}} + \int_0^{\widehat{x}(t)} s(x)dx + H(\widehat{x}(t),t)\frac{d\widehat{x}(t)}{dt},
\end{aligned}$$

since $H = D = 0$ at $x = 0$. Our assumption (3.8) means that

$$D(H,H_x) + H(\widehat{x}(t),t)\frac{d\widehat{x}(t)}{dt} = 0,$$

so the velocity $v = d\widehat{x}/dt$ is driven only by the diffusion term and we get

$$v = \frac{d\widehat{x}(t)}{dt} = -\frac{D(H,H_x)}{H} = -cH^4H_x^3. \tag{3.9}$$

Expressing the velocity in this manner apparently presents problems when dealing with the boundary condition $H = 0$ at $x = b$, giving a zero velocity at the right boundary, resulting in a glacier that will never move, which we know physically is not the case. However it is perfectly possible for v to be non-zero as long as H_x is infinite.

3.3.2 Method 2 - Normalisation Assumption

Method 2 assumes that an ice volume fraction remains constant in time, i.e.

$$\frac{d}{dt}\left(\frac{1}{\theta(t)}\int_0^{\widehat{x}(t)} H(x,t)dx\right) = 0 \tag{3.10}$$

$$\Rightarrow \frac{1}{\theta(t)}\int_0^{\widehat{x}(t)} H(x,t)dx = \mu,\ \text{say}, \tag{3.11}$$

where we define $\theta(t)$ to be the total volume of the ice, equation (3.6).

Differentiating 3.10, we get

$$-\frac{\theta'}{\theta^2}\int_0^{\widehat{x}(t)} H(x,t)dx + \frac{1}{\theta}\frac{d}{dt}\int_0^{\widehat{x}(t)} H(x,t)dx = 0,$$

and then by applying by Leibniz's integral rule and our boundary conditions

$$-\frac{\theta'}{\theta^2}\int_0^{\widehat{x}(t)} H(x,t)dx + \frac{1}{\theta}\left[D + \int_0^{\widehat{x}} sdx + H(\widehat{x},t)\frac{d\widehat{x}}{dt}\right] = 0, \tag{3.12}$$

where $\theta' = \frac{d\theta}{dt}$. From here we can then rearrange to get the velocity.

$$\begin{aligned}\frac{d\widehat{x}}{dt} &= \frac{\theta'}{H\theta}\int_0^{\widehat{x}(t)} H(x,t)dx - \frac{D}{H} - \frac{1}{H}\int_0^{\widehat{x}} sdx \\ &= \frac{\mu\theta'}{H} - cH^4H_x^3 - \frac{1}{H}\int_0^{\widehat{x}} sdx \\ &= \frac{1}{H}\left[\mu\theta' - \int_0^{\widehat{x}} sdx\right] - cH^4H_x^3\end{aligned} \tag{3.13}$$

Note that we have substituted the constant μ from Equation 3.11, and that the final term in the equation is the same as the velocity in method one. Note also that when $\widehat{x} = b(t)$, at the snout of the glacier, the first term disappears and the velocity reduces to the same as in method 1.

Equation (3.16) requires θ' ($= \frac{d\theta}{dt}$), from 3.7

$$\begin{aligned} \theta' &= \frac{d}{dt}\int_0^b H dx \\ &= \int_0^b s dx. \end{aligned} \tag{3.14}$$

The analysis of this method will be carried out in Chapter 5.

3.4 Snout Profile

From (3.9) we have the useful form

$$\begin{aligned} v = \frac{d\widehat{x}}{dt} &= -c(H^{4/3}H_x)^3 \\ &= -c\left[\frac{3}{7}(H^{7/3})_x\right]^3 \\ &= -\frac{27}{343}c\left[(H^{7/3})_x\right]^3. \end{aligned} \tag{3.15}$$

Which shows that v does not have this problem at the right hand boundary, since it is perfectly possible for $(H^{7/3})_x$ to be non-zero, as long as H_x is infinite. Hence this is the velocity for method 1, and method 2 becomes

$$v = \frac{1}{H}\left[\mu\theta' - \int_0^{\widehat{x}} s dx\right] - \frac{27}{343}c\left[(H^{7/3})_x\right]^3. \tag{3.16}$$

This point will be examined further in Chapter 4.

When expressing the velocity in this manner it is worth considering what will happen when we substitute the initial expression for H, from equation (3.4).

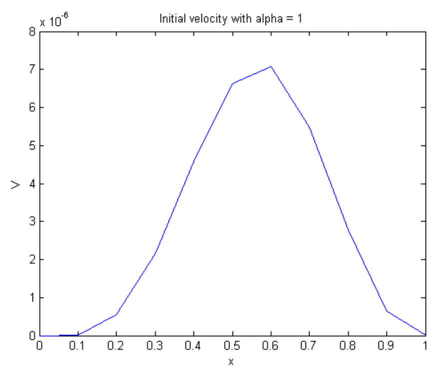
$$\begin{aligned} H^{\frac{7}{3}} &= (1-x^2)^{\frac{7\alpha}{3}} \\ (H^{7/3})_x &= 2x.\frac{7\alpha}{3}(1-x^2)^{\frac{7\alpha}{3}-1}. \end{aligned} \tag{3.17}$$

This has some interesting properties as $x \longrightarrow 1$, depending on the value of α.

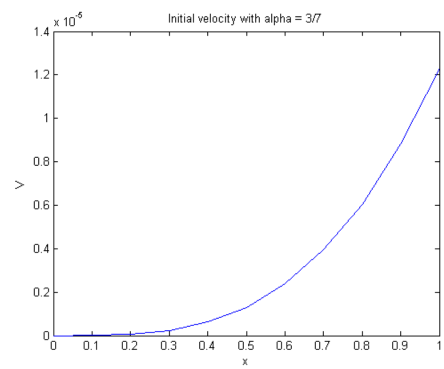
$$\begin{aligned} \text{Case 1: } \frac{7\alpha}{3} &> 1, \Rightarrow (H^{7/3})_x \text{ is zero} \\ \text{Case 2: } \frac{7\alpha}{3} &< 1, \Rightarrow (H^{7/3})_x \text{ is infinite} \\ \text{Case 3: } \frac{7\alpha}{3} &= 1, \Rightarrow (H^{7/3})_x \text{ is finite} \end{aligned}$$

In case 1, from equation (3.15), the initial velocity of the snout of the glacier is zero, and it is stationary; the chosen setting of $\alpha = 1$ satisfies this, as shown in Fig.3.2(a). In case 2, we get an infinite velocity, which is not physical and the model is incorrect. In case 3 we get a finite initial velocity value at the snout when $\alpha = \frac{3}{7}$, which would be the point when the right hand boundary starts to move, as observed in Fig.3.2(b). From this analysis we can expect H to be of this form asymptotically at the moment of initial movement. Note that H_x is infinite at the boundary. This point will be considered again when we seek an estimate of the time that the glacier will start to move.

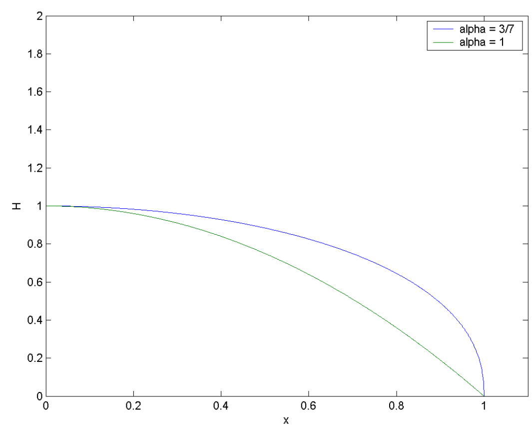
For a glacier under case 1, we would expect the changing shape of H over time to be due to the snow term. If it asymptotically reaches case 3 at the snout then the end point will move.



(a) Initial V under case 1 ($\alpha = 1$)



(b) Initial V under case 3 ($\alpha = 3/7$)



(c) Initial Ice Thickness under different α

Figure 3.2: *Analysis of initial velocity*

3.4.1 Surface Elevation

A flat bed model can be limiting when considering physical examples. For glaciers that are situated in the mountains the bed will almost certainly be sloped. Consider now the domain in Fig.3.3, where we have a linear slope. H is still defined as the thickness of the ice, but we have an additional variable h which is adds the surface elevation to the thickness of the ice.

With the addition of h, the initial equation (3.1) is slightly different, and

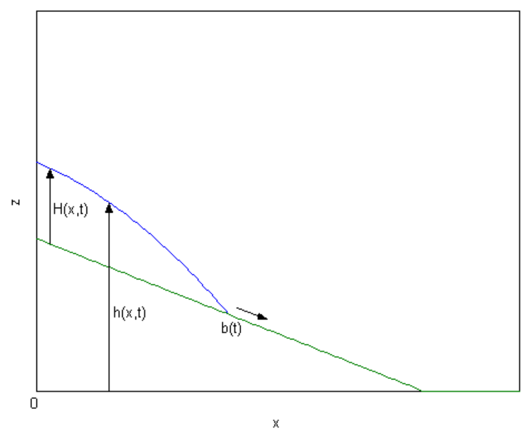


Figure 3.3: *Domain with surface elevation*

becomes the following:

$$\begin{aligned} \frac{\partial H(x,t)}{\partial t} &= \frac{\partial}{\partial x}\left[cH(x,t)^5 h_x(x,t)^3\right] + s(x) \\ &= (D(H, h_x))_x + s(x) \end{aligned} \tag{3.18}$$

where c is defined as in 3.1. Similar boundary conditions apply, as before we have $H = 0$ at $x = b(t)$ and now $\frac{\partial h}{\partial x} = 0$ at $x = 0$.

To calculate the grid velocity we are going to use the same assumption that is made in method 1 (section 3.3.1). Following through we end up with a similar equation

$$(v =)\frac{d\widehat{x}(t)}{dt} = -\frac{D(H, H_x)}{H} = -cH^4 h_x^3. \quad (3.19)$$

For the velocity to be finite at the snout where $H = 0$, we must have $h_x = \infty$. Since we cannot express equation (3.19) in the same form that we did before the problem at the right hand boundary still occurs.

In addition we also need to define the slope, Γ. This will be set to

$$\Gamma = -x + 5, \quad (3.20)$$

chosen so that the glacier is initially small in comparison with the slope. We now use these model descriptions to analyse and compute glacier behaviour.

Chapter 4

Computation from Method 1: Subdomain Assumption

4.1 Numerical Grid

Equation 3.1 is non-linear and thus difficult to solve analytically, so we seek a numerical approximation via a grid. As a 1D problem, the domain only needs to be divided up along the x-direction. Since the problem involves a moving boundary, a natural description is to use a moving grid. The grid points will need to be updated every time step, since as the glacier moves we expect the grid to follow, giving a moving grid problem.

The initial grid is chosen to be evenly spaced at the initial time, although there is potential to introduce a non-evenly spaced grid, particularly near the right hand boundary to give more information about the velocity and movement at the snout of the glacier.

4.2 Numerical Approximation

For the velocity in equation (3.15) we can use our grid from above to form an approximation using an upwind difference

$$v_i = -\frac{27}{343}c\left(\frac{H_i^{7/3} - H_{i-1}^{7/3}}{x_i - x_{i-1}}\right)^3. \tag{4.1}$$

At each time step the velocity can be calculated, from which we can then use equation (4.1) to update the x and H values. For the new x values we approximate Equation 3.15 using forward Euler time stepping,

$$\begin{aligned} \frac{d\widehat{x}(t)}{dt} &= v \\ \frac{\widehat{x}^{n+1} - \widehat{x}^n}{\Delta t} &= v \\ \widehat{x}^{n+1} &= \widehat{x}^n + v.\Delta t. \end{aligned} \tag{4.2}$$

To determine H we go back to the assumption we made in equation (3.8), and use the same time-stepping scheme. Note that the limits in (3.8) have been changed to allow the midpoint rule to be applied for computational simplicity.

$$\frac{\int_{j-1}^{j+1} H^{n+1}dx - \int_{j-1}^{j+1} H^n dx}{\Delta t} = \int_{j-1}^{j+1} s dx.$$

Then, using the midpoint rule we get

$$(x_{j+1}^{n+1} - x_{j-1}^{n+1})H_j^{n+1} - (x_{j+1}^n - x_{j-1}^n)H_j^n = \Delta t(x_{j+1}^n - x_{j-1}^n)s_j^n,$$

giving

$$H_j^{n+1} = \frac{(x_{j+1}^n - x_{j-1}^n)}{(x_{j+1}^{n+1} - x_{j-1}^{n+1})}(H_j^n + \Delta t s_j^n). \tag{4.3}$$

4.3 Results

What does the velocity do over time under the assumption in equation (3.8)? From Fig.4.1 we can see that the velocity builds up into a dome shape before the peak begins to move towards the right hand boundary, eventually reaching it and pushing the boundary of the glacier into movement. This resembles the solution of a non-linear differential equation that generates a shock, something which we consider further in Section 7.

How does the ice thickness behave under this velocity profile? From Fig.4.2(a), we can see that the glacier does not move past its initial endpoint until $t = 10000$, after the time that it has been reached by the shock. The change in ice thickness appears to be initially dominated by the build up of snow rather than the diffusion term, hence why the glacier appears to grow high before some of the snow term becomes negative and the diffusion can take over, pulling the glacier back down to a more reasonable shape. From a physical perspective this perhaps is not the most realistic of movements.

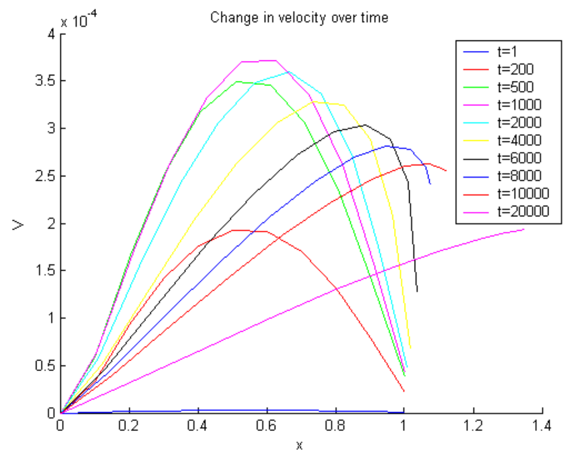
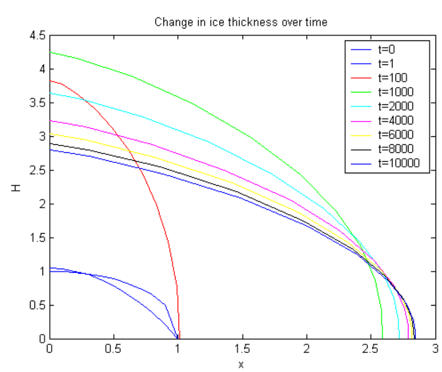
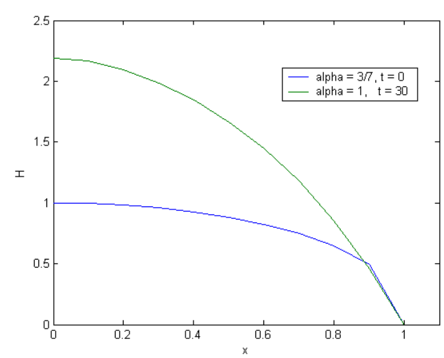


Figure 4.1: *Velocity under method 1*



(a) Change in ice thickness



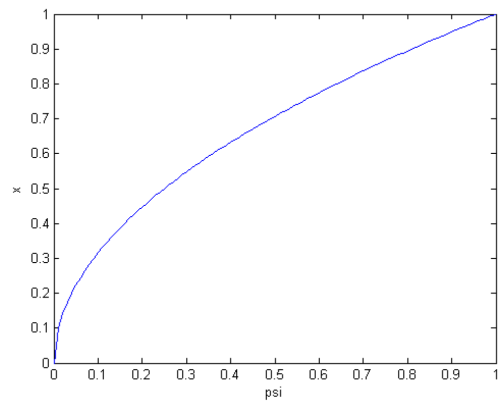
(b) Comparison with $\alpha = 3/7$

Figure 4.2: *Results of the subdomain assumption*

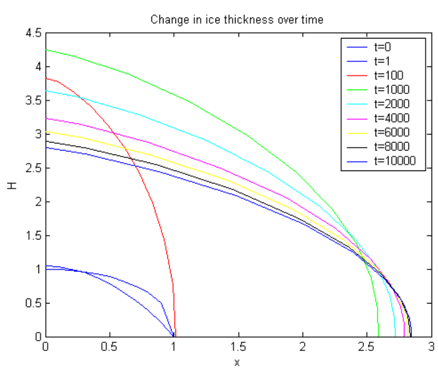
Up until now, the results have come from an initial H with $\alpha = 1$, so let us compare these results with those in Section 3.4. As you can see in Fig.4.2(b), the gradients at the snout are very similar, which is as we expected. Though we expect to see an infinite gradient at the boundary, we note that this is not possible numerically, due to the grid spacing.

4.4 Non-Evenly Spaced Grid

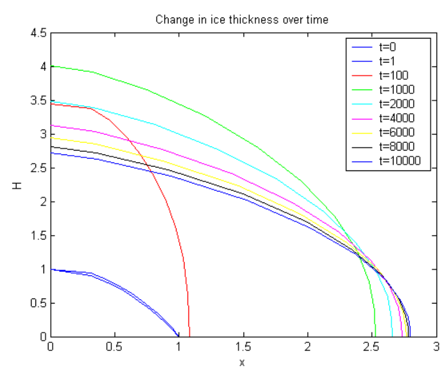
Since it is clear that the important area is the snout perhaps it might be an idea to focus the grid on this point to improve accuracy there. To do this we declare a new space variable, ψ, which takes on the role of the existing evenly spaced grid. Then we define $x = \sqrt{\psi}$. As you can see in Fig. 4.3(a), the region close to the snout at 1 is covered by more points under x. The main question is does this have an impact on the other variables. Looking at Fig.4.3, it would appear that the general shape is very similar, though the lack of information near the left hand boundary does cause a bit of inaccuracy, which is to be expected since this area is less well resolved. Perhaps the most interesting fact is that it appears that the increased resolution of the non-even grid, Fig.4.3(c) starts moving before the evenly spaced grid, judging by the location of the line $t = 100$. This might lead to an underestimated waiting time if calculated on the evenly spaced grid.



(a) Difference in grid spacing variables



(b) Even spaced grid



(c) Non-Even grid

Figure 4.3: *Computation using the non-evenly spaced grid*

Chapter 5

Computation from Method 2: Normalisation Assumption

Calculating the velocity by method 2 from section 3.3.2 requires a little more work than for the subdomain assumption. Recalling equation 3.16, we now have more terms to handle. To start with, the constant μ, which will be a vector, can be defined at the outset since this will not need to be updated. Any integrals can be estimated numerically by the trapezium rule via a sum, firstly approximating θ by

$$\theta = \sum_{j=0}^{b} 0.5(H_j + H_{j-1})(x_j - x_{j-1})$$

and then the integral up to each $\widehat{x}_i$ by

$$\int_0^{\widehat{x}_i} H dx = \sum_{j=0}^{i} 0.5(H_j + H_{j-1})(x_j - x_{j-1}).$$

This leads us to define μ numerically as

$$\mu_i = \frac{\sum_{j=0}^{i} 0.5(H_j + H_{j-1})(x_j - x_{j-1})}{\sum_{j=0}^{b} 0.5(H_j + H_{j-1})(x_j - x_{j-1})}$$

We can use a similar approximation to determine the integrals of the snow term, giving us

$$\begin{aligned}\theta' = \int_0^b s dx &= \sum_{j=0}^{b} 0.5(s_j + s_{j-1})(x_j - x_{j-1}), \text{ and} \\ \int_0^{\widehat{x}_i} s dx &= \sum_{j=0}^{i} 0.5(s_j + s_{j-1})(x_j - x_{j-1}).\end{aligned}$$

Putting this all together gives us a numerical expression for the velocity

$$\begin{aligned}v_i = \frac{1}{H_i}\Bigg[\frac{\sum_{j=0}^{i} 0.5(H_j + H_{j-1})(x_j - x_{j-1})}{\sum_{j=0}^{b} 0.5(H_j + H_{j-1})(x_j - x_{j-1})} \sum_{j=0}^{b} 0.5(s_j + s_{j-1})(x_j - x_{j-1}) \\ - \sum_{j=0}^{i} 0.5(s_j + s_{j-1})(x_j - x_{j-1})\Bigg] - \frac{27}{343} c \left[(H_i^{7/3})_x\right]^3\end{aligned} \tag{5.1}$$

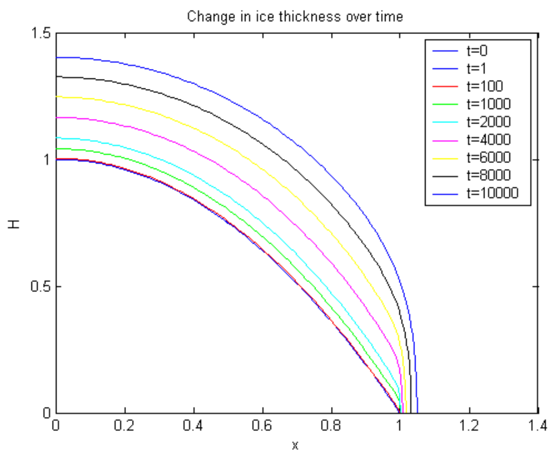
As with method 1 we need to update the x and H values, but additionally for method 2 the new value of θ also needs to be calculated. Note that x can be updated in the same way as in Equation 4.2, since that equation is the same. θ can be calculated using Equation 3.14,

$$\begin{aligned}\frac{\theta^{n+1} - \theta n}{\Delta t} &= \int_0^b s dx \\ \theta^{n+1} &= \theta^n + \Delta t \int_0^b s dx.\end{aligned} \tag{5.2}$$

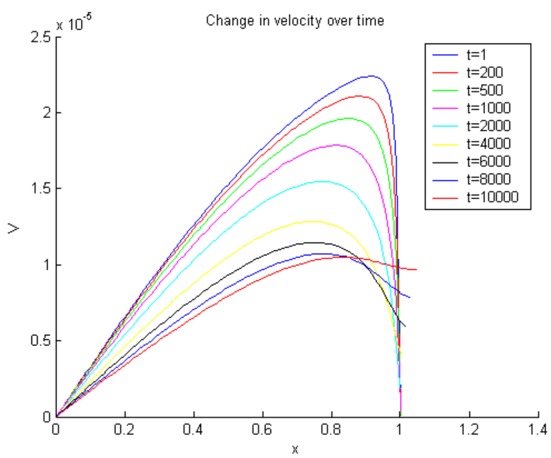
To update H, we note that since μ is constant in time, then it follows that at any time n

$$\begin{aligned}
\frac{1}{\theta(t_n)}\int_0^{\widehat{x}^n} H^n dx &= \frac{1}{\theta(0)}\int_0^{\widehat{x}^0} H^0 dx \\
\frac{1}{\theta(t_n)}\int_{x_{i-1}^n}^{x_{i+1}^n} H^n dx &= \frac{1}{\theta(0)}\int_{x_{i-1}^0}^{x_{i+1}^0} H^0 dx \\
\frac{1}{\theta(t_n)} H_i^n (x_{i+1}^n - x_{i-1}^n) &= \frac{1}{\theta(0)} H_i^n (x_{i+1}^0 - x_{i-1}^0) \\
H_i^n &= \frac{\theta(t_n)}{\theta(0)} H_i^0 \frac{(x_{i+1}^0 - x_{i-1}^0)}{(x_{i+1}^n - x_{i-1}^n)}. \qquad (5.3)
\end{aligned}$$

However when running this program, it became unstable very quickly, which turned out to be caused by the snow term. Reducing the coefficients in equation 3.5 to 0.0005 fixed the problem, the consequences being that movement was not as quick as in method 1. As you can see from Fig.5.1(a), this actually yields more realistic results in that the glacier reaches the required gradient at the end point and begins to move without a large build up of snow. Reducing the snow coefficient did not have the same effect in method 1, since this method is more dependent on the snow. In Fig.5.1(b) we again see evidence of a shock occurring, but unfortunately because the velocity (5.1) is considerably more complex, it does not lend itself to the shock theory described in Chapter 7 like the subdomain assumption.



(a) Ice Thickness



(b) Velocity

Figure 5.1: *Results from the normalisation assumption*

Chapter 6

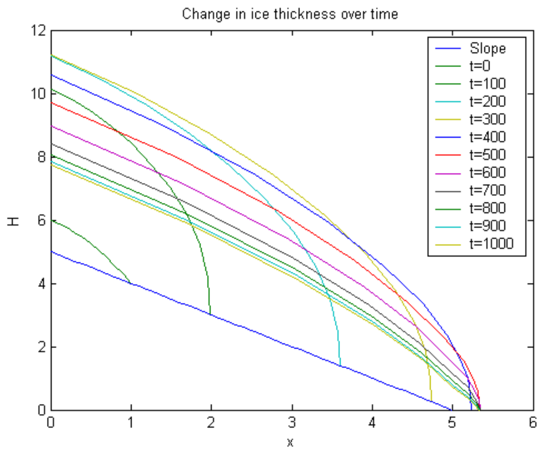
Surface Elevation Model

For the most part the theory for the surface elevation model is similar to Chapter 4, as we are making the same assumption for a different domain. The slope is defined as in equation (3.20), the ice thickness H will be the same as before, and h is defined as

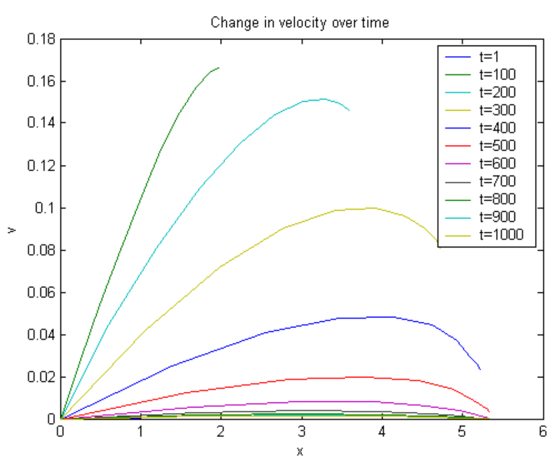
$$h = H + \Gamma.$$

As we would expect, the slope increases the velocity and causes the glacier to move quicker down the mountain, something which is clearly evident in Fig.6.1(a). However once the glacier is off the slope it seems to grind to a halt, which we see in the velocity plot, Fig.6.1(b). This is due to the linear snow term, since once x gets past the value 2 we see a negative contribution of snow (ablation), which becomes increasingly negative the further down we go.

Once the glacier has left the slope, the gradient decreases, so the diffusion term has less effect and from Chapter 4 we saw that the snow term is dominant under the subdomain assumptions.



(a) Ice Thickness



(b) Velocity

Figure 6.1: *Results with surface elevation*

6.1 Basal Sliding

At the end of chapter 2 we mentioned the concept of basal sliding. Depending on the characteristics of a glacier, this can account for a significant part of its movement. Since the models we have been using involve a vertically averaged velocity, the concept of basal sliding need only be considered as an additional velocity rather than a velocity at the base. Van Der Veen [4] proposed including the basal sliding as an additional part of the diffusion term. From 3.1

$$\frac{\partial H(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[cH(x,t)^5 H_x(x,t)^3 + H(x,t)V_{bs}(x,t)\right] + s(x) \tag{6.1}$$

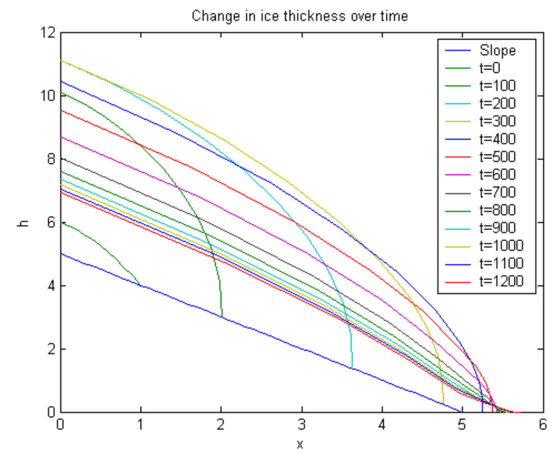
where V_{bs} is the additional term for basal sliding. Working through as in 3.3.1 we find that the velocity term becomes

$$(v =)\frac{d\widehat{x}(t)}{dt} = -cH^4H_x^3 - V_{bs}. \tag{6.2}$$

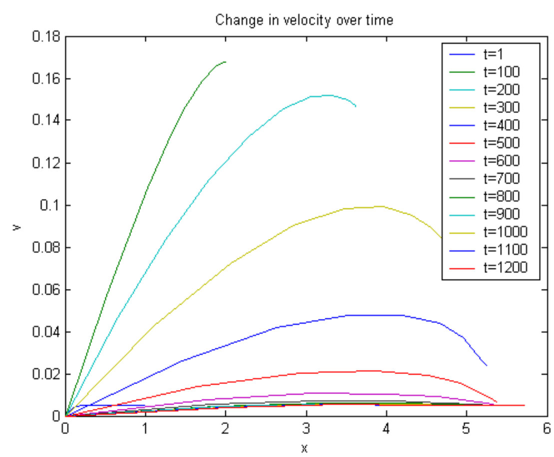
Now all we need is a suitable value to set the basal sliding velocity; Hutter [6] suggests a maximum of 0.005, which we shall use here.

Fig.6.2(a) shows what happens when we add this term into the equation. We can see that not much is different while the glacier is still on the slope besides a slight increase in how far down the slope the snout is. It is when the glacier reaches the bottom that things get a little more interesting. Where as before the glacier appeared to grind to a halt, the basal sliding causes the end to keep moving, pulling the rest of the glacier with it. This has some physical implications in that we see glaciers spreading out when they reach a valley at the bottom, usually known as Piedmont Glaciers.

From Fig.6.2(b) we note that with the addition of basal sliding the velocity never slows down to zero, and if left long enough the glacier would keep moving and stretching out until it becomes a thin sheet.



(a) Ice Thickness



(b) Velocity

Figure 6.2: *Results with surface elevation and Basal Sliding*

Chapter 7

A Shock Equation

From equation 3.9 recall that under the subdomain assumption, the velocity is given by $v = -cH^4H_x^3$. Although v is given explicitly it is useful to look at an equation satisfied by v. To derive such an equation we are interested in the time and space derivative; so differentiating v we get

$$v_x = -3cH^4H_x^2H_{xx} - 4cH^3H_x^4 \tag{7.1}$$

$$v_t = -3cH^4H_x^2H_{xt} - 4cH^3H_x^3H_t \tag{7.2}$$

In equations (7.1) and (7.2) we have the functions H_t and H_{xt} that we do not know. To deal with this, and keep the equations in terms of space derivatives, we can substitute equation (3.9) into the original equation (3.1).

$$\begin{aligned} H_t &= s - (Hv)_x \\ &= s - H_xv - Hv_x, \end{aligned} \tag{7.3}$$

which we can differentiate to get

$$H_{tx} = s_x - H_{xx}v - 2H_xv_x - Hv_{xx}. \tag{7.4}$$

These can be substituted into 7.2 to give

$$\begin{aligned} v_t &= -3cH^4H_x^2(s_x - H_{xx}v - 2H_xv_x - Hv_{xx}) - 4cH^3H_x^3(s - H_xv - Hv_x) \\ &= 3cH^5H_x^2v_{xx} + (10cH^4H_x^3)v_x + (3cH^4H_x^2H_xx + 4cH^3H_x^4)v - 3cH^4H_x^2s_x - 4cH^3H_x^3s \\ &= 3cH^5H_x^2v_{xx} - 10vv_x - v_xv - 3cH^4H_x^2s_x - 4cH^3H_x^3s, \end{aligned}$$

and finally rearranged in the form of a Burgers-like equation with extra source terms.

$$v_t + 11vv_x = 3cH^5H_x^2v_{xx} - 3cH^4H_x^2s_x - 4cH^3H_x^3s. \tag{7.5}$$

We shall use this equation to characterise the evolution of v and estimate a waiting time.

7.1 Numerical Approximation to a Burgers equation

With equation 7.5 we can form an approximation to the change in velocity over time. We expect this to reflect the change of velocity that we see in Fig. 4.1.

Each of the velocity derivative terms have been numerically approximated, again using an upwind difference,

$$v_t = \frac{v_i^{n+1} - v_i^n}{\Delta t} \tag{7.6}$$

$$vv_x = \frac{1}{2}\frac{(v_i^n)^2 - (v_{i-1}^n)^2}{\Delta x} \tag{7.7}$$

$$v_{xx} = \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{(\Delta x)^2}. \tag{7.8}$$

The snow derivative term also appears in equation (7.5), but since s is a linear function (equation (3.5)), the derivative is constant, in our case

$$s_x = -0.25.$$

Substituting all of this into equation 7.5,

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} + \frac{11}{2}\frac{(v_i^n)^2 - (v_{i-1}^n)^2}{\Delta x} = 3cH^5H_x^2\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{(\Delta x)^2} + \frac{3}{4}cH^4H_x^2 - 4cH^3H_x^3 s, \tag{7.9}$$

which can be rearranged to give an explicit equation for v_i^{n+1}.

In order to implement 7.9 we need an initial state for velocity, which we can approximate from 4.1, by

$$v_i^0 = -\frac{27}{343}c\left(\frac{H_i^{7/3} - H_{i-1}^{7/3}}{x_i - x_{i-1}}\right)^3. \tag{7.10}$$

Note that since equation (7.5) depends on H (and as a result x), these variables will also need to be updated every time loop, which can be done using the same methods as we used in Chapter 4.

Plotting the solution to this equation, Fig. 7.1, it is encouraging to see that the velocity changes in a similar manner to what we saw in Fig. 4.1.

7.2 Characteristics

Using equation (7.5) as a check that our method produces the correct results is useful, but we can also use the equation to estimate when the shock occurs. To do this we use characteristics theory to observe that by the chain rule

$$v_t + \frac{dx}{dt}v_x = \frac{dv}{dt}.$$

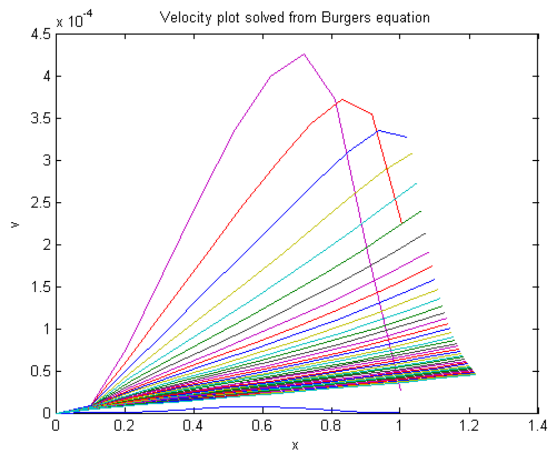


Figure 7.1: *Numerical solution to Burgers equation*

Comparing this with 7.5, we deduce that on characteristics

$$\frac{dx}{dt} = 11v \tag{7.11}$$

$$\frac{dv}{dt} = 3cH^5H_x^2v_{xx} - 3cH^4H_x^2s_x - 4cH^3H_x^3s. \tag{7.12}$$

7.2.1 Homogeneous Case

Now we have a system of ODEs, but before we try to solve the full equation, we can look at the homogeneous case,

$$v_t + 11vv_x = 0, \tag{7.13}$$

to get an idea of what a pure shock might do. For this equation

$$\frac{dx}{dt} = 11v, \tag{7.14}$$

$$\frac{dv}{dt} = 0. \tag{7.15}$$

Under this condition we see that

$$v = v_0, \text{ say,} \tag{7.16}$$

which, when substituted into equation (7.14) implies that

$$x = v_0 t + x_0. \tag{7.17}$$

This gives us the set of characteristics which we plot in Fig.7.3(a), and we can see that the characteristics cross towards the right. In a characteristics plot for a conservation law of the form of equation (7.13), any time lines cross a shock is generated, which moves forward in time. It is interesting to note that the shock occurs where the gradient of initial velocity was steepest, (see Fig.3.2(a)). This is in accordance with conservation law theory which says that the shock occurs when $v_0'(x)$ is a maximum. In addition, we can plot x against v, Fig.7.3(b), where we see the velocity overturning.

From here we would like to know the point of first crossing, since this is where the shock begins and can help us predict the waiting time we are looking for. We consider the envelope of the equations taken from equation (7.17), with x_0 as the parameter,

$$F(x, t, x_0) = x - v_0 t - x_0 = 0. \tag{7.18}$$

On the envelope

$$\frac{\partial F}{\partial x_0} = -\frac{\partial v_0(x_0)}{\partial x_0} t - x_0 = 0. \tag{7.19}$$

which leads us to

$$t = -\frac{1}{v_0'(x_0)}. \tag{7.20}$$

The envelope forms, and the shock appears at the earliest possible time satisfying (7.20), i.e.

$$t_min = \frac{1}{max[-v_0'(x_0)]}. \quad (7.21)$$

Now we have a time that the shock will occur in the homogeneous case, but we are also interested in the shock speed since this will help us predict the time the shock arrives at the boundary.

One way of calculating the shock speed is to look at the conservation property (Whitham, 1974). This work was used by C.P. Reeves, and states that we can replace the overturned part of the curve by a vertical line such that the areas A and B are the same, see Fig.7.2[11]. When the vertical line reaches the boundary, it is equivalent to saying that the shock has also reached the boundary. It is at this point that the boundary will begin to move. The theory is hard to put into practice though, as we normally visualise the equal areas, making this difficult to compute.

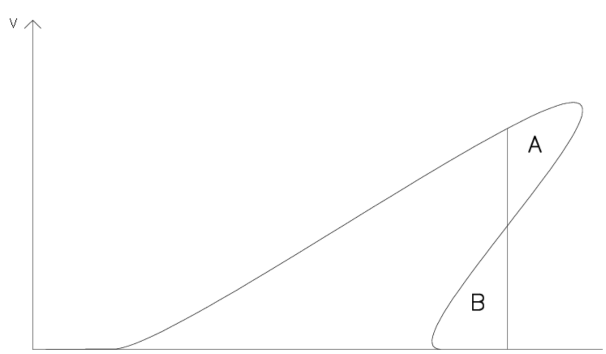
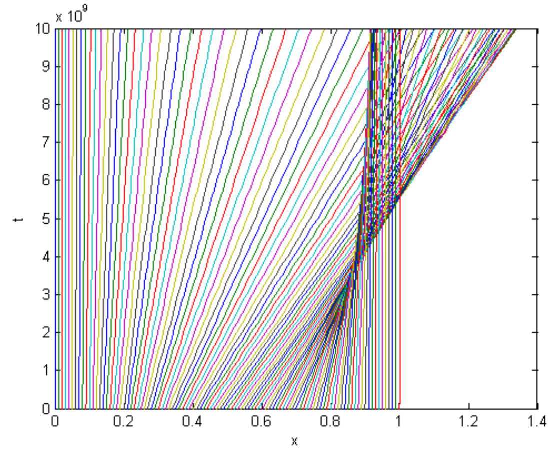
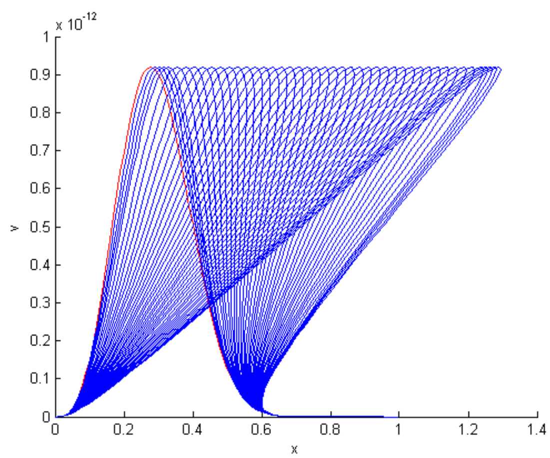


Figure 7.2: *Equal Area Construction*

We can also get a decent idea of what is going on through a 3D plot of x, v and t. The first of these, Fig.7.4(a) plots the same curves as in the 2D



(a) Characteristic plot



(b) Shock through characteristics

Figure 7.3: *2D plots of Characteristics*

Fig.7.3(b), but it is interesting to see with the added dimension the overturn developing. In addition Fig.7.4(b) shows how each individual point moves through time, and while the shape is the same, we can see which points move the most.

7.2.2 Inhomogeneous Case

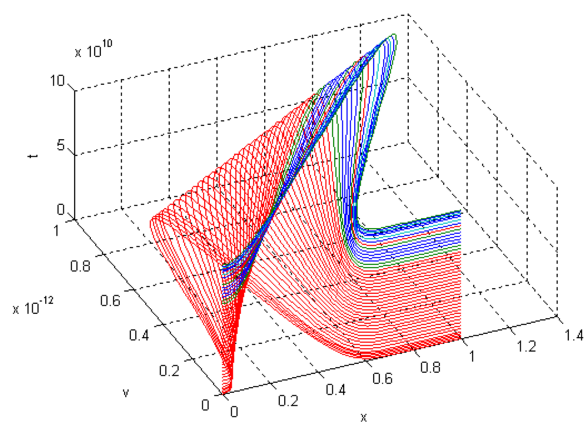
Now to try an find a solution to the general case. Firstly, the right hand side involves $H(x,t)$, an unknown quantity which changes with time, so an extra equation needs to be added to cope with this. Again by the chain rule

$$\frac{dH}{dt} = H_t + \frac{dx}{dt}H_x.$$

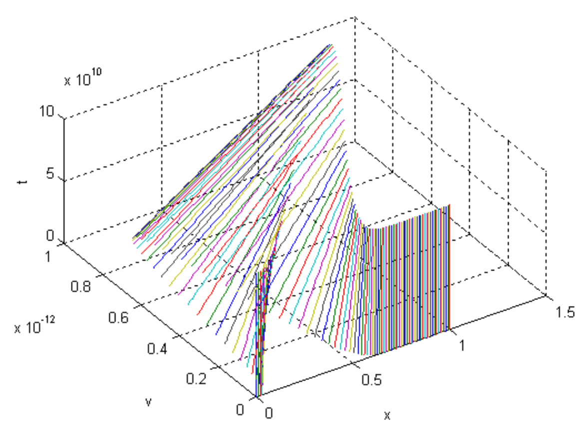
We can substitute in the terms we know from 7.14 and 3.1, giving

$$\frac{dH}{dt} = \left[cH^5 H_x^3\right]_x + s + 11vH_x. \tag{7.22}$$

Along with equations 7.14 and 7.15 we have a system of ODEs that we can compute. However when computing the system appeared to blow up, which we could fix by scaling the ice thickness H by a small factor $p = 0.000000002$, found by trial and error. Fig.7.5(a) shows the characteristics plot with curved characteristics leading to a shock. The corresponding velocity plot, Fig.7.5(b), show a much larger overturn, as well as a significant increase in velocity. Instead of falling to the right, as we see in Fig.7.3(b), the velocity grows at the same time as toppling over. We can also look at the 3D plot, Fig.7.5(c), where we can clearly see the increase in velocity through time. The equal area construction method would also work for the overturn in this



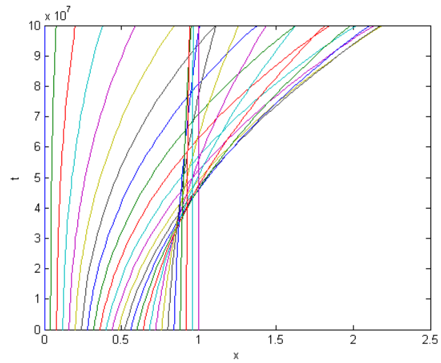
(a) Progression of shock through time



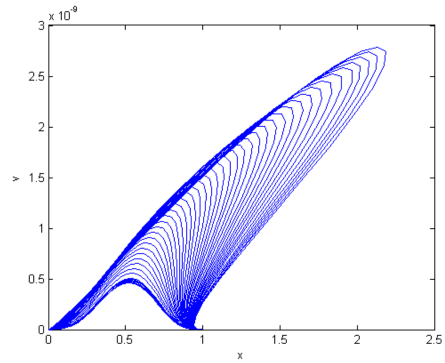
(b) Progression of shock points through time

Figure 7.4: *3D plots of Characteristics*

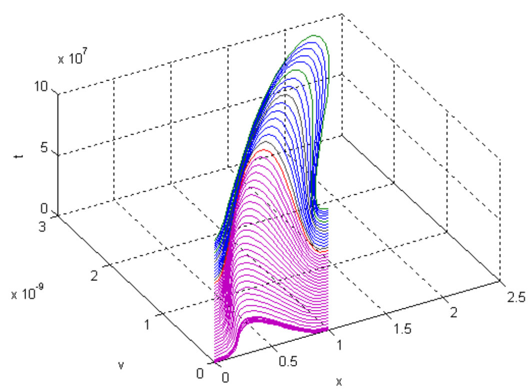
case. Due to the scaling the results here are speculative, and their value needs to be investigated.



(a) Characteristics Plot



(b) Velocity Plot



(c) Progression of shock through time

Figure 7.5: *Computation solutions to the inhomogeneous case*

Chapter 8

Conclusions and Further Work

8.1 Summary

This dissertation has looked at a number of techniques for modelling a one-dimensional glacier model using a moving grid. In addition we observed that glaciers experienced a waiting time and required certain circumstances before they began to move. As such we looked at combining the work of Roberts [1] on the 1D model, and Stojsavljevic [10] on waiting times for PDEs.

Firstly we took the flat bed model and used two methods of extracting the grid velocity. The first was a simpler solution, assuming a subdomain held the same properties as the whole domain. This was a physical approach, which gave results that appeared to be dominated by the snow term, something mentioned to look at as further work.
At this point we examined the effect of a non-evenly spaced grid, which gave

very slight different results. This was put to one side as something to consider later. It is suspected that on a model with more grid points this will not matter, but for a sparse grid there are benefits, if small.

The second method was a more mathematical approach, normalising the volume of ice to estimate a grid velocity. Unfortunately this method became unstable, probably due to the Euler time step which depends on Δt. Efforts to achieve stability were hampered by computational power. This method also did not admit the formation of a Burgers equation as in method 1. We decided that while this method worked in theory, the results only worked for a small amount of snow, so for our linear snow term the normalisation assumption was not the most beneficial. Once the source term has been readdressed the method could be reassessed.

Next we looked at changing the grid from a flat bed to a linear slope. We used the subdomain assumption, so the methodology was already set in place. The results were as expected with an increased velocity on the slope before slowing down once the glacier reaches the flat plane. We introduced the concept of basal sliding to this model, which increased the velocity, and once off the slope the glacier kept sliding. The effects of basal sliding in our model are limited due to the vertically averaged velocity.

Finally, we applied characteristics theory to the subdomain methods on a flat bed. First we formed and solved a Burgers Equation with source terms numerically and we saw similar results, which acted as a reinforcement on the work we had already done. Then we applied the method of characteristics to get a system of ODEs that we solved using Matlab's inbuilt solver. Firstly the homogeneous case gave good results that we were able to use to get an explicit formula for the shock formula. The inhomogeneous problem gave some output, but only after scaling down the ice thickness to avoid blow up. We noticed that the velocity increases while overturning, which was not seen in the homogeneous case. These results are dubious.

8.2 Further Work

Throughout this dissertation there were many considerations as to which areas to look at, most of which had to be put to one side. First of all there was the issue of the computational problems encountered in chapter 5. Continuation on the non-evenly spaced grid could also help give more information at the snout.
Some of the further work that takes the model a step further has been listed here as a potential for anyone interested in this area of research.

8.2.1 Snow Term

The snow term has shown all sorts of differences between methods. In Chapter 4 we observed that the snow term was completely dominant, in Chapter 5 we saw that it made the computation unstable and had to be turned down, and in chapter 6 we saw that the snow term ground the ice to a halt as soon as it left the slope. This is something which should be addressed before any of the other pieces of work, particularly as it appears to have a large effect on results. Possible solutions would be to find a different set of coefficients to give a more realistic quantity of snow, use a non-linear function, which would stop the snow getting increasingly negative, or another option is to have seperate terms for snow accumulation and ablation.

8.2.2 Maritime Boundary Conditions

Payne's paper [5] considers different boundary conditions at the snout of the glacier, steering away from $H = 0$. Physically this occurs when a glacier reaches the edge of a cliff or enters the ocean, and for cases where the glacier mostly sits on the water there is the added problem of buoyancy. Payne proposes a maritime boundary condition of the form

$$\frac{\partial v}{\partial x}|_{shelffront} = A\left[\frac{1}{4}\rho_i g\left(1-\frac{\rho_i}{\rho_w}\right)\right]^n h^n, \tag{8.1}$$

where ρ_w is an additional variable introduced for the density of the water.

8.2.3 Retreat and Break-up

Our model works well for glaciers that are advancing, but what happens when the melting increases and the glacier retreats, something that we are seeing more and more under global warming. This would require a negative velocity, so care needs to be taken that the grid points do not overlap one another.

In addition glaciers can also break-up, where large sections of ice completly break off from the main part and drift away or melt. This presents all sorts of problems with the ice thickness vanishing at points other than the boundary, and also we might see a positive H_x gradient in certain areas, which would affect the velocity. One method of avoiding points overlapping would be to remove some points when bits of the glacier break off.

8.2.4 Two Dimensional Model

The model we have been analysing throughout was only a 1D model, so a logical next step would be to take the model into 2D. This could mean one of two things;

The first is that we consider velocity throughout the height of the glacier instead of vertically averaging. When velocity varies in height we might see quite different movement, This would require grid points in height in addition to the x direction. We would expect the top and bottom to be moving quicker than the middle section, as such the impact of basal sliding can be analysed more effectively. This would also allow the uneven surface of the ground to be taken into consideration, and how much loose debris there is. Fig.8.1 from K.A. Lemke[12] shows her thoughts from a geology prospective, where there is a maximum sliding velocity depending on the quantity of debris, so being able to translate this into mathematical form would yield interesting results.

Alternatively we could keep the velocity averaged vertically and consider the domain in the y-plane, as opposed to a cross section in the x-plane as we have currently. This does require the addition of boundary conditions at the

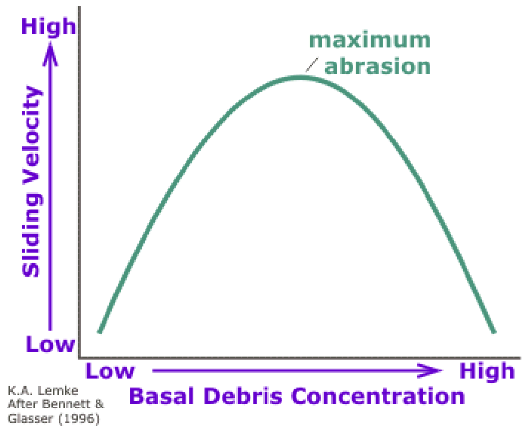


Figure 8.1: *Effect of Abrasion on Basal Sliding*

sides of the glacier, and the type would depend on if the sides meet a wall (no flux condition) or if they just curve to the ground, ($H = 0$). The model itself will take the form

$$H_t(x, y, t) = \underline{\nabla}.(H(x, y, t)^5 \underline{\nabla} H(x, y, t)^3) + s(x, y). \tag{8.2}$$

The results of such a model are expected to be of a similar form to the 1D model, and results such as equation (3.7) are believed to still hold.

For a particularly adventurous project a full blown 3D model is also possible, combining both the vertical velocity and the y-plane.

Bibliography

[1] Rhiannon Roberts: Modelling Glacier Flow, *University of Reading MSc Dissertation*, 2007.

[2] http://www.geo.cornell.edu/geology/eos/iceflow/model_description.html, Cornell EOS Project (last checked on 17.08.2009).

[3] B. Rae, D. Irving, B. Hubbard, J. McKinley: Preliminary investigations of centrifuge modelling of polycrystalline ice deformation. *Annals of Glaciology,* **Vol. 31**, 2000.

[4] C.J. Van Der Veen: *Fundamentals of Glacier Dynamics*, Taylor & Francis, 1999.

[5] A.J. Payne, A. Vieli: Assessing the ability of numerical ice sheet models to simulate grounding line migration. *Journal of Geophysical Research,* **Vol. 110**, 2005.

[6] K. Hutter: *Theoretical Glaciology, Material Science of Ice and the Mechanics of Glaciers and Ice Sheets*, Springer, 1983.

[7] P. Singh: *Snow and Glacier Hydrology*, Springer, 2001.

[8] M. Hambrey, J. Alean: *Glaciers*, Cambridge University Press, 1992.

[9] J. Diaz: *The Mathematics of Models for Climatology and Environment*, Springer, 1996.

[10] Jovan Stojsavljevic: Investigation of waiting times in non-linear diffusion equations using a moving mesh method. *University of Reading MSc Dissertation*, 2007.

[11] C.P. Reeves: Moving Finite Elements and Overturning Solutions. *University of Reading PhD Thesis*, 1991.

[12] http://www.uwsp.edu/geO/faculty/lemke/lemke.html, Karen Lemke Homepage (last checked on 19.08.2009).

[13] Dr Alex Lukyanov: Advection Equations and Conservation Laws. *University of Reading Course Notes*, 2008/2009.

[14] http://amsglossary.allenpress.com/glossary, AMS Glossary (last checked on 09.06.2009).